Functional Analysis
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IMSc
Normed Linear Spaces

Hello, welcome to this course on Functional Analysis.

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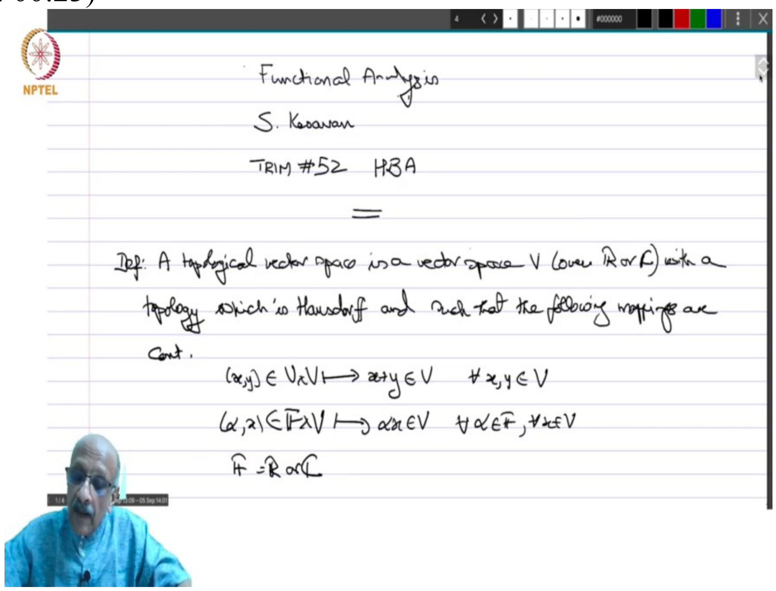

I will be following fairly faithfully the book Functional Analysis by myself, which appears in the TRIM series, number 52 of the Hindustan book agency. So, this is the book which we will be following. So, functional analysis is essentially the marriage of linear algebra and analysis; we do analysis on vector spaces. And to do that a vector space needs to be endowed with a topology which has to be compatible with the linear structure. So, more precisely we have the following definition:

Definition. A topological vector space is a vector space V over R or C (we will always only have these two fields) with a topology which is Hausdorff and such that the following mappings are continuous:

Vector Addition: $(x, y) \in V \times V \longmapsto x + y \in V, \forall x, y \in V.$

Scalar Multiplication: $(\alpha, x) \in F \times V \longmapsto \alpha x \in V, \forall \alpha \in F, \forall x \in V.$

$F = R$ or C.

So, we need that these two mappings are continuous.

The standard example of topological vector space is what we call normed linear space.

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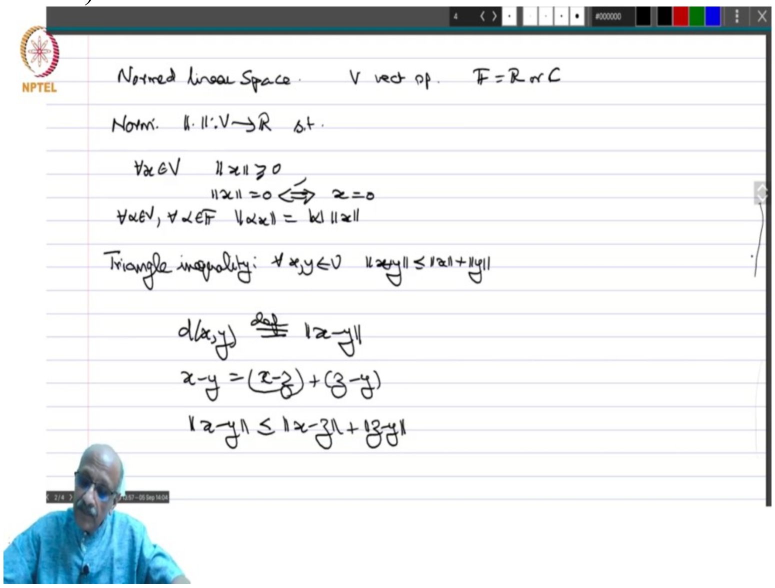

Normed linear space. This is a vector space which is equipped with a norm.

What is a norm?

Norm. Let V be a vector space over R or C. A norm is a function from V to R, such that

$(i) \forall x \in V$, we have $¿ \vee x \vee ¿ \geq 0$,

$(ii) \|x\| = 0 \Longleftrightarrow x = 0$,

$(iii) \forall x \in V, \forall \alpha \in F$ we have $\|\alpha x\| = |\alpha| \vee ¿ x \vee ¿$; $F = R \vee C$.

(iv) The triangle inequality: $\forall x, y \in V$, we have $\|x + y\| \leq \|x\| + ¿ |y| \vee ¿$.

Whenever we are confronted with the problem of verifying whether given function defines a norm or not, the first three properties will be more or less obvious, and most of the effort, if any, would go in verifying this last statement, namely the triangle inequality. So, once a vector space with a norm would be called a normed linear space. So, given a normed linear space we can define a metric $d(x, y) = ¿ |x - y| \vee ¿$. It is clear that $d(x, y)$ is non-negative and $d(x, y) = 0$ if and only if $x = y$. Now, if you have $x, y, \wedge z$ then $x - y$ can be written as $x - y = (x - z) + (z - y)$ and therefore by the triangle inequality, we get $\|x - y\| \leq \|x - z\| + ¿ |z - y| \vee ¿$. Therefore, the distance function d satisfies the usual triangle inequality for a metric; and that is why we have the same name for these two inequalities.

Therefore, automatically a normed linear space gets a topology defined by this norm which is a nice metric topology; and that is called the norm topology of this vector space. So, now let us see whether this norm topology makes those two functions continuous. Since we are dealing with metric spaces; it is enough to check continuity via convergence of sequences.

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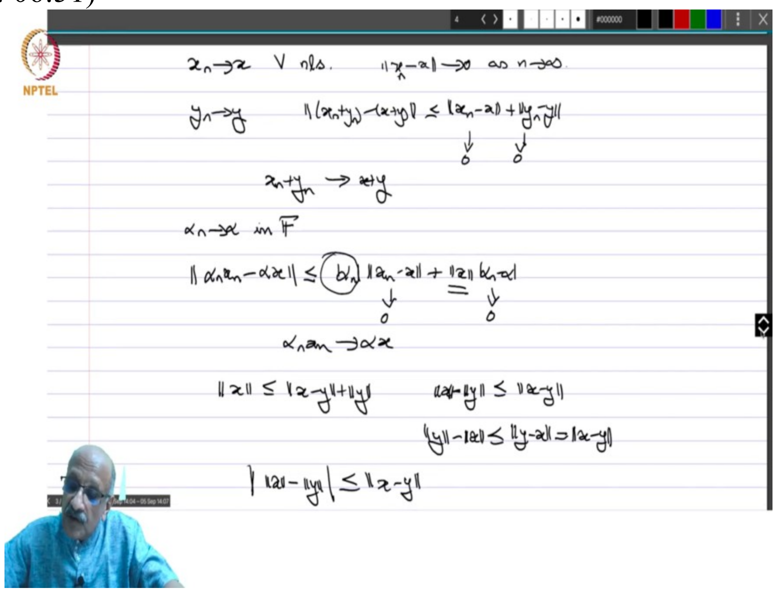

Now, what do we mean by $x_n \to x$ in a normed linear space V.

Definition of convergence: $x_n \to x$ if $\|x_n - x\| \to 0$ as $n \to \infty$.

Now if you have $x_n \to x$, $y_n \to y$, then you can write that

¿.

So, $x_n + y_n \to x + y$; so, we have that vector addition is indeed continuous.

Similarly, if $x_n \to x$ in V and $\alpha_n \to \alpha$ in F (which is always Ror); then by adding and subtracting the appropriate thing we can have

$$\|\alpha_n x_n - \alpha x\| \leq |\alpha_n| \|x_n - x\| + \|x\| |\alpha_n - \alpha|.$$

Since $\alpha_n \to \alpha$, $\|x\| |\alpha_n - \alpha| \to 0$ and since $x_n \to x$, $\|x_n - x\| \to 0$and α_n being a convergent sequence is automatically bounded, therefore $|\alpha_n| \|x_n - x\| \to 0$. So, you get $\alpha_n x_n \to \alpha x$. Thus scalar multiplication is also continuous and therefore a normed linear space is automatically a topological vector space.

The norm itself is a continuous function in a vector space, because if you are given two vectors x and y, by writing $x=(x-y)+y$ and using the triangle inequality:

$$\|x\| \le \|x-y\| + \|y\| \text{ hence } \|x\| - \|y\| \le \|x-y\|,$$

Similarly, $\|y\| - \|x\| \le \|y-x\|$, but $\|y-x\| = |-1|\|x-y\| = \|x-y\|$.

Therefore, you get $\big|\|x\| - \|y\|\big| \le \|x-y\|$. This is very useful and a useful inequality to remember; and this shows that the norm itself is a continuous function in a normed linear space.

So, what is you have in a normed linear space and the associated norm topology with symmetric topology? You have sequences which are Cauchy and if every Cauchy sequence converges, you say that this space is complete.

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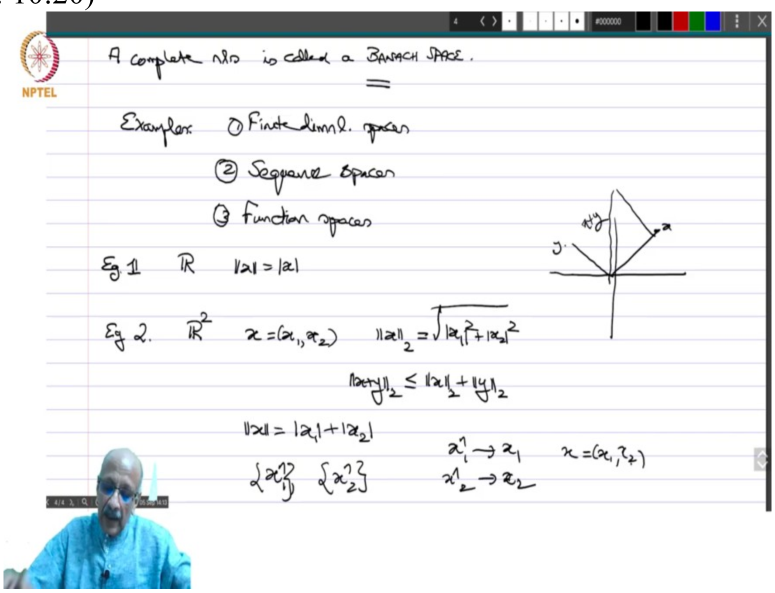

A complete normed linear space is called a Banach space; i.e., we have a vector space on which we have defined a norm that gives you a metric topology called the norm topology, and if this topology is complete then the normed linear space is called a Banach space.

So, now it remains to give some examples of normed linear spaces. So, I am going to give three classes of examples:

(i) finite dimensional;

(ii) sequence spaces;

(iii) function spaces.

Therefore, this is a very rich class and you will see several examples of function spaces during the course. To start with, this is a good way of classifying the vector spaces and looking at the various examples.

So, let us start with finite dimensional spaces.

Example 1. Let us just take R. In throughout, I am not going to say R or C; I will do the calculations for R and you can easily check that the complex case is identical almost. In case there are some special changes to be made; I will then explain in that situation. So, I will deal with the real vector spaces for most of the time, and most of those results will carry over to the complex case without problem.

So, R is a one dimensional vector space over itself, and we define the norm of x as the usual modulus of x. Then of course it is now trivial that all the three properties, the norm $¿x\vee¿$ is equal to 0 if and only if x equals to 0. And the triangle inequality is of course well known $|x+y|\leq\vee x\vee¿+¿y\vee¿$. And R is a complete metric space as you know and therefore this is an example of a Banach space.

Example 2. Let us take another example. This time I will deal with R^2; so given any vector x in R^2, it will have two coordinates x_1 and x_2. And I am going to define

$$¿|x|\vee¿_2=\surd(x_1^2+x_2^2)¿$$

Again the first three properties, non-negativity, equal to 0 if and only if x equals 0; and $¿|\alpha x|\vee¿$ equals $|\alpha|\vee|x|\vee¿$ are all trivial from this. So, what about the triangle inequality? So to see the triangle inequality, we call that $¿|x|\vee¿$ is nothing but the magnitude of x_1, x_2. So, for any two vectors $x, y\in R^2$, $x+y$ represents the diagonal of the parallelogram formed by x and y. So, now its a old theorem from high school that two sides of a triangle are in length greater than the third side. So that immediately tells you that $¿|x+y|\vee¿_2\leq¿|x|\vee¿_2+¿|y|\vee¿_2¿¿¿$. In fact, the name triangle inequality comes only from this case, because this is the generic case; and generalizations have been done from this and therefore this shows that this is a norm.

We can give one more norm on R^2, say, $\|x\|_1=|x_1|+¿x_2\vee¿$. And in this case its even easier to check the triangle inequality because it just comes from the triangle inequality of the mod.

Now, in both these cases if you take a Cauchy sequence (x_n); then x_n^1 will also be a Cauchy sequence and x_n^2 (the sequence of second coordinate) will also be a Cauchy sequence. Therefore, x_n^1 will converge to some x_1; x_n^2 will converge to some x_2. And we denote that $x=(x_1, x_2)$, and it is immediate to see that

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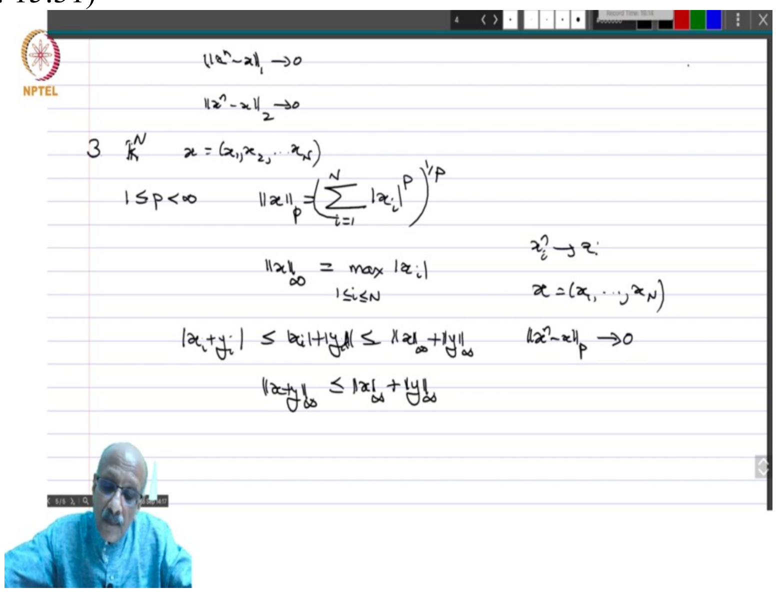

$\|x_n - x\|_1 \to 0$ and $\|x_n - x\|_2 \to 0$ because of the component wise convergence, the vector convergence also takes place. So, R^2 with either of these norms is a Banach space.

Example 3. Now, we will generalize all of this and then let us go to R^N; Let us go to R^N. So any vector x, you write $x=(x_1, x_2, \dots, x_N)$; then let $1 \le p < \infty$. Then we define

$$\|x\|_p = \left(\sum_{\{i=1,2,\dots N\}} |x_i|^p \right)^{1/p}$$

And I am also going to define another norm:

$$\|x\|_\infty = max_{i=1,2,\dots N}\{|x_i|\}$$

Now, it is easy to see again that the first three properties of a norm are immediately satisfied. The triangle inequality is easy for $\|x\|_\infty$, because you have $|x_i + y_i| \le |x_i| + |y_i| \le \|x\|_\infty + \|y\|_\infty$ for all $i = 1, 2, \dots N$. So if I take the maximum on the left hand side; I get $\|x+y\|_\infty \le \|x\|_\infty + \|y\|_\infty$. Therefore, this becomes a normed linear space.

Now, our main job is to show the triangle inequality is true for each of $\|x\|_p$ for all $1 \le p < \infty$.

Once this is done, we see that all of these define norms on R^N. So, we have an uncountable family of norms defined here and in each case if you took a Cauchy sequence; it automatically means that the coordinate sequences are also Cauchy. Therefore they will all converge since R is complete, C is complete; and therefore component wise you will have $x_{n,i} \rightarrow x_i$. Consequently if I define ¿$(x_1, x_2, \ldots, x_N)$; then you will get $\|x_n - x\|_p \rightarrow 0$. These are very easy to check and therefore our job will be complete.

So, R^N with any of these norms will be a Banach space provided we have shown that the triangle inequality is true for $\|x\|_p$; $1 \leq p < \infty$. For $p = 1$, we have already seen it, so for $1 < p < \infty$, we want to show that $\|x\|_p$ is indeed a norm and which we will now do.