Functional Analysis
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Lecture 12
Geometric Version (Contd…)

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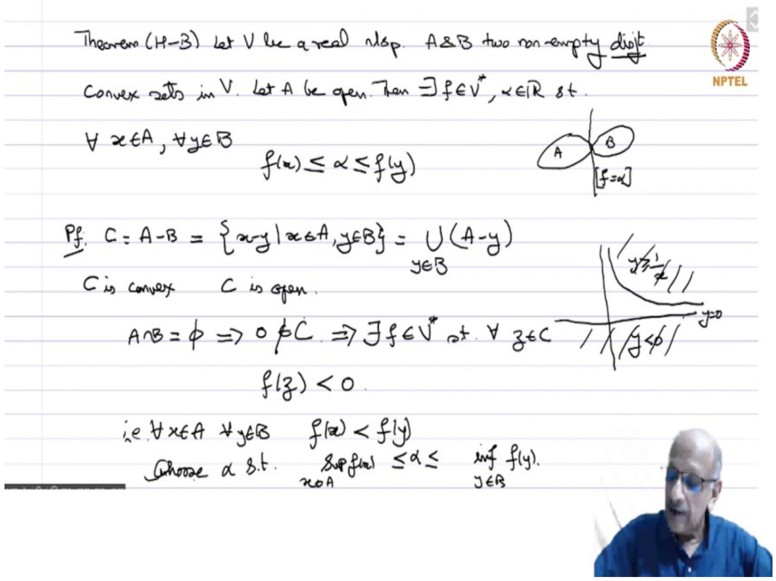

Theorem Let V be real normed-linear space, A and B be two nonempty disjoint convex sets in V. Let A be open. Then there exist $f \in V^{¿}$ and $\alpha \in R$ such that, for all $x \in A, y \in B$, we have $f(x) \leq \alpha \leq f(y)$.

So, $[f=\alpha]$ is a closed hyper plane and it separates the two convex sets i.e., you have two disjoint convex sets A and B, then you can find hyper plane $[f=\alpha]$ that separates them.

Proof. Take $C = A - B = \{x - y : x \in A, y \in B\} = \cup_{y \in B}(A - y)$. It is immediate to see that C is convex. Also, the translation of the open A by y remains open and then arbitrary union of open sets is open and therefore C is open.

Finally, $A \cap B = \phi$ implies that $0 \notin C$. So, now we go to the previous proposition. We have a convex open set and we have a point which is not in it. This implies that there exists an $f \in V^{¿}$ such that for all $z \in C$, you have $f(z) < f(0) = 0$. Thus, for all $x \in A, y \in B$, $f(x) < f(y)$. Choose α such that $¿_{x \in A} f(x) \leq \alpha \leq \inf_{y \in B} f(y)$. Then we are done.

Let us take an example. For instance, in R^2, take the rectangular hyperbola i.e., $y \geq \frac{1}{x}$. So that is one convex set and then the other set is $y < 0$ (that is lower half plane). So, this is the other non-empty convex set. You want to put a hyper plane between them. In fact, you will see that $y = 0$ is the hyper plane which separates these two. You cannot have a strict separation because these two sets asymptotic. Though they are disjoint the points become closer and closer to each other so they cannot be separated by any single plane. So, now we would like to see when you can do a strict separation.

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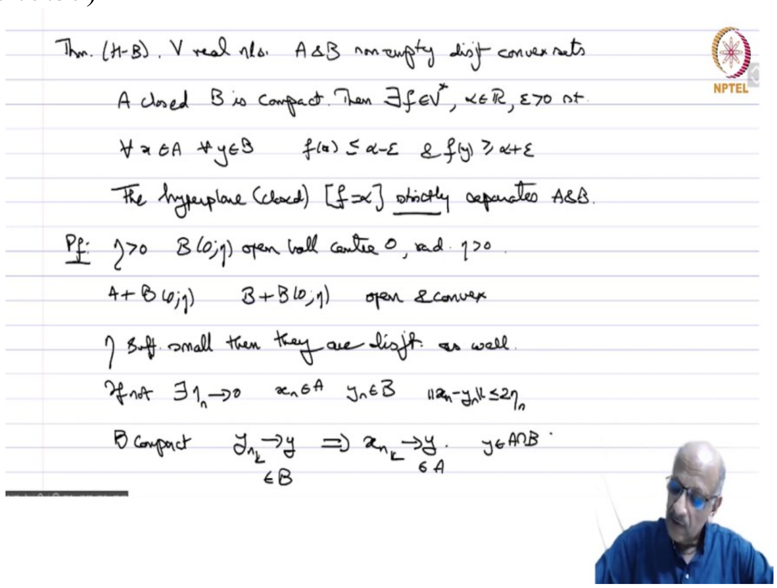

Theorem Let V be a real normed-linear space. A and B nonempty disjoint convex sets, A is closed and B is compact. Then there exists an $f \in V^{¿}$, $\alpha \in R$ and $\epsilon > 0$ such that $\forall x \in A, \forall y \in B$, $f(x) \leq \alpha - \epsilon < \alpha + \epsilon \leq f(y)$.

So, you see that they are away from alpha on both sides and therefore you say that the closed hyper plane $[f = \alpha]$ strictly separates A and B. So both the sets have to be closed and one of them is compact. In the previous example, neither set was compact and that is why we could not separate them.

Proof. Let $\eta > 0$. Then you take $B(0, \eta)$ is the open ball centre at the origin and radius η and you then take $A + B(0, \eta), B + B(0, \eta)$. Now, both the sets are open and convex, convex because each is a sum of two convex sets and open because $B(0, \eta)$ is open and just like last time you can write $A + B(0, \eta) = \cup_{x \in A} [x + B(0, \eta)]$ and the translation of open sets are open and union of open sets is open and therefore this will be open.

If η is sufficiently small then they $A + B(0, \eta), B + B(0, \eta)$ are disjoint as well, why? If not there exists $\eta_n \to 0, x_n \in A, y_n \in B$ and $\|x_n - y_n\| \leq 2\eta_n$. Therefore, since B compact there exists a convergent subsequence $y_{n_k} \to y$. But $(x_n), (y_n)$ must converge to the same limit because they are less than $2\eta_n$ which goes to 0. So, this implies that $x_{n_k} \to y$. But $y_{n_k} \in B$ and A is closed and therefore, $y \in A$. So $y \in A \cap B$ and that is a contradiction because A, B are disjoint. So, if you take η sufficiently small you can also ensure that they are disjoint.

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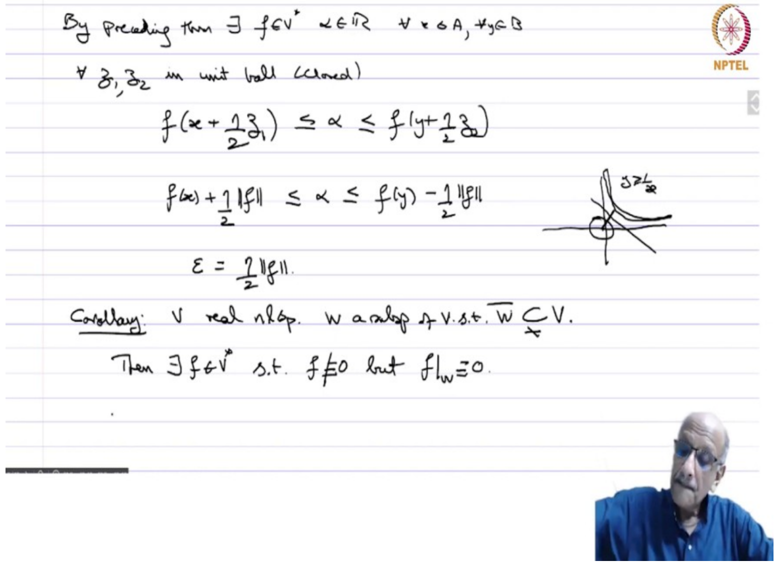

So, you have two disjoint convex open sets which are non empty. So, by the preceding theorem there exists a closed hyperplane $f \in V^*$ **and** $\alpha \in R$ such that for all $x \in A, y \in B$ and for all z_1, z_2 in the closed unit ball such that $\left(x+\frac{\eta}{2} z_1\right) \leq \alpha \leq f\left(y+\frac{\eta}{2} z_2\right)$. You can take this for all z_1, z_2, $f(x)+\frac{\eta}{2}\|f\| \leq \alpha \leq f(y)-\frac{\eta}{2}\|f\|$ and that will prove the theorem by taking $\epsilon=\frac{\eta}{2}\|f\|$. This proves the strict separation theorem completely.

So, for instance if you had the same set i.e., $\{(x, y): y \geq \frac{1}{x}\}$ and then you take a small ball centre at the origin so that they are disjoint. Then, of course, you can find a plane separating these two. Now we are going to have a corollary of this which is so important this is probably the most often applied version of the Hahn Banach theorem, this how the Hahn Banach theorem gets applied most of the time, so it is really very important corollary.

Corollary Let V be a real normed-linear space and W be a subspace of V such that $\overline{W} \subsetneq V$. Then there exists a $f \in V^*$ that $f \neq 0$, but f restricted to W is identically 0.

Proof.

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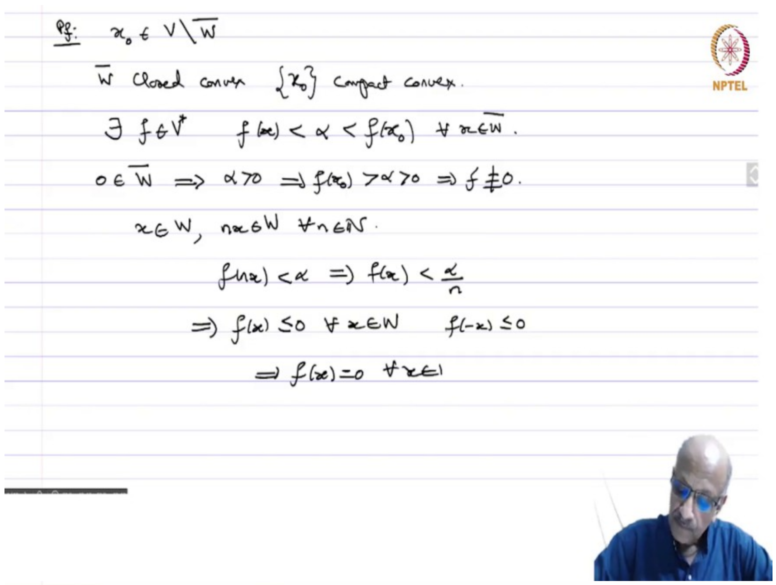

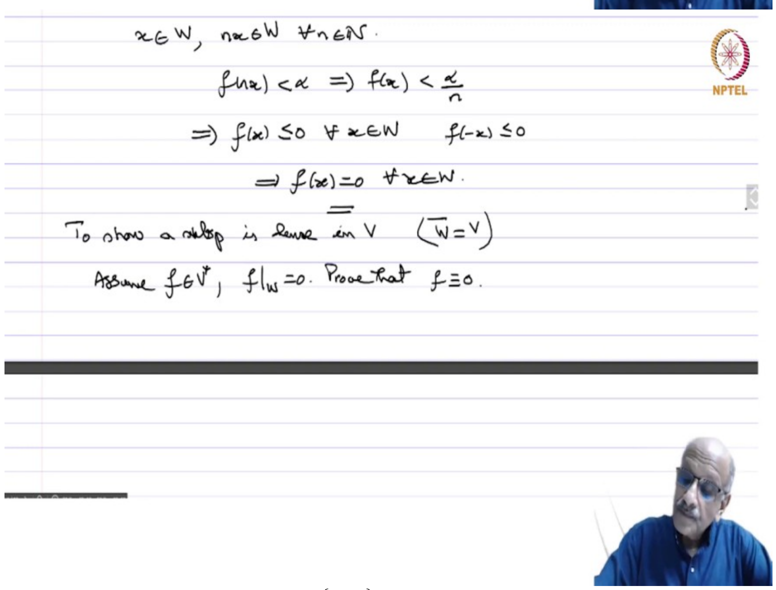

Let $x_0 \in V \setminus \overline{W}$. $\overline{W}$ is closed convex and $\{x_0\}$ is compact and it is a singleton so it is trivially convex. So, there exists $f \in V^{¿}$ such that $f(x) < \alpha < f(x_0), \forall x \in \overline{W}$. $0 \in \overline{W}$ implies $\alpha > 0$, which implies $f(x_0) > \alpha > 0$, so this implies $f \neq 0$.

On the other hand, if $x \in W$, then we have $nx \in W, \forall n \in N$. So, we have $f(nx) < \alpha$, which implies $f(x) < \frac{\alpha}{n}$. So, you let $n \to \infty$, this implies that $f(x) \leq 0, \forall x \in W$. But, then $-x \in W$, so $f(-x) \leq 0$ and this implies that $f(x) = 0, \forall x \in W$. Tthis completes the proof.

How do we use this thing? So, to show a subspace is dense in V i.e., $\overline{W} = V$.

Assume $f \in V^{¿}$, f restricted W is 0. Prove that $f = 0$.

So, this is the way we use the above theorem, because if f is not identically 0 then you know, if $\overline{W} \neq V$, you can always find a f which is not identically 0. But if you take f vanishing on W and show that it implies that f vanishes everywhere that will prove that the subspace is dense, so this is how we prove the density of subspaces.

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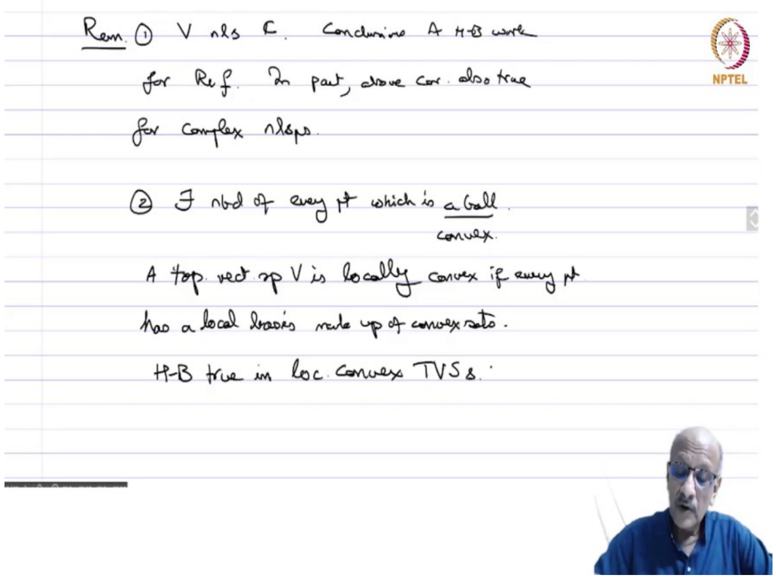

So, couple of remarks.

Remark 1. Let V be a normed-linear space over **C**. Then, you know that linear functionals are essentially given by their real part. Therefore, conclusions of Hahn-Banach work for real part of f. So, in particular, the above corollary is also true for complex normed-linear spaces.

We will give another proof in the exercises using the extension form of the theorem.

The second remark.

Remark 2. In all the proofs we used the fact that there exists a neighbourhood of every point which is a ball, more than that the set is convex.

We say that a topological vector space V is locally convex if every point has a local basis made of convex sets. Norm linear spaces are trivially locally convex because the neighbourhoods are all balls, balls are all convex sets and therefore. TheHahn Banach theorem goes true in locally convex topological vector spaces.

So, I will stop here with this and as an application of the Hahn Banach theorem thing we will study next what is meant by vector-valued integration. So, if you have a function from the real line into the vector into a vector space we want to give a meaning to the integral and that is thing which we will see next.