Functional Analysis
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Lecture 13
Vector Valued Integration

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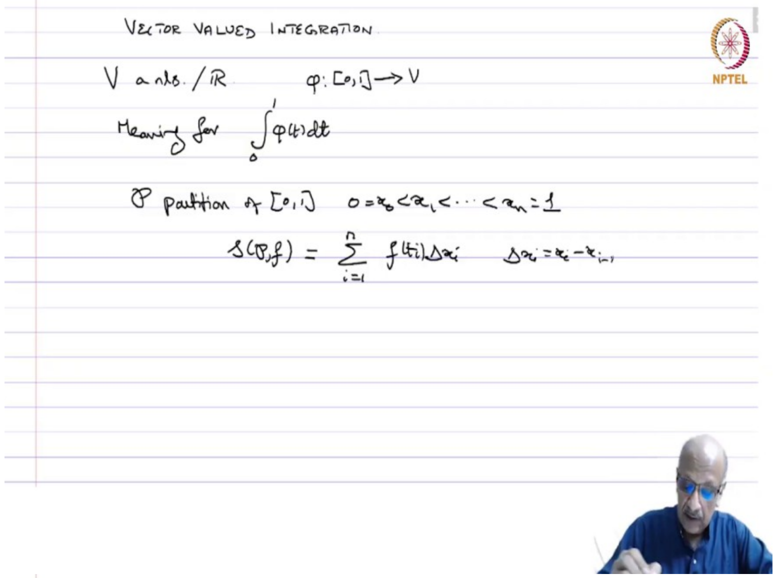

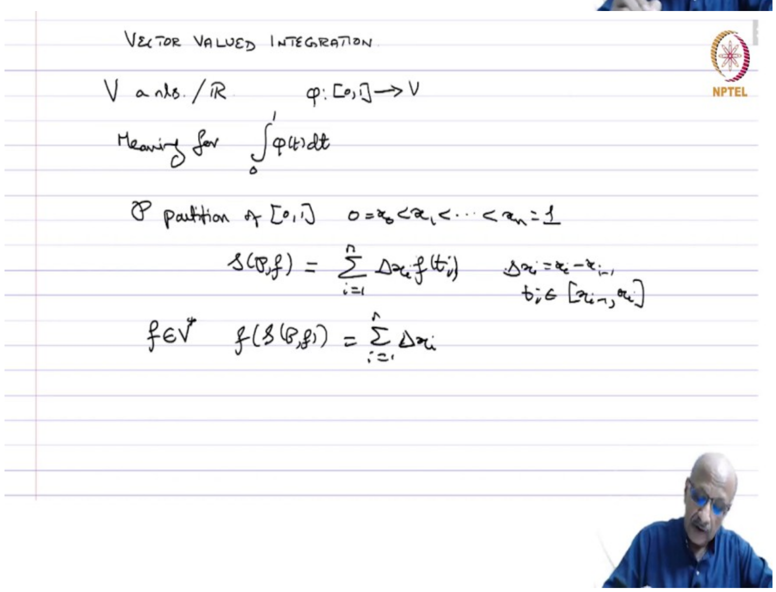

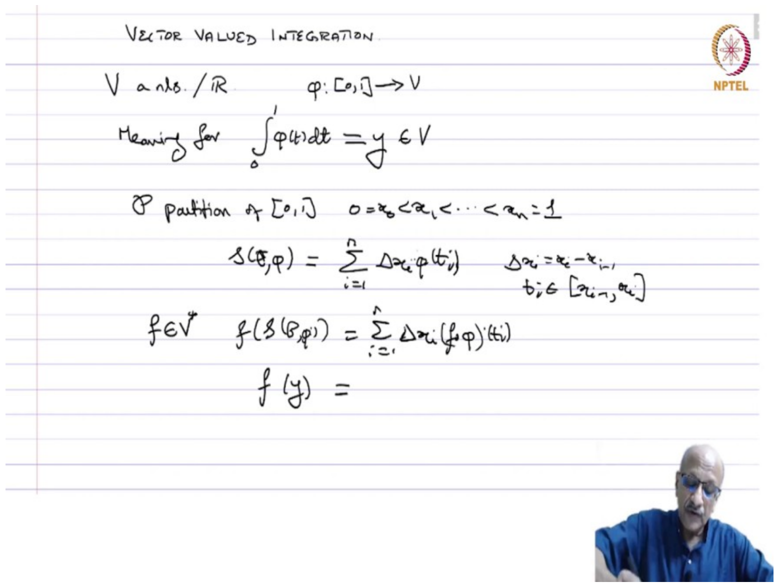

Now, we will discuss vector valued integration. So, let V a normed-linear space, we will discuss over **R** and let us take $\phi:[0,1]\mapsto V$ be a given continuous function. We will talk of continuous functions, but let us say, so we want to give a meaning to the expression $\int_{[0,1]} \phi(t)\,dt = y \in V$. How do you define such an integral?

So, suppose you had just a real valued function, what would you do? you take a partition of $[0,1]$, say, $P=\{0=x_0, x_1, \ldots, x_n=1\}$ and then you would assign $S(p,\phi)$ which is the Riemann sum associate with this partition, which will be $S(p,\phi)=\sum_{i=1,2,\ldots,n} \phi(t_i)\Delta x_i$, where $\Delta x_i = x_i - x_{i-1}, t_i \in [x_{i-1}, x_i]$. Then using some suitable limit process we would define the integral. Suppose we do the same thing. Now, of course, Δx_i must be written in the front because $\phi(t_i)$ is a vector and Δx_i is a scalar. So we will write here $S(p,\phi)=\sum_{i=1,2,\ldots,n} \Delta x_i \phi(t_i)$ where ti belongs to the interval $[x_{i-1}, x_i]$. So, this notation is a bit faulty, but it does not matter, it does not take into account which t's we are talking about.

So, if we take a limit and we are able to define the integral then if $f \in V^{¿}$ then $f(S(p,\phi))=\sum_{i=1,2,\ldots n} \Delta x_i f(\phi(t_i))$ and then we pass on to the limit.

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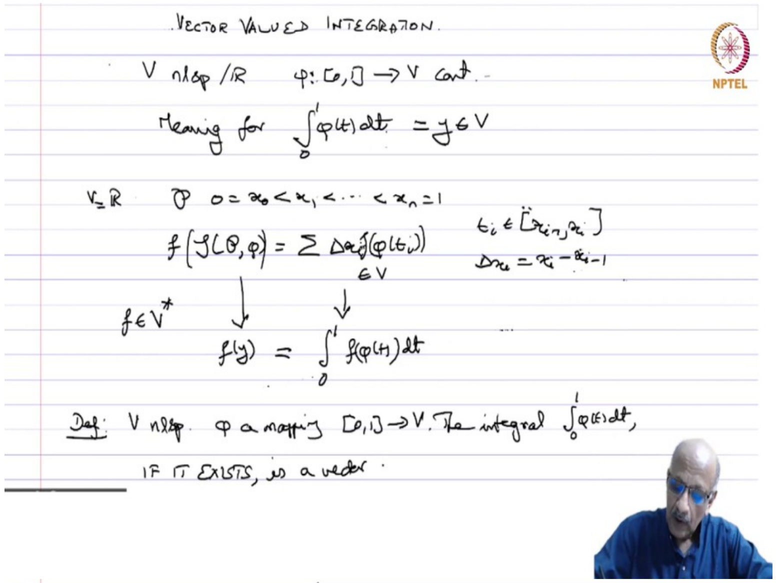

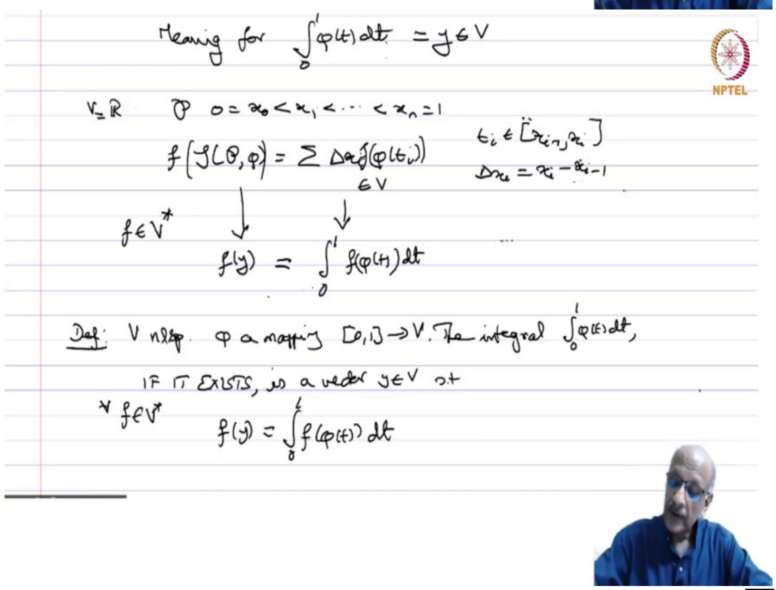

now when you pass to the limit $f(S(p,\phi)) \to f(y)$. On the other hand, $\sum_{i=1,2,\ldots n} \Delta x_i f(\phi(t_i))$ is nothing but the Riemann sum for the continuous real valued function $f \circ \phi$ and therefore this should converge to integral $\int_{[0,1]} f(\phi(t))\,dt$. And therefore, these two should be equal i.e., $(y) = \int_{[0,1]} f(\phi(t))\,dt$.

We are going to make a definition.

Definition. Let V be a normed-linear space and $\phi : [0,1] \mapsto V$ be a mapping.

So, we assume that $[0,1]$ is given in the Lebesgue measure, so I am not specifying it here.

The integral $\int_{[0,1]} (\phi(t))dt$, if it exists, is a vector $y \in V$ such that for every $f \in V^{¿}$ we have $f(y) = \int_{[0,1]} f(\phi(t))dt$. So, it should be a vector such that for any $f \in V^{¿}$ the integral should be equal to this real value of the above integral. So, now let us prove the following proposition.

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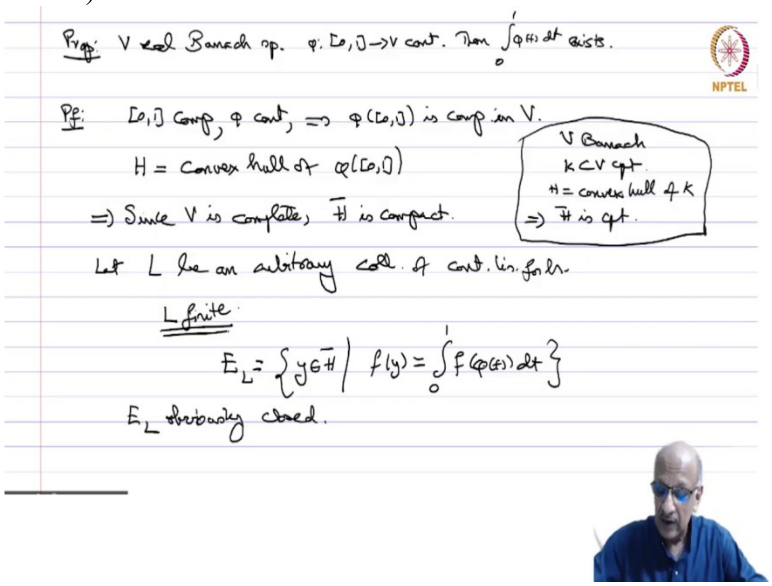

Proposition. Let V be a real Banach space and $\phi:[0,1] \mapsto V$ continuous. Then $\int_{[0,1]} \phi(t)dt$ exists.

Proof. $[0,1]$ is compact, ϕ is continuous implies $\phi([0,1])$ is compact (continuous image of a compact set is compact) in V. Now, you take H to be the convex hull of $\phi([0,1])$ (it means that it is the smallest convex set which contains this image). Since V is complete, $\overline{H}$ is compact. If you have a Banach space and you have a compact set K then the closure of the convex hull of K is also compact. In fact, because of the completeness, it is enough to show that $\overline{H}$ is totally bounded and that can be done. Now, let L be an arbitrary finite collection of continuous linear functional. You define $E_L = \{y \in \overline{H} : f(y) = \int_{[0,1]} f(\phi(t))dt\}$. So, this y satisfies the condition of being the integral $\int_{[0,1]} \phi(t)dt$ only for finite number of linear functions, not for everything. Our aim is to find a y which does it for every linear functional.

E_L is obviously closed.

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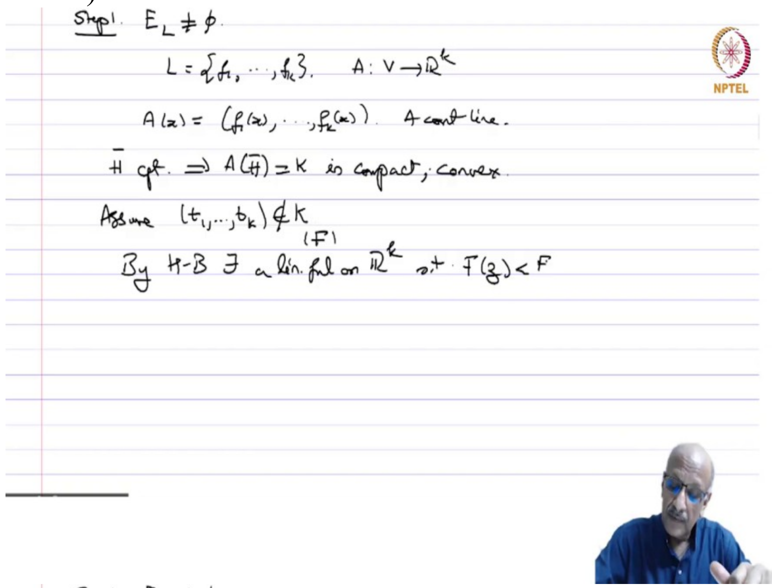

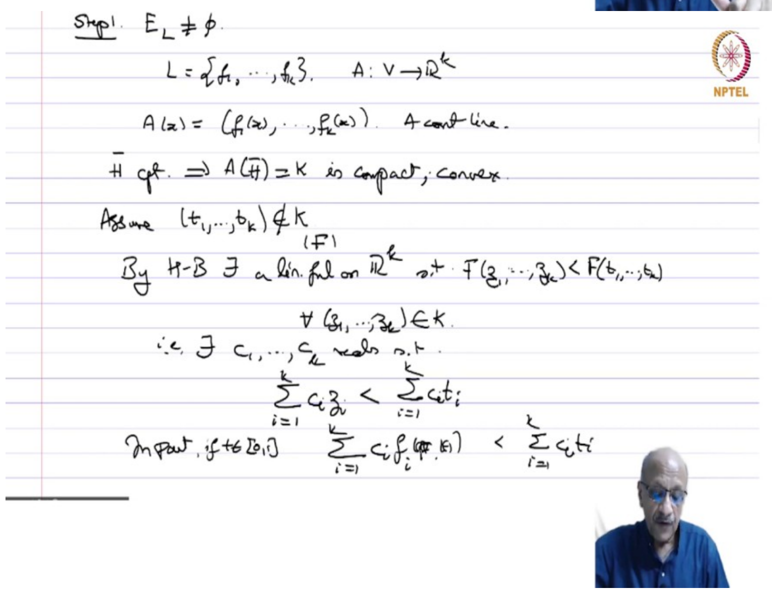

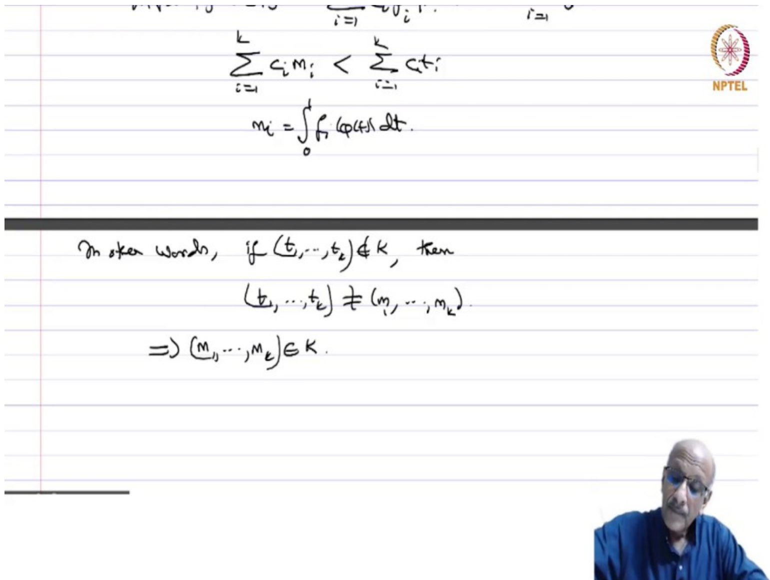

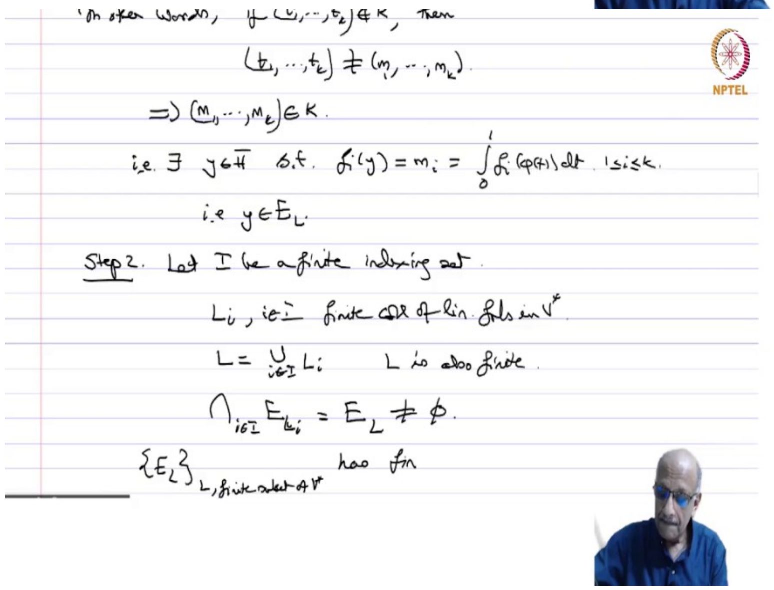

Step 1. $E_L \neq \phi$. We are going to show that this is never an empty set. Given any finite collection you can always find a common vector such that the equality holds. So, let us take $L = \{f_1, \ldots f_k\}$. Now I am going to define $A : V \mapsto R^k$ linear map such that $A(x) = (f_1(x), \ldots, f_k(x))$. So, A is obviously continuous because each of these coordinates is continuous. A is continuous linear and $\overline{H}$ is compact. This implies that $K \coloneqq A(\overline{H})$ is compact and convex. Assume that there is a vector $(t_1, \ldots t_k) \notin K$. So, K is a compact, convex set and $\{t_1, \ldots, t_k\}$ is singleton which is also compact. So, by Hahn Banach, there exists a linear functional F on R^k such that $F(z_1, \ldots, z_k) < F(t_1, \ldots, t_k); \forall (z_1, \ldots, z_k) \in K$. But what is a linear functional on R^k? Dual of R^k is itself and a linear functional is just a linear combination of these things i.e., there exists

$C_1,\dots C_k \in R$ such that $\sum_{i=1,2,\dots k} C_i z_i < \sum_{i=1,2,\dots k} C_i t_i$. In particular if $t \in [0,1]$, you have $\sum_{i=1,2,\dots,k} C_i f_i(\phi(t)) < \sum_{i=1,2,\dots,k} C_i t_i$. So, now let us integrate this. Therefore,
$\sum_{i=1,2,\dots,k} C_i m_i < \sum_{i=1,2,\dots,k} C_i t_i$, where $m_i = \int_{[0,1]} f_i(\phi(t))dt$. In other words, if $(t_1,\dots,t_k) \notin K$, then $(t_1,\dots,t_k) \neq (m_1,\dots,m_k)$. So, this implies that $(m_1,\dots m_k) \in K$. But what is K? K is nothing but the image $A(\overline{H})$. Therefore, there exists $y \in \overline{H}$ such that $f_i(y) = m_i = \int_{[0,1]} f_i(\phi(t))dt, \forall 1 \leq i \leq k$ and thus, $y \in E_L$. So, EL is nonempty.

Step 2. Let I I be a finite indexing set and then you take $L_i, i \in I$ finite collection of linear functionals in $V^{¿}$. Then you take $L = \cup_{i \in I} L_i$, so L is also finite. And it is very easy to check that $\cap_{i \in I} E_{L_i} = E_L$. Therefore this is not empty by Step 1. Therefore, we have shown that $\{E_L : L\, finite \subset of\, V^{¿}\}$ has finite intersection property.

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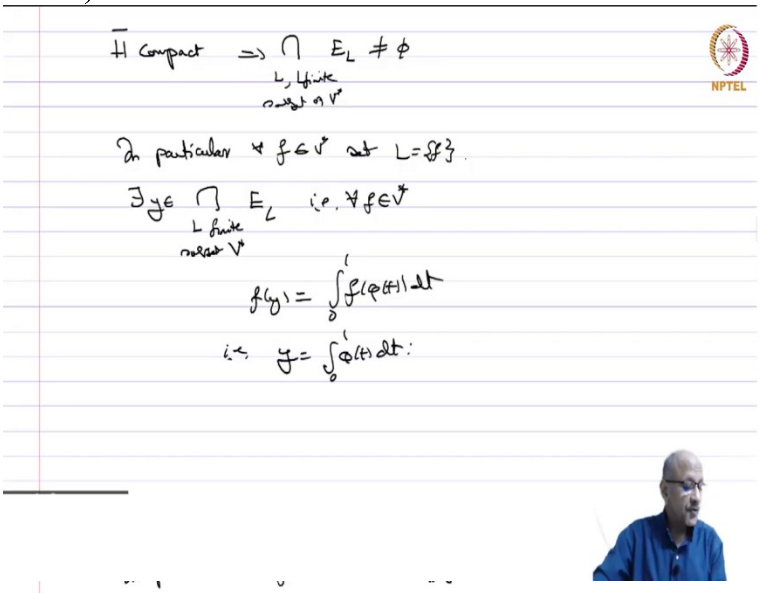

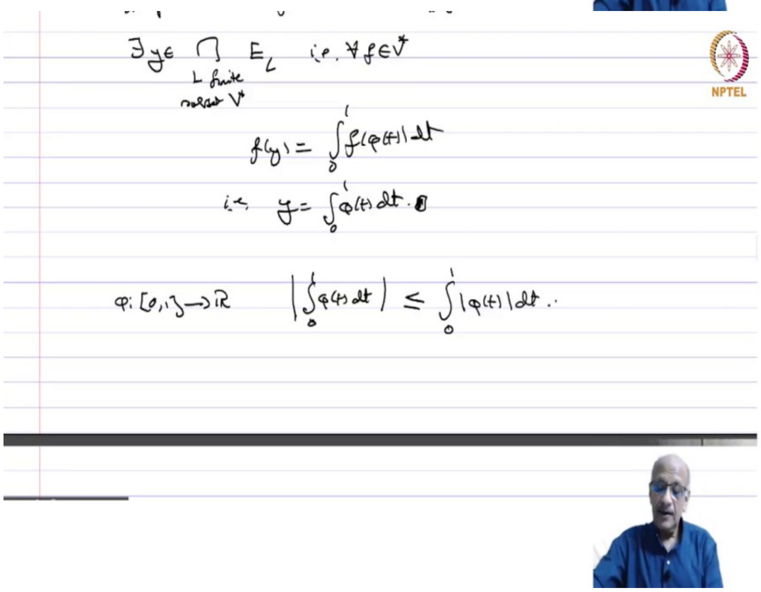

But $\overline{H}$ is compact implies $\cap_{L, L\, finite \subset of\, V^*} E_L \neq \phi$. In particular, for every $f \in V^*$, you set $L = \{f\}$.

Therefore, there exists a $y \in \cap_{L, L\, finite \subset of\, V^*} E_L$ i.e., $\forall f \in V^*$ we have $f(y) = \int_{[0,1]} f(\phi(t))dt$ i.e.,

$y = \int_{[0,1]} \phi(t)dt$.

So, this proves that for continuous functions you always have the integral, very good.

One of the important properties of the integral in one dimensions is that, if you have a $\phi : [0,1] \mapsto R$ **phi from 0 1 to R,** then $\left| \int_{[0,1]} \phi(t)\, dt \right| \leq \int_{[0,1]} |\phi(t)|\, dt$. We want to generalize this to vector valued integration because this is a very very important estimate, so whenever we want to estimate the norm of an integral, this is the first step which we will do.

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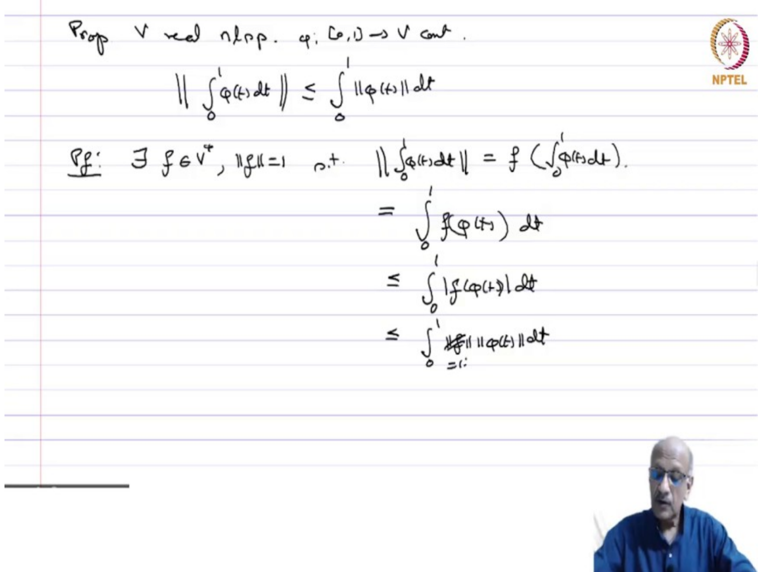

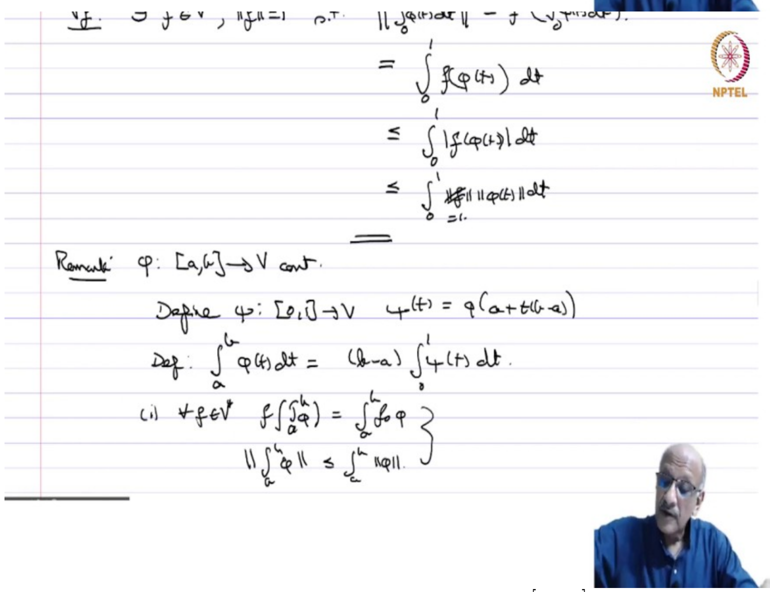

Proposition Let V be real normed-linear space, $\phi:[0,1]\mapsto V$ to be continuous. Then, $\left\|\int_{[0,1]} \phi(t)\,dt\right\| \le \int_{[0,1]} ¿|\phi(t)|\vee dt$.

Proof. Again, this is an application of the Hahn Banach theorem. There exists $f \in V^{¿}$ with $\|f\|=1$ such that $\left\|\int_{[0,1]} \phi(t)\,dt\right\| = f\,¿$.

So, we have done integral over $[0,1]$, but you can do this in any interval.

Remark. Suppose $\phi:[a,b]\mapsto V$ to be continuous. How will I define the integral in the same way?

Define $\psi:[0,1]\mapsto V$ where $\psi(t)=\phi(a+t(b-a))$ and we define $\int\limits_{[a,b]}\phi(t)\,dt=(b-a)\int\limits_{[0,1]}\psi(t)\,dt$ (by change of variable) and you can check both the properties, namely,

1. $\forall f\in V^{¿}, f ¿$

2. $\left\|\int\limits_{[a,b]}\phi(t)\,dt\right\|\leq\int\limits_{[a,b]} ¿|\phi(t)|\vee dt$. Both these you can check yourself.

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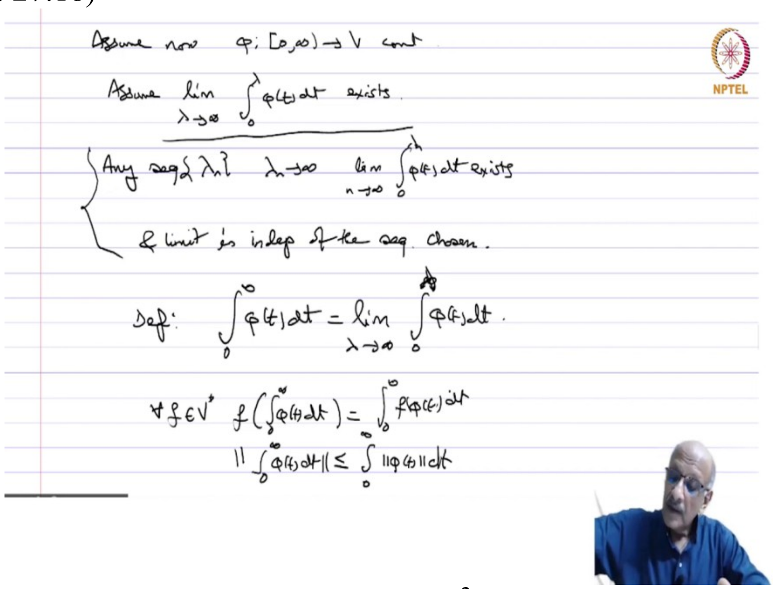

Now assume $\phi:[0,\infty)\mapsto V$ to be continuous, and $\lim\limits_{\lambda\to\infty}\int\limits_{[0,\lambda]}\phi(t)\,dt$ exists. What does this mean? You take any sequence (λ_n) with $\lambda\to\infty$ then $\lim\limits_{n\to\infty}\int\limits_{[0,\lambda_n]}\phi(t)\,dt$ exists and limit is independent of the sequence chosen. So, this is the meaning of the statement. Then we define $\int\limits_{¿}\phi(t)\,dt=\lim\limits_{\lambda\to\infty}\int\limits_{[0,\lambda]}\phi(t)\,dt$. Once again, for every $f\in V^{¿}$, $f ¿$ and $\left\|\int\limits_{[0,\infty)}\phi(t)\,dt\right\|\leq\int\limits_{¿} ¿|\phi(t)|\vee dt$. You can automatically see that all these exist and this will be the norm. You can define the integrals over other kinds of infinite intervals in a similar way.

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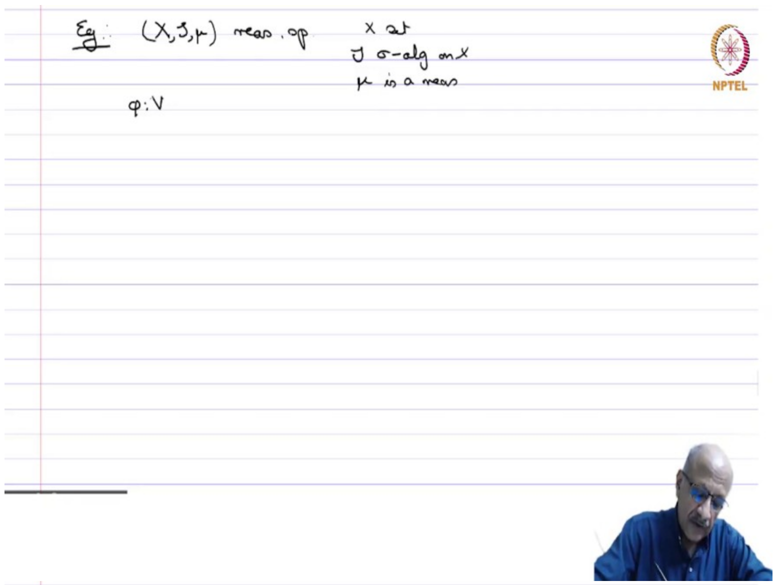

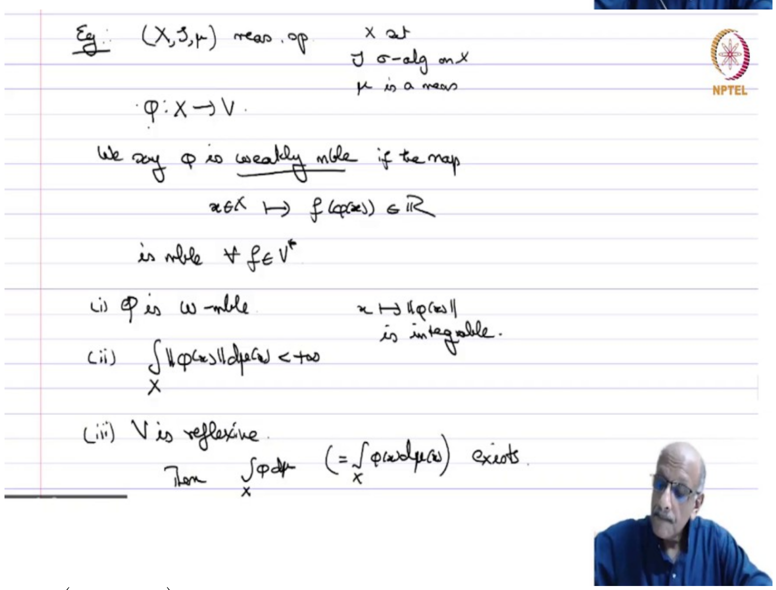

Example Let us take (X, S, μ) be a measure space. So, that means X is a set, S is a sigma algebra on X and μ is a measure and then we have the Lebesgue measure of function. So, now you look at $\phi : X \mapsto V$. We say ϕ is weakly measurable if the map $x \in X \mapsto f(\phi(x)) \in R$ is measurable for every $f \in V^{¿}$. So, now let us assume two things, (i) we assume that ϕ is weakly measurable, (ii) $\int_X \|\phi(x)\| d\mu(x) < \infty$, (iii) V is reflexive. Then $\int_X \phi(x) d\mu(x)$ exist.

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$F \in V^*$

Define for $f \in V^*$

$$\lambda(f) = \int_X f(\varphi(x))\,d\mu(x)$$

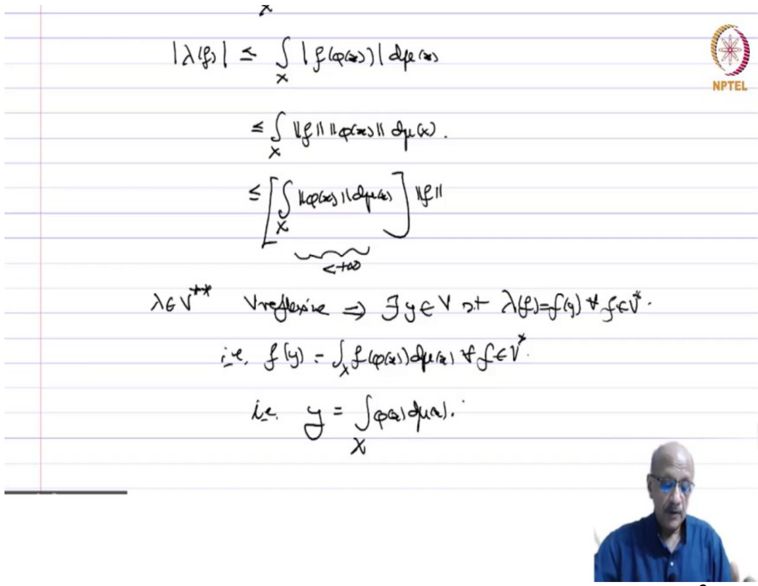
$$|\lambda(f)| \le \int_X |f(\varphi(x))|\,d\mu(x)$$

$$\le \int_X \|f\|\,\|\varphi(x)\|\,d\mu(x).$$

$$\le \Big[\underbrace{\int_X \|\varphi(x)\|\,d\mu(x)}_{<+\infty}\Big]\|f\|$$

$\lambda \in V^{**}$ V reflexive $\Rightarrow \exists\, y \in V$ s.t. $\lambda(f) = f(y)\ \forall f \in V^*$.

i.e. $f(y) = \int_X f(\varphi(x))\,d\mu(x)\ \forall f \in V^*$.



So, let us define, for $f \in V^{¿}$, $\lambda(f) = \int_X f(\phi(x)) \, d\mu(x)$. So, $|\lambda(f)| \le \int_X |f(\phi(x))| \, d\mu(x) \le \int_X \|f\| \|\phi(x)\| \, d\mu(x) \le \left[\int_X \|\phi(x)\| \, d\mu(x) \right] \|f\| < \infty$. Therefore, $\lambda \in V^{¿*¿¿}$. V is reflexive that means lambda has to be the evaluation function. This implies that there exists $y \in V$ such that $\lambda(f) = f(y), \forall f \in V^{¿}$ and that is exactly $f(y) = \int_X f(\phi(x)) \, d\mu(x), \forall f \in V^{¿}$.