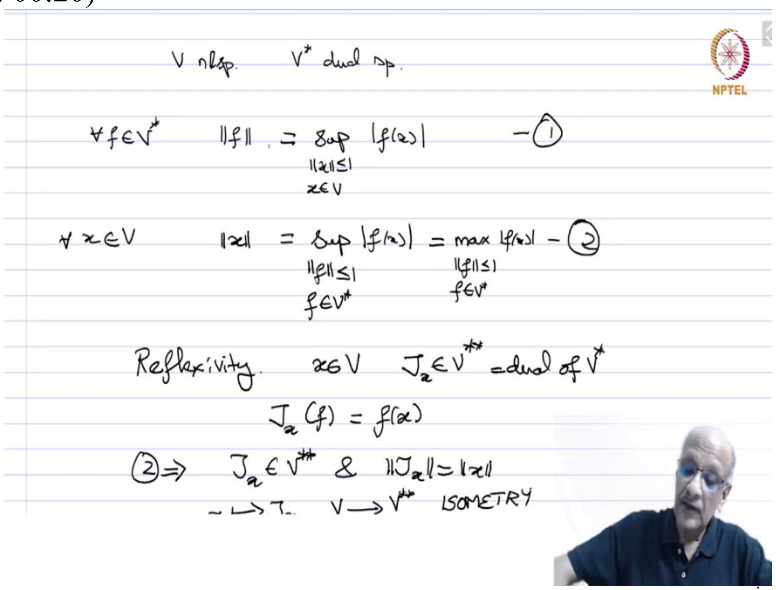
Functional Analysis
Professor S. Kesavan
Department of Mathematics
Institute of Mathematics Science
Lecture 2.4
Reflexivity

(Refer Slide Time: 00:20)



Let us look at the following two relations. Let V be a nonlinear space and V^* be the dual space.

So, for every $f \in V^*$, $\|f\| = \sup_{\|x\|\le 1, x\in V} |f(x)| \ldots.(1)$.

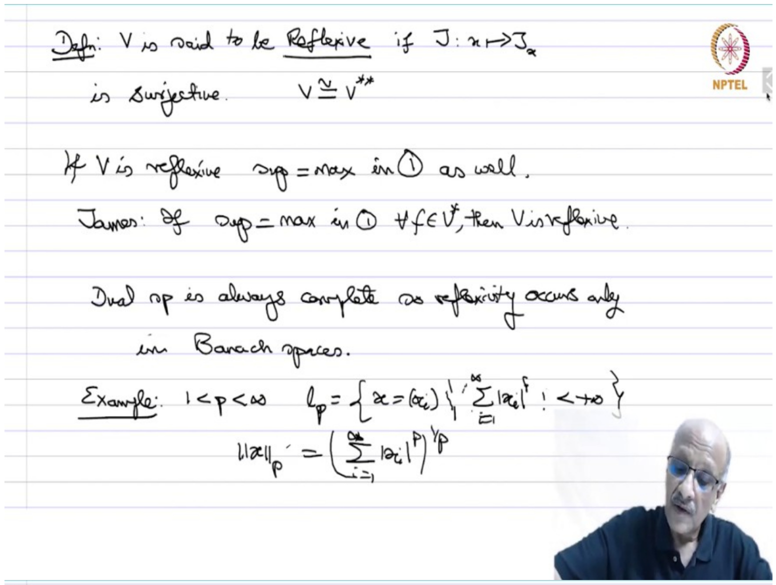
For every $x \in V$, (we just saw in the last corollary of the Hahn Banach theorem)

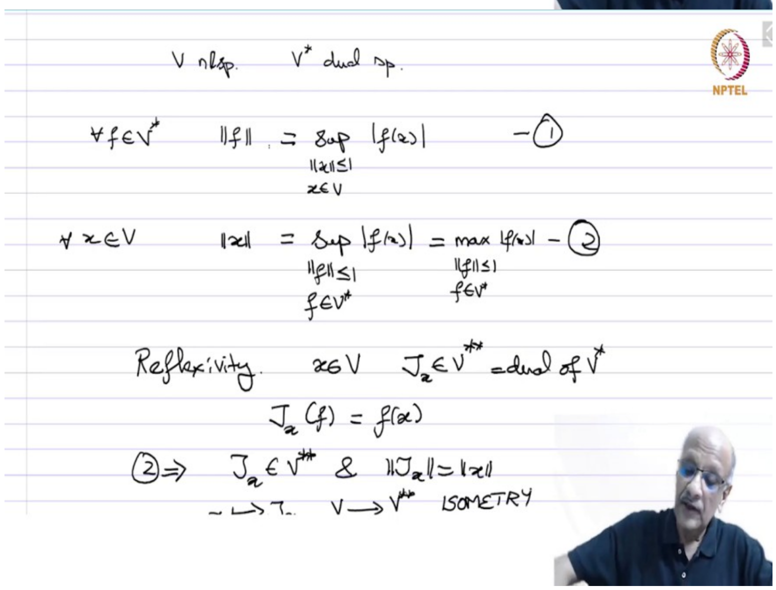
$\|x\| = \sup_{\|f\|\le 1, f\in V^*} |f(x)| = \max_{\|f\|\le 1, f\in V^*} |f(x)| \ldots\ldots(2)$.

So, it says in the first one, the supremum need not be attained, whereas, in the second one the supremum is always attained and therefore, it is a maximum this is the starting point of a very interesting concept in functional analysis this is called reflexivity.

So, let us take $x \in V$ and I define $J_x \in V^{**} = \text{dual of } V^*$ by $J_x(f) = f(x)$ (the evaluation of f at x). Then what does (2) imply? (2) implies that $J_x \in V^{**}$ and in fact, $\|J_x\|_{V^{**}} = \|x\|_V$. So, the mapping $J : V \mapsto V^{**}$ defined by $J(x) = J_x$ is linear and preserves norms and so it is called an isometry. In particular, it is one to one and therefore, it will map V into a subspace of V^{**}.

(Refer Slide Time: 04:00)





Definition. V is said to be reflexive if the map J is surjective.

What does this mean? Every element of $V^{¿*¿¿}$ i.e., every continuous linear functional on the dual space $V^{¿}$ is actually nothing but an evaluation functional which comes from V and it is just evaluation at that point. So, this is what we mean by saying that the space is reflexive and therefore, this mapping is then one to one, onto and continuous because it is an isometry it is continuous both ways. And then you have that V is can be identified with $V^{¿*¿¿}$. So, such a space is called a reflexive space and it is this particular mapping J which should be surjective. It is not enough if V is isomorphic to $V^{¿*¿¿}$ by some other mapping; we want this particular mapping J, namely, the evaluation functional mapping, has to be surjective.

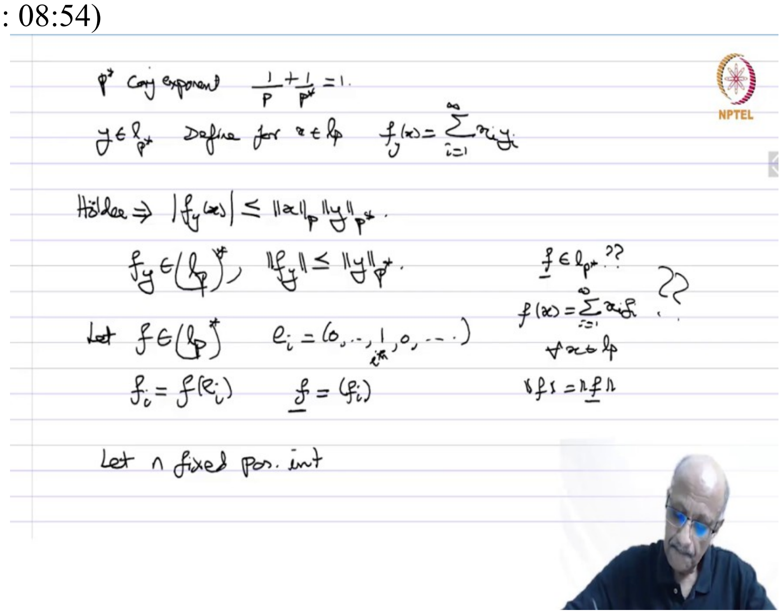
If the space is reflexive, you apply relation (2) for $V^{¿}$ instead of V, then you will get $V^{¿*¿¿}$ will be just V again, hence, because of the isometry J_x you will get that the supremum is attained in (1) also. If V is reflexive, then $¿=max$in (1) as well. Now, there is a deep theorem of James if sup equals max in (1) for all $f \in V^{¿}$, then V is reflexive. So, this is a necessary and sufficient condition. This we will not prove. So, this is a very deep theorem in this.

Now, the dual space is always complete. So, reflexivity occurs only in Banach spaces.

If V equal $V^{¿*¿¿}$ and if isomorphic through this mapping J, then automatically since $V^{¿*¿¿}$ is complete, V has to be complete. So, the notion of reflexivity is only there for Banach spaces.

Example Let $1<p<\infty$and then you look at l_p which is set of all sequences (x_i) such that $\sum_{i=1,2\ldots,\infty} |x_i|^p < \infty$and you have $\|x\|_p = \left(\sum_{i=1,2\ldots,\infty} |x_i|^p \right)^{\frac{1}{p}}$. This is a Banach space and we have seen this.

(Refer Slide Time: 08:54)



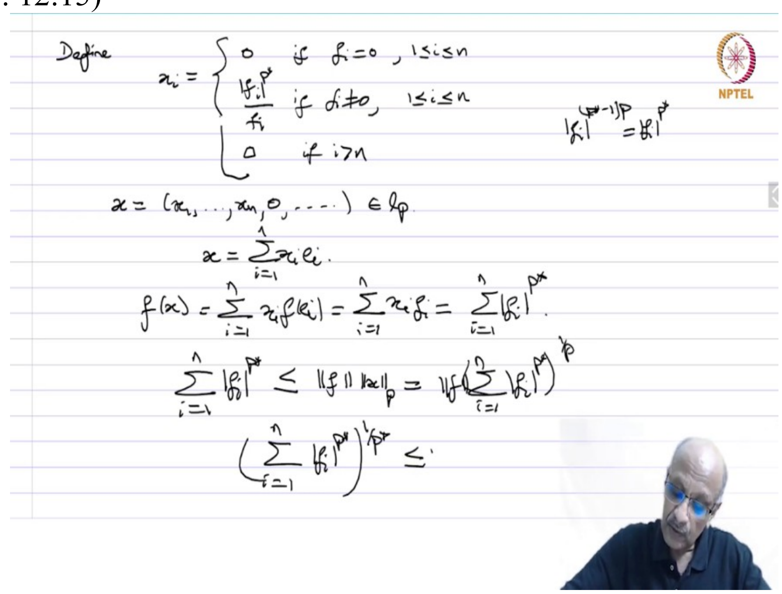
Now, I am going to compute what its dual is and identify that. So, let us take $p^{¿}$ is the conjugate exponent, what does it mean? $\frac{1}{p}+\frac{1}{p^{¿}}=1$. Let $y \in l_{p^{¿}}$. For $x \in l_{p,}$ define $f_y(x) = \sum_{i=1,2,\ldots,\infty} x_i y_i$. I am doing this in the real case, if it is a complex case, it will be $x_i \overline{y_i}$. So, that is a convention which we take. So, we are doing everything in case of reals, but you can change it with $\overline{y_i}$ and whatever I am going to say will go through and therefore, you we will just work with this. Then, is this well defined? Yes, because of Holder's inequality, $|f_y(x)| \le \|x\|_p \|y\|_{p^{¿}}$.

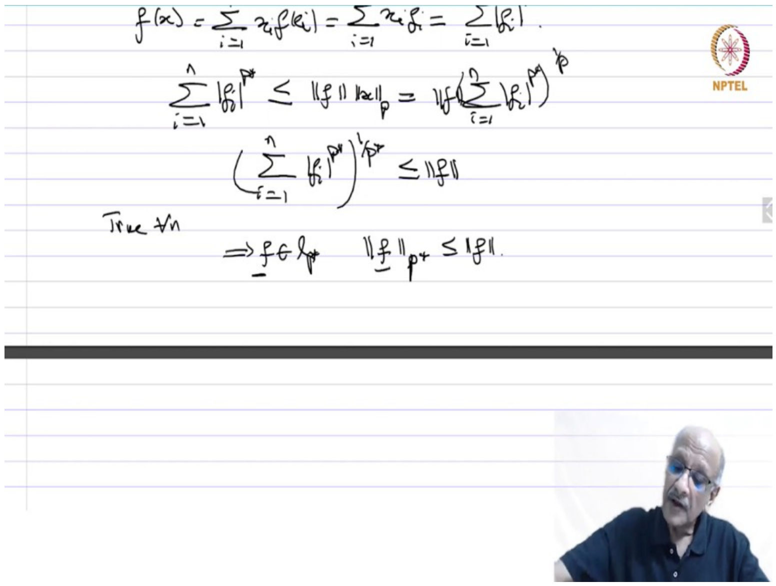
So, $f_y \in l_p^{¿}$and $\|f_y\| \leq \|y\|_{p^{¿}}$. I want to now show that every continuous linear functional on l_p occurs in this fashion and in fact this inequality is an equality.

So, let us take let now $f \in l_p^{¿}$. It is a arbitrary element of the dual. Now let (e_i) be the sequence which is 0 everywhere, 1 in the i-th place and you define $f_i = f(e_i)$. So, now you have a sequence $\underset{\sim}{f} = (f_i)$. So, I have a candidate for an element in $l_p^{¿}$. So, the questions I am going to ask are, if $\underset{\sim}{f} \in l_{p^{¿}}$ and can we say $f(x) = \sum_{i=1,2,\ldots\infty} x_i f_i, \forall x \in l_p$ and then thirdly, $\|\underset{\sim}{f}\|_{p^{¿}} = ¿|f| \vee ¿$. So, these three questions if you answer then we will say that in fact, the dual of l_p is nothing but $l_{p^{¿}}$. So, that is why we have put this notation here.

Let n be a fixed positive integer.

(Refer Slide Time: 12:15)





We define (x_i) in the following manner:

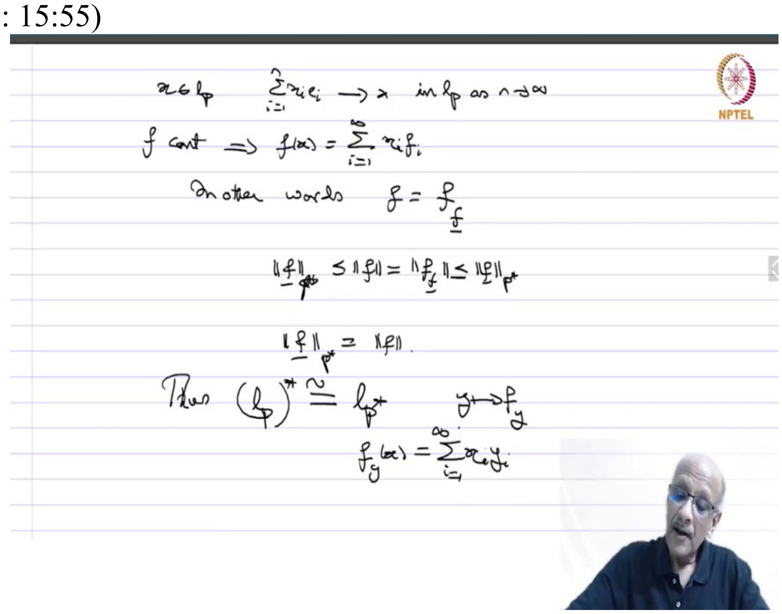
$$x_i = 0, \text{ if } f_i = 0, 1 \le i \le n \text{ and } x_i = \frac{|f_i|^{p^*}}{f_i} \text{ if } f_i \ne 0, 1 \le i \le n \text{ and } x_i = 0, \text{ if } i > n.$$

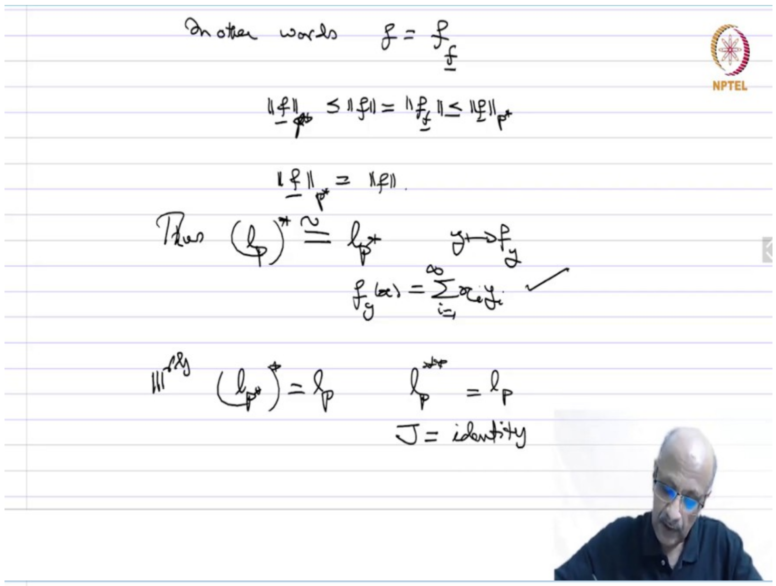
So, if you look at x, it is in fact of the form $x = (x_1, x_2, \ldots x_n, 0, 0, \ldots) \in l_p$ and further you can say $x = \sum_{i=1,2,\ldots\infty} x_i e_i$. Therefore, since f is linear,

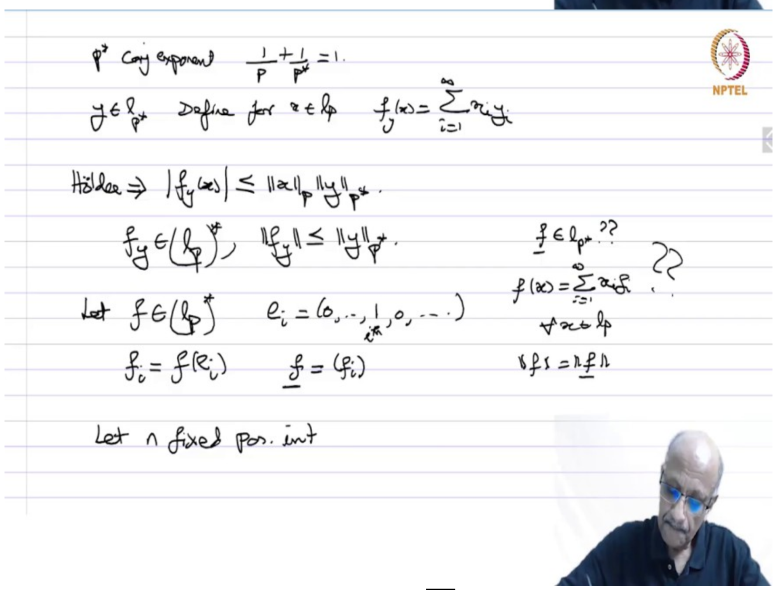
$$f(x) = \sum_{i=1,2,\ldots\infty} x_i f(e_i) = \sum_{i=1,2,\ldots\infty} x_i f_i.$$

But let me put what is the definition of x_i. $f(x) = \sum_{i=1,2,..n} |f_i|^{p^*}$. Therefore, $\sum_{i=1,2,\ldots n} |f_i|^{p^*} \le \|f\| \|x\|_p = \|f\| \left(\sum_{i=1,2,\ldots n} |f_i|^{p^*} \right)^{\frac{1}{p}}$. This implies $\left(\sum_{i=1,2,\ldots n} |f_i|^{p^*} \right)^{\frac{1}{p^*}} \le \|f\|$. Now, this is true for all n. So, this implies that $\underline{f} \in l_{p^*}$ and $\|\underline{f}\|_{p^*} \le \|f\|$.

(Refer Slide Time: 15:55)







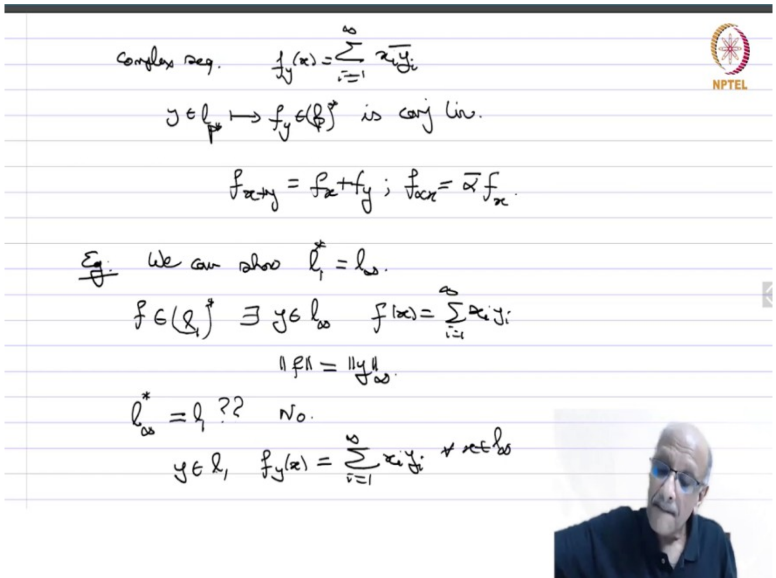
Further, if you take $x \in l_p$ then if you look at $\sum_{i=1,2,..n} x_i e_i \to x$ in l_p as $n \to \infty$.Because of the continuity of f, we have $f(x) = \sum_{i=1,2,...\infty} x_i f_i$.

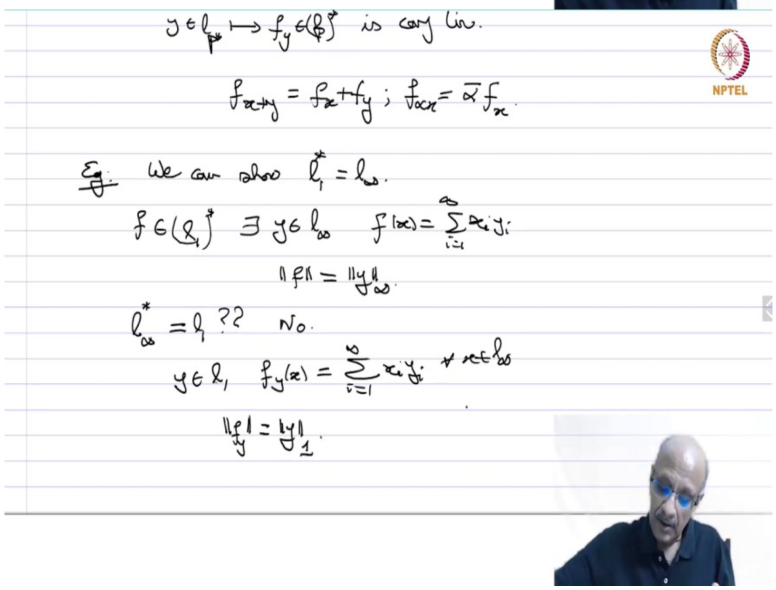
In other words, f is nothing but $f_{\underline{f}}$. So we have shown that $\|\underline{f}\|_{p^*} \le \|f\| = \|f_{\underline{f}}\| \le \|\underline{f}\|_{p^*}$ and therefore, the two norms are equal and so, you have $\|\underline{f}\|_{p^*} = \|f\|$..

Thus we have that $l_p^* \cong l_{p^*}$. lp via the relation $y \mapsto f_y$.

So now you can do the same game with p^*. So, similarly, $l_{p^*}^* \cong l_p$. So $l_p^{**} = l_p$ and in fact the mapping $J = identity$. Therefore, l_p is reflexive. So, this is an example of a reflexive space.

(Refer Slide Time: 19:08)



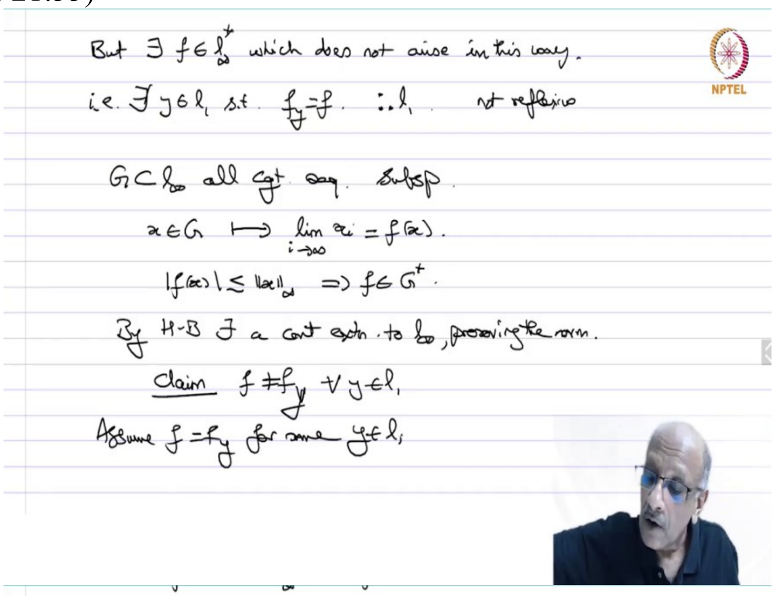


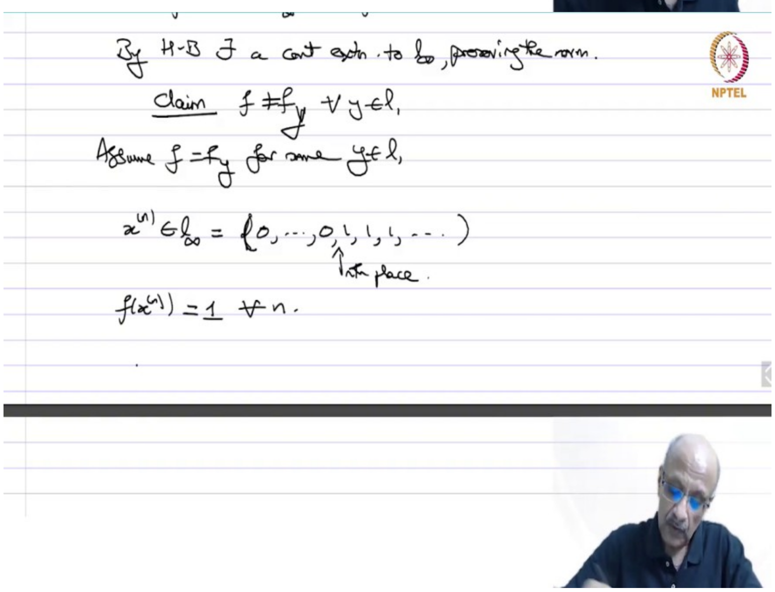
In the previous thing, as I said, we did everything for real. So, if we are looking at complex sequences then you will define $f_y(x) = \sum_{i=1,2,\ldots\infty} x_i \overline{y}_i$ and then the mapping $y \in l_{p^{¿}} \mapsto f_y \in l_p^{¿}$ is conjugate linear. What does it mean? You have $f_{x+y} = f_x + f_y ; f_{\alpha x} = \overline{\alpha} f_x$. This is a conjugate linear map, but it is still an isomorphism, the mapping J is identity and it is on to everything else goes through.

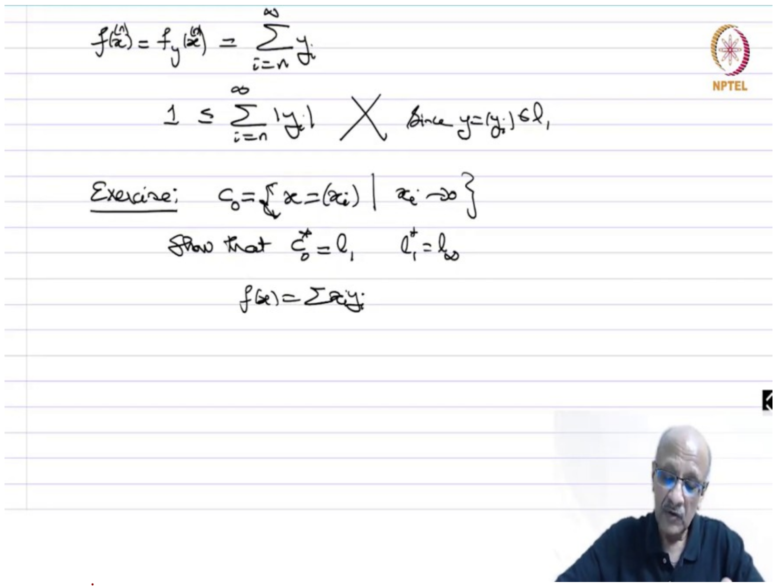
Example In the same way you can show we can show that $l_1^{¿} = l_\infty$. So, given a continuous linear functional $f \in l_1^{¿}$, there exists a $y \in l_\infty$ such that $f(x) = \sum_{i=1,2,\ldots\infty} x_i y_i$ and $\|f\| = \|y\|_\infty$.

Now, if you start with $l_{\infty}^{¿}$, is $l_{\infty}^{¿}=l_1$? Answer is no. So, if I have $y\in l_1$, I can define $f_y(x)=\sum_{i=1,2,\ldots\infty} x_i y_i, \forall x\in l_\infty$. This is a continuous linear functional by Holder inequality and in fact, you will have $\|f_y\|=\|y\|_1$.

(Refer Slide Time: 21:55)







But, but there exists $f \in l_{\infty}^{¿}$ which does not arise in this way i.e., there does not exist $y \in l_1$ y in l1 such that $f_y = f$. Therefore, l_1 are not reflexive. Later we will see that this also means l_∞ is not reflexive.

So, we want to produce a continuous linear functional on l_∞ which will not come in this way. So, let us take $G \subseteq l_\infty$ set of all all convergent sequences. So, this is a subspace. If $x \in G$, then you map it to $\lim_{i \to \infty} x_i =: f(x)$. Then $|f(x)| \leq \|x\|_\infty$ and therefore, $f \in G^{¿}$. So, by Hahn Banach, there exists a continuous extension to l_∞ preserving the norm, let us continue to call that as f. Claim $f \neq f_y, \forall y \in l_1$. So, you cannot produce $y \in l_1$ which comes like this, let us assume $f = f_y$ for some $y \in l_1$. So, now we want to get a contradiction.

So, you look at the following sequence $x^{(n)} \in l_\infty = (0, \ldots 0,1,1,1, \ldots)$ upto the n-th place it is 0. So, this is a convergent sequence limit is 1. So, $f(x^{(n)}) = 1, \forall n$. So, if $f = f_y$, then $f(x^{(n)}) = f_y(x^{(n)}) = \sum_{i=n,\ldots\infty} y_i$. So $1 \leq \sum_{i=n,\ldots\infty} y_i$ yi. This is a contradiction, since $y = (y_i) \in l_1$.

If it is in l_1, the tail of a convergent series should be going to 0 instead it is always greater equal to 1. So, there do exist continuous linear functionals on l_∞, which do not come from l_1, and therefore, l_1 is not a reflexive space.

Exercise $C_0 = \{x = (x_i) : x_i \to 0\}$. We saw this was a close subspace of l_∞. So, it is a Banach space. Show that $C_0^{¿} = l_1$ and $l_1^{¿} = l_\infty$. So, C_0 and l_∞ are not the same obviously, this is strictly proper close subspace of l_∞ and therefore, this is another a direct proof that these two spaces are not the same, I mean that l_1 is not reflexive.

So, $C_0^{¿}=l_1$ and $l_1^{¿}=l_\infty$ and therefore, you have all these are examples of non reflexive spaces. But l_p is reflexive for all $1<p<\infty$. We will see many other ways of proving this non reflexivity of the spaces. So, with this I will stop the analytic version and so discussion of reflexivity and which was also a consequence of analytic version of the Hahn Banach theorem. And so, our next topic which we will take up now are the geometric versions of the Hahn-Banach theorem.