Functional Analysis
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Lecture 14
Exercises - Part 1

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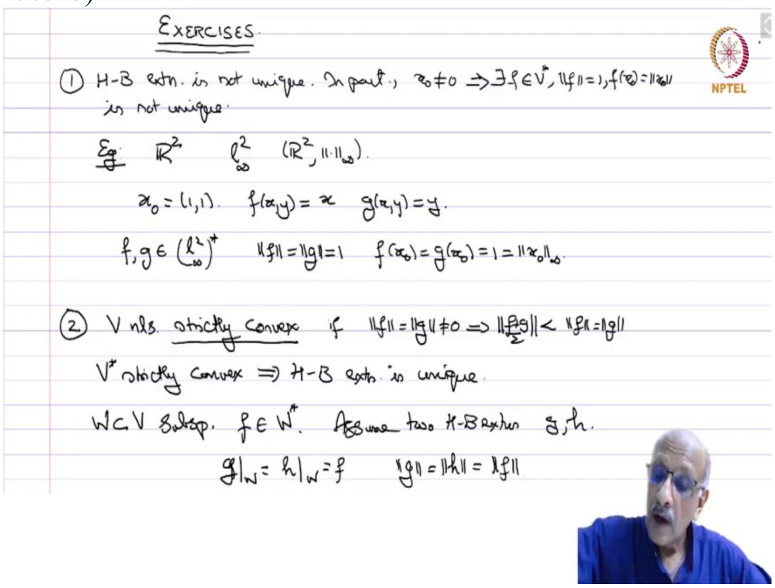

Exercise 1. The first one Hahn Banach extension is not unique. In particular, $x_0 \neq 0$ implies that there exists $f \in V^{¿}$, $\|f\|=1$, $f(x_0)=¿|x_0|\vee¿$ is not unique. So, let us give an example. Let us take $l_\infty^2=(R^2,\vee¿.\vee|_\infty)$ **and l2 infinity, so, this is R2 with the norm infinity.** Let us take $x_0=(1,1)$ and $f((x,y))=x, g((x,y))=y$. Any linear functional is continuous in finite dimension. So, $f,g \in (l_\infty^2)^{¿}$ and then it is very easy to check that $\|f\|=\|g\|=1$ (because $|f((x,y))| \leq |x| \leq \|(x,y)\|_\infty$ and $|g((x,y))| \leq |y| \leq \|x,y\|_\infty$ and then you take $(1,0)$ or $(0,1)$ as a test vector then you will get the maximum is achieved.

So, $f(x_0)=g(x_0)=\|x_0\|_\infty$ and therefore this is not unique.

So, when can we say the extension is unique? So, that is the next exercise.

Exercise 2. We say a normed linear space V is strictly convex, if $\|f\|=\|g\| \neq 0$ implies $\left\|\frac{f+g}{2}\right\|<\|f\|=¿|g|\vee¿$. If $V^{¿}$ is strictly convex, then Hahn Banach extension is unique.

Let W be a subspace and let $f \in W^{¿}$. Assume that we have two Hahn Banach extensions g, h. So, you have g restricted to W equals h restricted to W, which is equal to f and $\|g\| = \|h\| = \|f\|$. So, these are the two Han Banach extensions we have.

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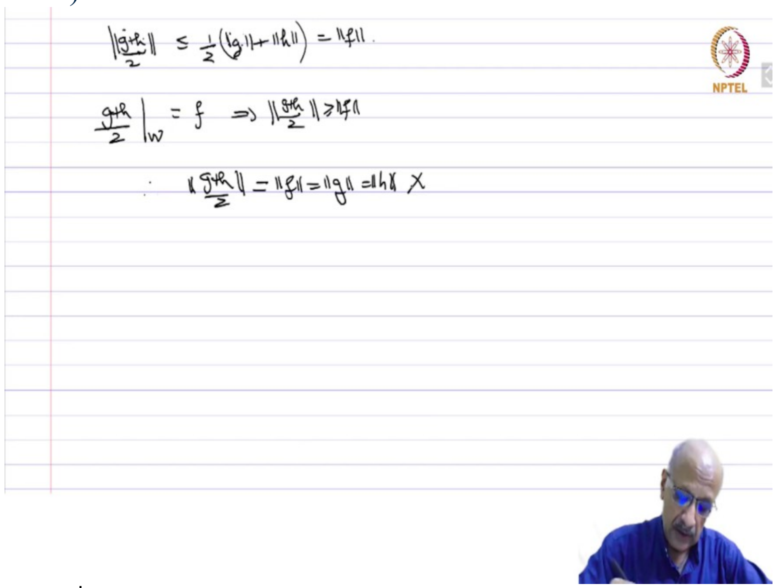

Then you look at $\left\|\frac{g+h}{2}\right\| \leq \frac{1}{2}(\|g\| + \|h\|) = ¿|f| \vee ¿$. On the other hand, $\frac{g+h}{2}$ restricted to W equals f (since both of them are equal to f) and therefore, $\left\|\frac{g+h}{2}\right\| \geq \|f\|$(because it is an extension). Therefore, $\left\|\frac{g+h}{2}\right\| = \|f\| = \|g\| = ¿|h| \vee ¿$ and that contradicts the fact that $V^{¿}$ is strictly convex. Therefore, you have the Hahn Banach theorem extension is unique.

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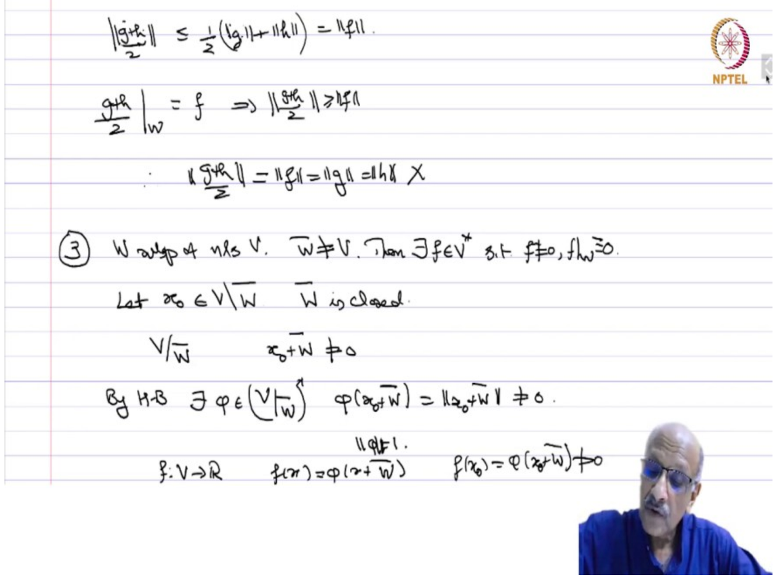

Exercise 3 Let us take a subspace W of a normed-linear space V and $\overline{W} \neq V$. Then there exists a $f \in V^{¿}$ such that $f \neq 0$ and f f restricted to W is identically 0.

So, this is the way of proving subspaces are dense. So, we have proved this using the fact that you had the separation of convex sets. Now, we want to do it using the extension method.

Proof. Let $x_0 \in V \setminus \overline{W}$. Then you can consider $\frac{V}{\overline{W}}$ and you have $x_0 + \overline{W} \neq 0$. So, by Hahn Banach, there exists a $\phi \in \left(\frac{V}{\overline{W}}\right)^{¿}$ such that $\phi(x_0 + \overline{W}) = \|x_0 + \overline{W}\| \neq 0$ and $\|\phi\| = 1$. Now, we will define $f: V \mapsto R$. We define $f(x) = \phi(x + \overline{W})$. So, in particular, $f(x_0) = \phi(x_0 + \overline{W}) \neq 0$. So, $f \neq 0$. Is it a continuous?.

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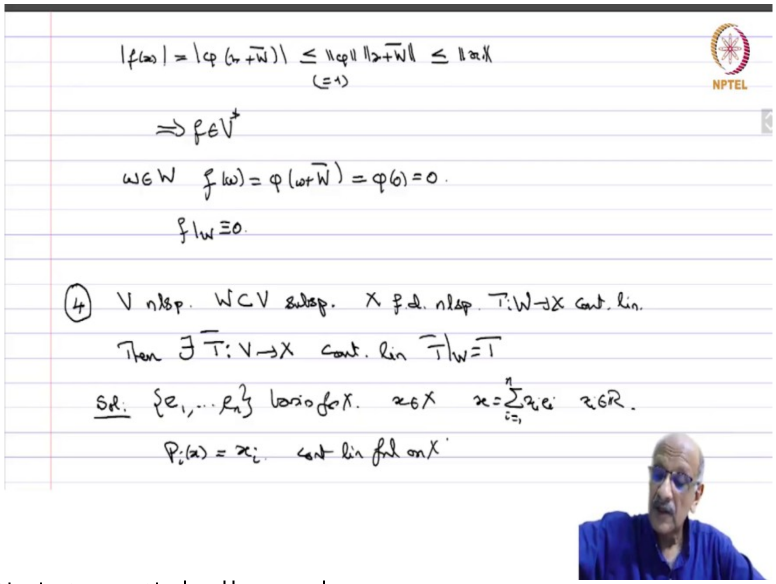

What about $\left|f(x)\right| = \left|\phi(x+\overline{W})\right| \le \left\|\phi\right\| \left\|x+\overline{W}\right\| \le \vee |x| \vee$¿. Therefore, f is continuous i.e., $f \in V$¿. Finally, if $w \in W$, we have $f(w) = \phi(w+\overline{W}) = \phi(0) = 0$. Hence, f restricted to W is identically 0.

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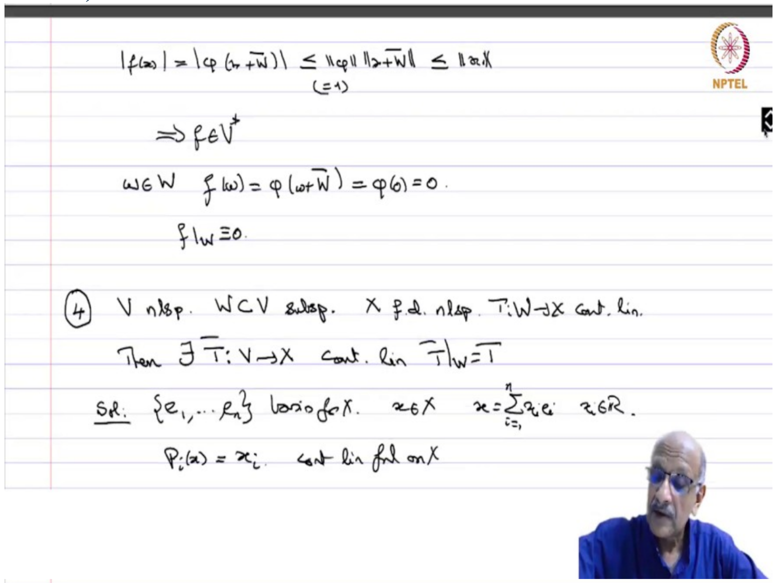

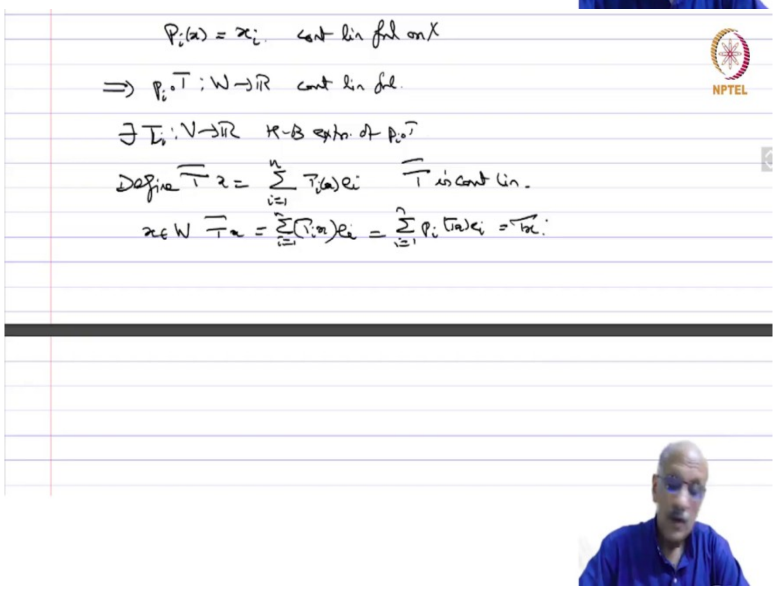

Exercise 4. Let V be a normed-linear space and W be a subspace in V and X be a finite dimensional normed linear space. Let $T:W\mapsto X$ be continuous. Then there exists $\overline{T}:V\mapsto X$ continuous linear continuous, and $\overline{T}$ restricted to W is nothing but T.

So, previously we extended linear functional in the Hahn Banach theorem. Now, we are trying to extend linear operators, but in the case where the Range space is finite dimensional.

Solution. You take a basis $\{e_1,..e_N\}$ for X. So, for any $x\in X$, you can write $x=\sum_{i=1,2,\ldots,N} x_i e_i$, $x_i\in R$. Now you define $P_i(x)=x_i$ (the projection), then this is a continuous linear functional that is immediate. This implies $P_i\circ T:W\mapsto R$ is a continuous linear functional. So, there exists

$T_i : V \mapsto R$ Hahn Banach extension of $P_i \circ T$. Now you define $\overline{T}(x) = \sum_{i=1,2,\ldots N} T_i(x) e_i$. Then, of course, $\overline{T}$ is continuous linear and if you have $x \in W$, you have $\overline{T}(x) = \sum_{i=1,2,\ldots,N} T_i(x) e_i = \sum_{i=1,2,\ldots,N} P_i(T(x)) e_i = T(x)$. So, this proves.

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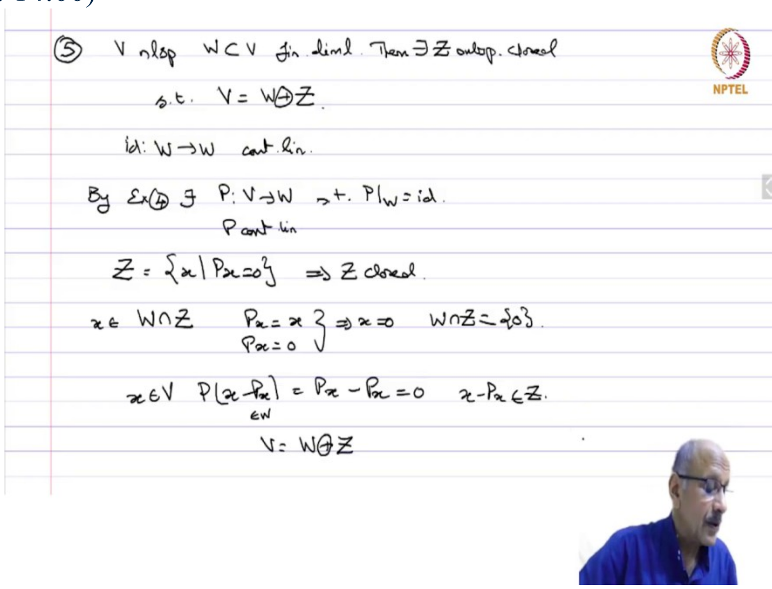

Exercise 5. Let V be normed-linear space and $W \subseteq V$ finite dimensional subspace. Then there exists a subspace Z which is closed such that $V = W \oplus Z$.

Now, W is finite dimensional so it is already closed. Now, given any subspace in linear algebra, you know, we can extend the basis and therefore write V as a direct sum of W and Z. But we are asking something more here we want Z to be also closed.

Now, this is a non-trivial problem generally. Given a closed subspace can you find another closed subspace such that the direct sum of these two is in fact the whole space? This is a very important question in function analysis, we are answering it partially here, namely, if W is finite dimensional.

Solution. Take the identity map $Id : W \mapsto W$. This is continuous, linear and since W is finite dimensional, therefore, by Exercise 4, there exists a $P : V \mapsto W$ such that P restricted to W is Id. P is of course continuous. Now, you define $Z = \{x : P(x) = 0\}$. So, Z is closed. Suppose $x \in W \cap Z$. Then you have that $P(x) = x \wedge P(x) = 0$. This implies that $x = 0$. Therefore, $W \cap Z = \{0\}$. Now, you take any $x \in V$. Then

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Thus, $x - P(x) \in Z$. Therefore, $V = W \oplus Z$.

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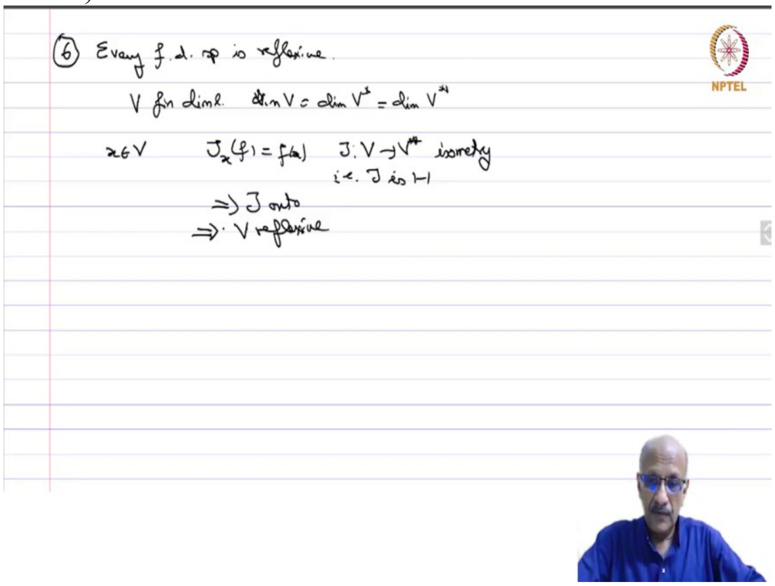

Exercise 6. Every finite dimensional space is reflexive.

Solution. Let V be finite dimensional. Then $dim\,V = dim\,V^{*} = dim\,V^{**}$. Now, J is the canonical mapping. So, for $\in V$, $J_x(f) = f(x)$. So, $J(x) = J_x$ J is a mapping from V to V^{**}, which is an isometry i.e., J is 1-1 also. V, V^{**} are of same dimension implies J is onto and therefore, we have these V is reflexive. So, every finite dimensional space is automatically reflexive.

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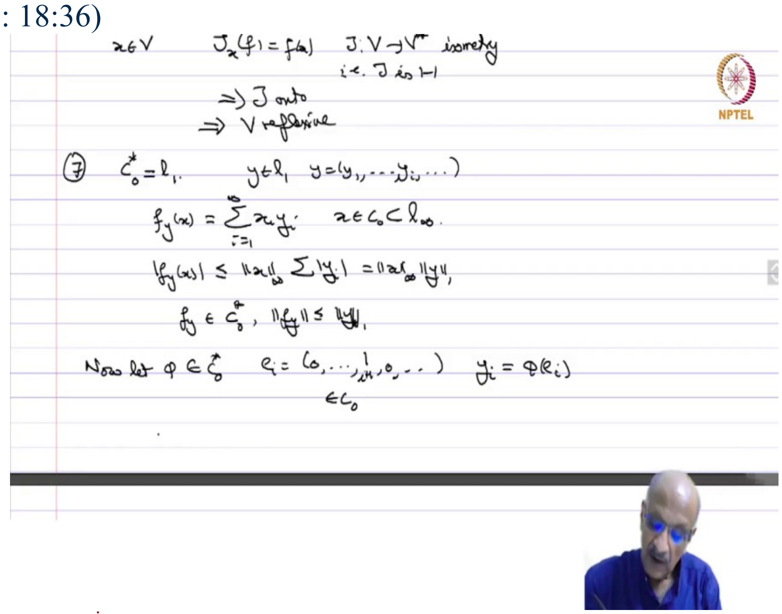

Exercise 7. Show that $C_0^{*} = l_1$.

Solution. You take any $y = (y_1, y_2, \ldots.) \in l_1$ and define $f_y(x) = \sum_{i=1,\ldots,\infty} x_i y_i, x \in C_0 \subset l_\infty$..

So, $\left|f_y(x)\right| \le \|x\|_\infty \|y\|_1$. Therefore, $f_y \in C_0^{¿}$ and $\|f_y\| \le \|y\|_1$. Now, let $\phi \in C_0^{¿}$. We want to show that every element in $C_0^{¿}$ occurs in this way. You take $e_i = (0 \ldots .,1,0,0, \ldots \ldots) \in C_0$ and then you take $y_i = \phi(e_i)$.

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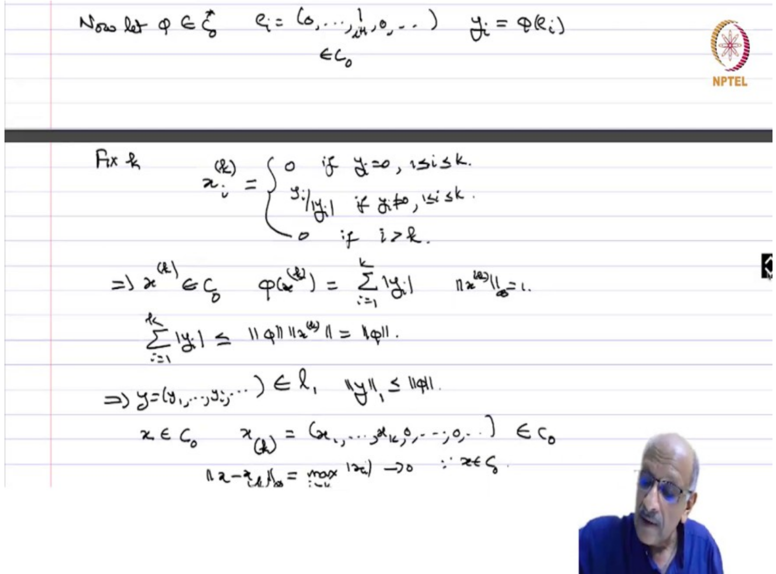

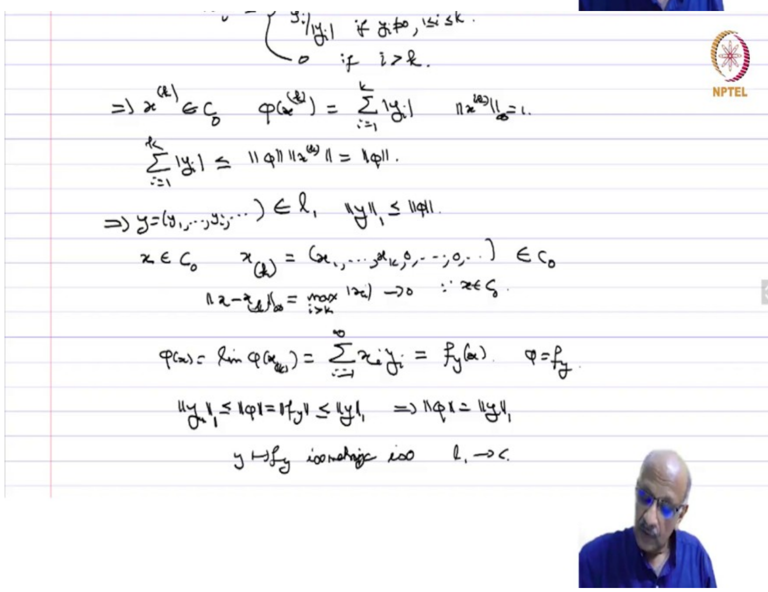

Now you fix a positive integer k and define

$$x_i^{(k)} = 0 \; if \; y_i = 0, 1 \le i \le k. \; x_i^{(k)} = \frac{y_i}{|y_i|} \; if \; y_i \ne 0, 1 \le i \le k. \; x_i^{(k)} = 0 \; if \; i > k.$$

Therefore, automatically $x^{(k)} \in C_0$ andit converges to 0. Now, $\phi(x^{(k)}) = \sum_{i=1,\dots k} |y_i|$ and $\|x^{(k)}\|_\infty = 1$.Therefore, $\sum_{i=1,\dots,k} |y_i| \le \|\phi\| \|x^{(k)}\|_\infty \le \|\phi\|.$ So, this is true for all k and therefore, $y \in l_1$ and $\|y\|_1 \le \|\phi\|$. Now let $x \in C_0$ and you define $x_{(k)} = (x_1, \dots x_k, 0, 0, \dots) \in C_0$and $\|x - x_{(k)}\|_\infty = \max_{i>k} |x_i| \to 0$ since $x \in C_0$ (this is exactly the place where $l_\infty^{*} = l_1$does not work).

Therefore, you have that $\phi(x)=\lim \phi(x_{(k)})=\sum_{i=1,2,..\infty} x_i y_i = f_y(x)$ i.e., $\phi = f_y$. Therefore, $\|y\|_1 \le \|\phi\| = \|f_y\| \le \|y\|_1$. Thus, $\|y\|_1 = \|\phi\|$. So, $y \mapsto f_y$ is an isometric isomorphism from l_1 to $C_0^{¿}$ and therefore, $C_0^{¿}$ is identified with l_1.