Functional Analysis

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Lecture 2.5

Geometry Version

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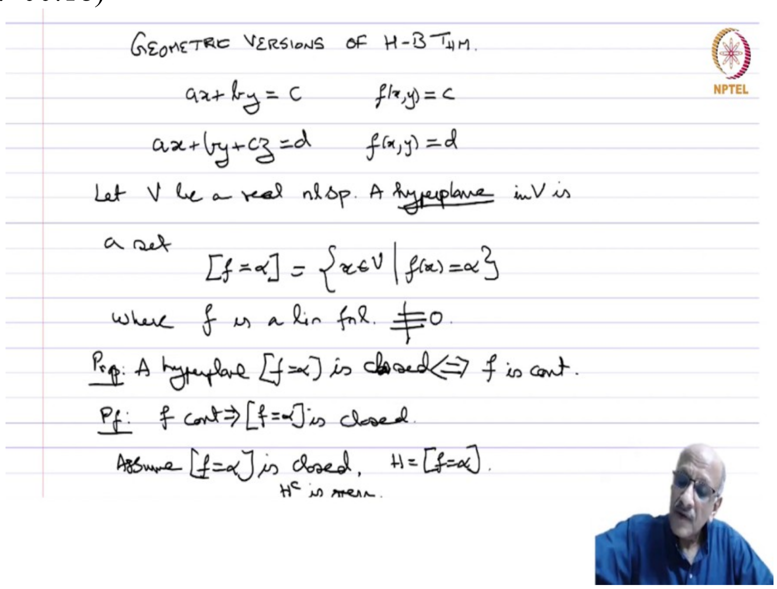

We will now talk about geometric versions of Hahn Banach. So geometric versions means, we are talking about separation of convex sets by means of hyperplanes. What do we mean by a hyperplane? So, this is a generalisation of the notion of a straight line in the plane or a plane in R^3 and so on.

So, what is a straight line?

$ax+by=c$ is a straight line in the plane R^2 and this can be written as $f(x,y)=c$ where $f(x,y)=ax+by$ and that is a linear functional on R^2. Similarly, a plane is $ax+by+cz=d$ and this can be written as $f(x,y,z)=d$ where $f(x,y,z)=ax+by+cz$ and that is again a linear functional in R^3. Therefore, in general, let V be real norm linear space. A hyperplane in V is a set $[f=\alpha]=\{x\in V: f(x)=\alpha\}$ where f is a linear functional which is not identically 0. So, α is a constant. So, we are not talking about continues linear functional; we are only saying hyperplane is given by the constant sets of a linear functional. So, we have the following proposition.

Proposition. A hyperplane $[f=\alpha]$ is closed if and only if f is continuous.

Proof. One way is obvious. If f continuous then clearly $\{x \in V : f(x) = \alpha\}$ is a closed set. So, $[f = \alpha]$ is closed. So, we want now prove the converse. Assume $[f = \alpha]$ is closed. So, let us call this set as H. So, $H = [f = \alpha]$. So, this means that H^c is open and since f is not identically 0, H^c is non empty. Why? If $\alpha = 0$,then there exists x_0 such that $f(x_0) \neq 0$ and hence $x_0 \in H^c$. If $\alpha \neq 0$, then $0 \in H^c$. So, in either case you have that the complement of the hyperplane is a non empty open set.

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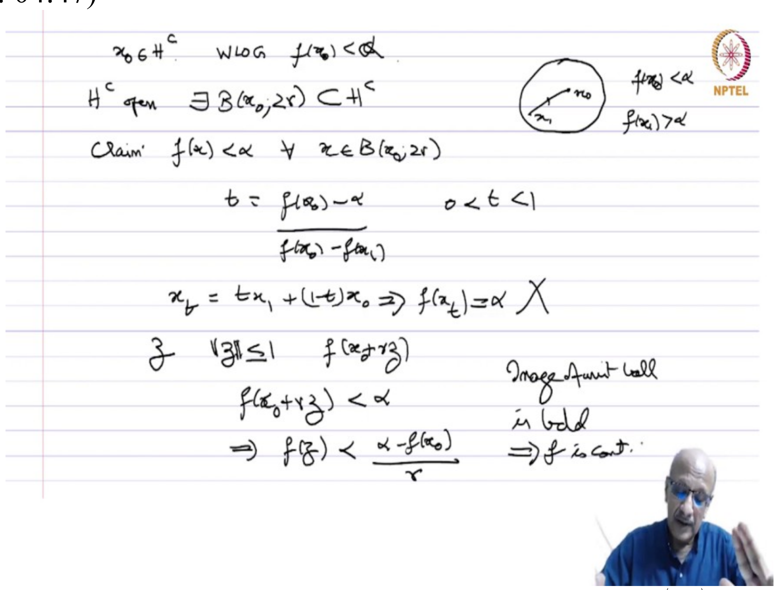

So now, we want to show that f is continuous. So, let $x_0 \in H^c$. So, $f(x_0) \neq \alpha$. So, without loss of generality, we will assume that $f(x_0) < \alpha$. Now, H^c is open. So, there exists a ball $B(x_0, 2r) \subseteq H^c$, this is just the definition of openness.

So, we claim $f(x) < \alpha, \forall x \in B(x_0, 2r)$. Let there exists $x_1 \in B(x_0, 2r)$ such that $f(x_1) > \alpha$. Set $t = \frac{f(x_0) - \alpha}{f(x_0) - f(x_1)}$. Then $0 < t < 1$. Now, let $x_t = t x_1 + (1 - t) x_0$. Then you have $f(x_t) = \alpha$, which is a contradiction.

So, now you take any z such that $\|z\| \leq$ and then $f(x_0 + rz) < \alpha$. This implies that, for all such z, $f(z) < \frac{\alpha - f(x_0)}{r}$ and therefore, the image of the unit ball is bounded. This implies that f is continuous. So, a closed hyperplane is a level set or set where f takes a constant value and f is a continuous linear functional.

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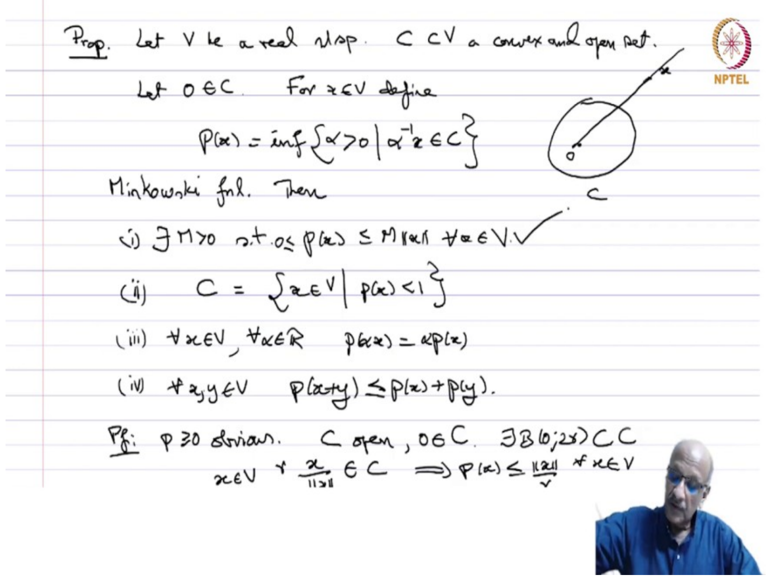

So now, we are going to through a proposition which is a very useful one.

Proposition. Let V be real normed-linear space and $C\subseteq V$ be a convex and open set. Let $0\in C$. For $x\in C$, define $P(x)=inf\{\alpha>0:\alpha^{-1}x\in C\}$. Then,

1. there exist an $M>0$ such that $0\leq P(x)\leq M\|x\|, \forall x\in V$.
2. $C=\{x\in V:P(x)<1\}$
3. $\forall x\in V, \forall \alpha\in R, P(\alpha x)=\alpha P(x)$
4. $\forall x, y\in V, P(x+y)\leq P(x)+P(y)$.

Proof $P\geq 0$ is obvious. Now, C is open and $0\in C$. Therefore, there exists a ball $B(0,2r)\subseteq C$ and therefore, if you take any $x\in V$, you have $r\frac{x}{¿|x|\vee ¿\in C¿}$. So, $\frac{r}{¿|x|\vee ¿¿}$ can be thought of as an α^{-1}, such that $\alpha^{-1}x\in C$. Therefore, $P(x)$ is the infimum of all such numbers. So, this implies that $P(x)\leq\frac{\|x\|}{r}, \forall x\in V$. So, you can take $\frac{1}{R}$ as M and so, this proves the first one.

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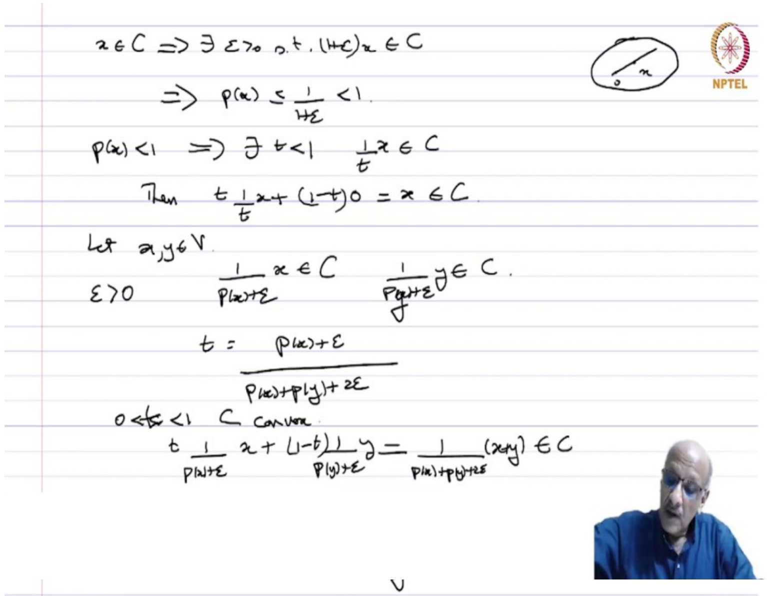

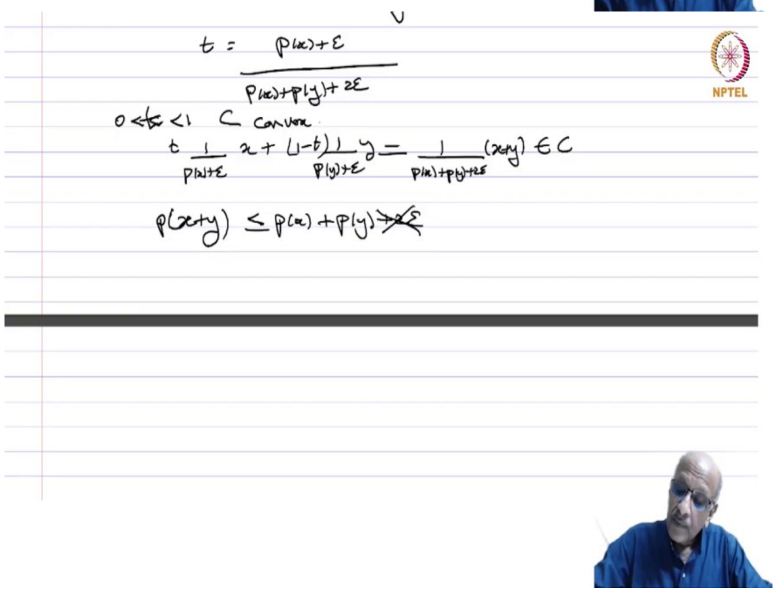

So now, let us see the second one. So, let $x \in C$. Since C is a convex set, there exists an $\epsilon > 0$ such that $(1+\epsilon)x \in C$. Therefore, $P(x) \leq \frac{1}{1+\epsilon} < 1$. Conversely, let us assume that $P(x) < 1$. Then what does it mean? $P(x)$ is the infimum of all t such that $\frac{1}{t}x \in C$. So, since the infimum is strictly less than one there exists a member of that set where you are taking the infimum which lies between these two numbers. This implies, there exists a $t < 1$ such that $\frac{1}{t}x \in C$. So, then C is convex and $0 \in C$, therefore, $t.\frac{1}{t}x + (1-t)0 = x \in C$. So, we prove the second one.

Now, the third one is obvious I will not bother about it, it is just straightforward right from the definition you have this.

So, let us now prove the last condition. So, let $x, y \in V$. So, by the definition of $P(x)$, $\frac{1}{P(x)+\epsilon} x \in C$ and $\frac{1}{P(y)+\epsilon} y \in C$. Now you take $t = \frac{P(x)+\epsilon}{P(x)+P(y)+2\epsilon}$. Then $0<t<1$. Since C is convex, therefore, $t \frac{1}{P(x)+\epsilon} x + (1-t) \frac{1}{P(y)+\epsilon} y = \frac{x+y}{P(x)+P(y)+2\epsilon} \in C$. So, this means that $P(x+y) \leq P(x)+P(y)+2\epsilon$. 2ϵ is arbitrary, so this can go and so, you have proved this condition. So, this proves all the properties of the Minkowski function.

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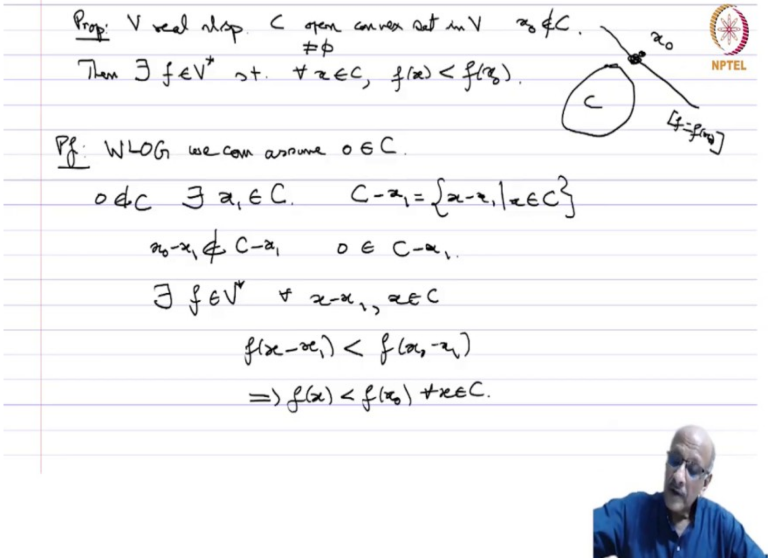

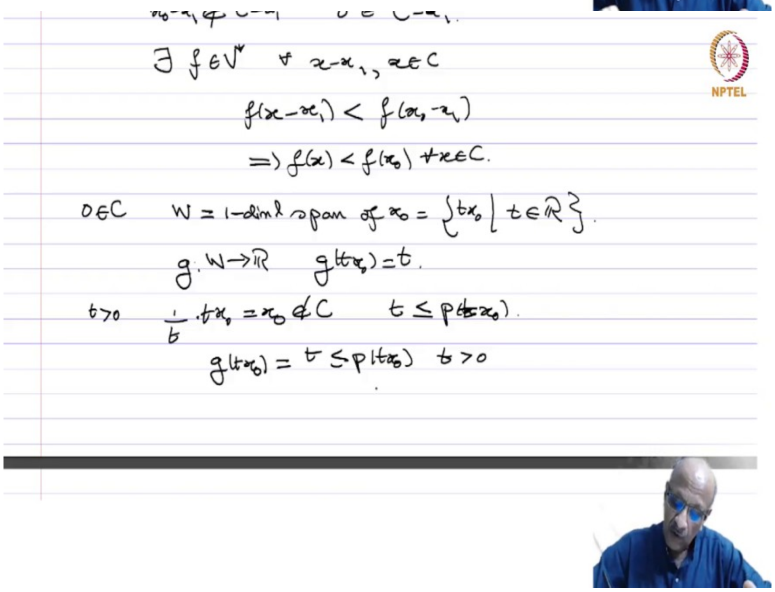

We will now use this to prove a proposition.

Proposition V be a real normed-linear space and C be nonempty open convex set in V and assume and $x_0 \in C$. Then, there exist $f \in V^{¿}$ such that $\forall x \in C, f(x) < f(x_0)$.

Proof. Without loss of generality we can assume $0 \in C$ 0 belongs to C, why? Suppose $0 \notin C$, So, there exists some $x_1 \in C$. Now you will consider the set $C - x_1 = \{x - x_1 : x \in C\}$. Then $x_0 - x_1 \notin C - x_1$ and $0 \in C - x_1$. So, we have the previous situation, namely, 0 is in the convex set and this $C - x_1$is nothing but you have just translated the origin. So, the convex set remains convex there is no change in that. So, by the proof for previous case, there exists $f \in V^{¿}$ such that, $\forall x - x_1, x \in C$, we have $f(x - x_1) < f(x_0 - x_1)$ and by linearity this means that

$f(x) < f(x_0), \forall x \in C$. So, there is no loss of generality in assuming that $0 \in C$. So, $0 \in C$ and now we take $W = span\{x_0\} = \{t x_0 : t \in R\}$. Now, we define $g : W \mapsto R$ as $g(t x_0) = t$. This defines a linear functional on this. Let $t > 0$. Now, $\frac{1}{t} t x_0 \notin C$. Thus, $t \leq P(t x_0)$. $g(t x_0) = t \leq P(t x_0)$, if $t > 0$. This is trivially true if $t \leq 0$. So, this is true in fact for all t.

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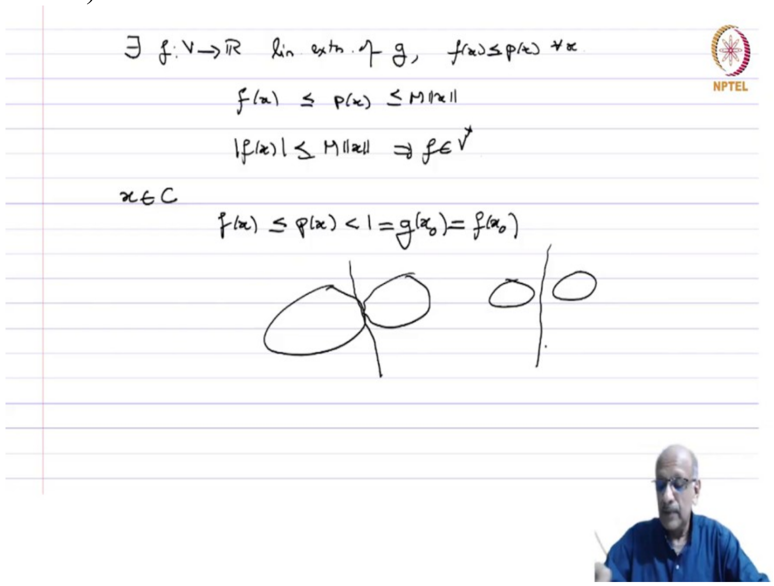

Recall that P P satisfies the conditions which we prove the very first Hahn Banach theorem and therefore, we get there exists an $f: V \mapsto R$, linear extension of g and $f(x) \leq P(x), \forall x \in V$. Now, $f(x) \leq P(x) \leq M \vee |x| \vee$¿. This is also true for $-x$ and therefore, $|f(x)| \leq M \|x\|$. and this implies $f \in V^{¿}$ (it is linear and it is continuous therefore, it belongs to $V^{¿}$). Then if $x \in C$, we have $f(x) \leq P(x) < 1$ (as $x \in C$)¿ $g(x_0) = f(x_0)$ and this completes the proof.

So we will now use this theorem to prove some separation theorems of convex sets. So, when we have two convex sets which are disjoint, we will show that you can separate them by means of hyperplane and sometimes under some other conditions you can actually strictly separate them. So, we have to measure this disjointeness properly. So, we will do that.