Basic Calculus - 1

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Lecture 5 - Part 1

Limits of functions - Part 1

Well, this is lecture 5 of Basic Calculus 1. In the last two lectures, we had introduced the notion of functions and then given some examples of functions. In fact, that consists of a repertory. Whenever we need some functions, we will take from them, or even join them together depending on different domains. That is how they will be helpful for us. Today, we will be considering the central idea of this calculus course, which is called the limits of functions. It concerns about the nearness of a point and how this affects corresponding functional values.
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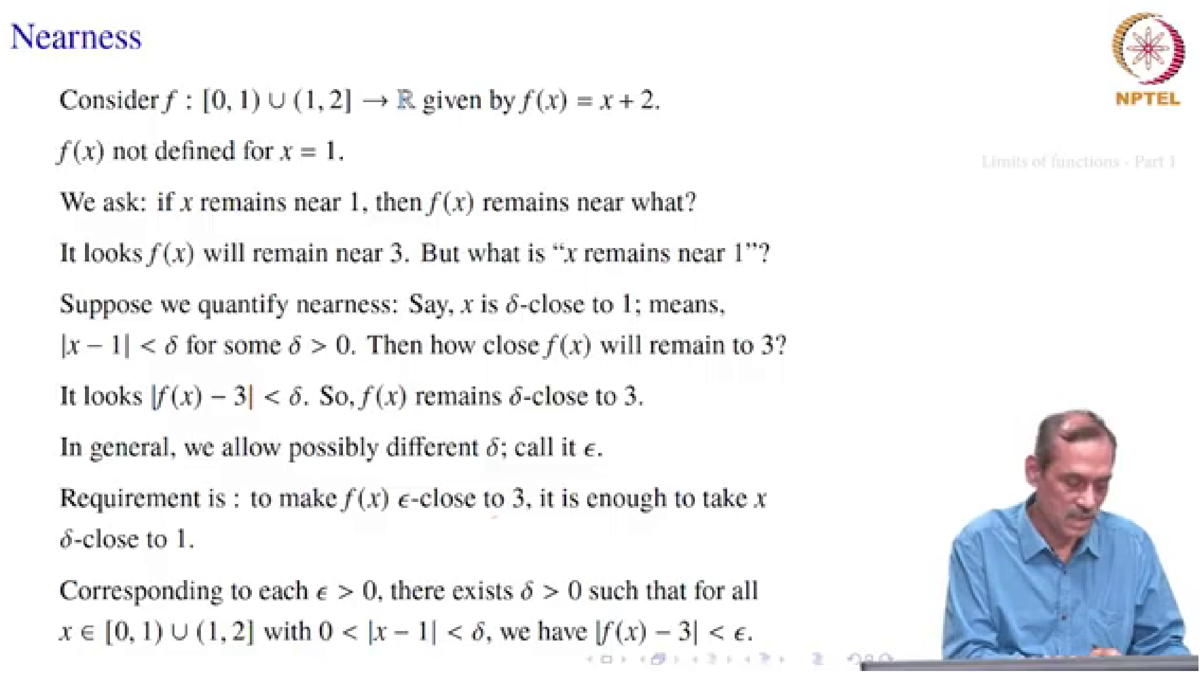

Let us start with one simple example. Let f be a function from $[0, 1) \cup (1, 2]$ to $\mathbb{R}$. The domain is the interval $[1, 2]$ except the point 1. Suppose f is defined on this, and it is taking real values. It is defined by $f(x) = x + 2$. It is a fairly simple function. Let us consider this function.

Now the question we are going to ask is this. First, notice that $f(x)$ is not defined for $x = 1$. We can think of x to be remaining near 1 on either side: either it is less than 1 or it is bigger than 1, but it cannot be 1 since f is not defined there. The point 1 is not in the domain. So, suppose, x remains near 1. And then where does its functional values remain? Well, had $f(1)$ been defined with the same function, it would have been 3. So, naturally we would think that if x remains near 1, then $f(x)$ might remain near 3. That is how it looks.

But what do we mean by this nearness? Only after confirming what this nearness we are talking about, we can think of whether this $f(x)$ remains near 3 or not. Well, that is a difficult thing.

Because we do not know what is the meaning of 'near' in real numbers. What happens is: if you take 1 and you choose a scale, say, 0 is here and this is 1, then you may think 'near' to mean that x is somewhere here. But suppose I have taken a bigger scale like: this is 1, but this is not 0, this may be, say, half. Then, it is twice that scale. Then that quantity, which we thought earlier may or may not be any more near.

So, what exactly we mean by remaining near this? This is what we should first formalize. Instead of general 'nearness', we may think of 'how near or how close'. Suppose, I choose a point, say, it is 0.9, somewhere close to 1. I may think of it is near 1. But with some tolerance slightly more than 0.1, within the radius of point 1, I might accept all of them to be near, that is, 0.91, 0.92 and so on, all those things can be taken as near 1. On the other side, if my nearness is something like 0.1 (some quantification of nearness), then I would think 1.09 is also there; it is also near 1. This is what we mean by telling how near, we say δ-close. So, x is δ-close to 1 means the distance between x and 1 should be less than δ. Obviously, I should take positive δs only when we think of δ-close.

So, our question will be something like if x remains in this interval, which is $1 - \delta$ to $1 + \delta$, but x is never equal to 1, then $f(x)$ remains close to which number? And in 'how close'? That is what we want to see. That means how close $f(x)$ will remain to 3? It looks like it will be same δ because it is $x + 2$. It is just shifts by two units. When x is near 1, it is δ-close to 1, $f(x)$ should be δ-close to 3. It looks like $|x - 3|$ should be less than δ. It is so because we can find it out. If you substitute $x + 2$ here, then 3 gets cancelled, you get the same inequality $|x - 1| < \delta$.

That is what we want to see in general, and not only for this function. So, the question is if x is δ-close to 1, then $f(x)$ is how close to 3? In this case we see that it is δ; but in general it might not be that. It is just a shifting here, that is what is happening. Suppose $f(x)$ is x^2. Then it may not be that δ. So, let us allow different closeness in the values of $f(x)$, call it ϵ. Our requirement is: if x is remaining δ-close to 1 then $f(x)$ should remain ϵ-close to that number, in this case, it is 3.

In fact, our requirement is the other way around. We say that to make $f(x)$ epsilon close to 3, I should take x delta close to 1. In this general sense of epsilon and delta, that is really our requirement. We will say that $f(x)$ remains close to 3 in epsilon sense if x remains close to 1 in this delta sense. We would think something like this: to make $f(x)$ and 3 to be epsilon close, should choose x and 1 delta close. We will thus say that corresponding to each $\epsilon > 0$, I do not know what is that ϵ, then I should be able to find out a δ such that x and 1 should be δ-close. Of course, x should be in the domain of the function; here, $|x - 1| < \delta$ and 1 is excluded. So, this is how it reads now: corresponding to each $\epsilon > 0$, there exists $\delta > 0$ such that for all x in our domain of f with $0 < |x - 1| < \delta$, this is your delta closeness of x to 1, then we should have $|f(x) - 3| < \epsilon$; that is, $f(x)$ will be epsilon close to 3. So, this is our requirement. We really abstract it to any function, not only for this function, and we should give a formal definition now.

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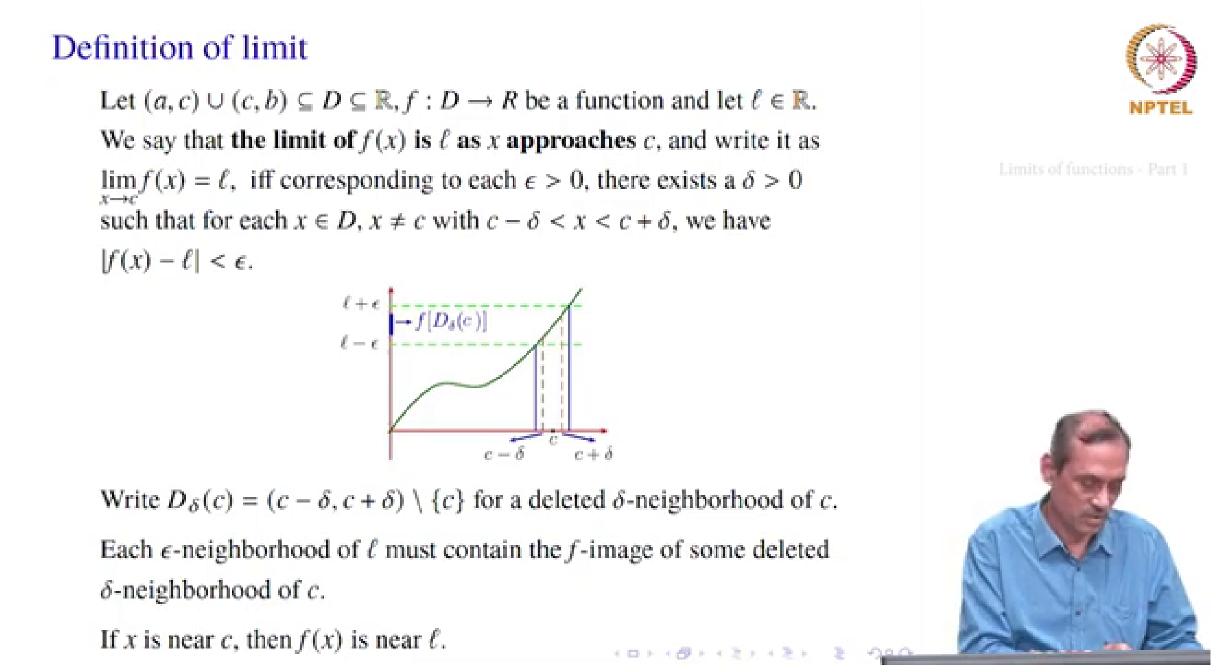

Let us say our domain of the function f is D such that it has a neighborhood around c inside D; possibly c is excluded; we are not interested in that value of c or whether f is defined at c or not. To the left of it, there is an open interval which is contained in D and to the right of c there is also an open interval which is contained inside D. This condition should hold, because we need all points δ-close to that c. So, suppose ℓ is a real number. We would say that **the limit of** $f(x)$ **is** ℓ **as** x **approaches** c. Look at these bold letters, this is what we are going to define. In the earlier case, our limit was 3, the limit of $f(x)$ is ℓ, $\ell = 3$ in that case, as x approaches c. So, x is here and the values of $f(x)$ are on the y-axis. The point c is here somewhere, x is remaining near c and ℓ is somewhere here. We say that $f(x)$ is lying near ℓ. That is what we say; the limit of $f(x)$ is ℓ as x approaches c; and we write this way: $\lim_{x \to c} f(x) = \ell$.

Now, you can read it the same way "limit of $f(x)$ as x tends to c is equal to ℓ if and only if the earlier conditions should be satisfied. "Corresponding to each $\epsilon > 0$" says that you choose how close $f(x)$ should be to ℓ. Corresponding to that you will find a closeness in c that there exists a $\delta > 0$ such that for each $x \in D$ (x should be obviously in the domain) with $x \neq c$, $c - \delta < x < c + \delta$. So, it is within this, which we can write as $D_\delta(c)$, we may write this set as $D_\delta(c)$, where this D is for 'deleted', for c has been deleted, and everything else in the open interval remains: $(c - \delta, c + \delta)$ but $x \neq c$. For all those x, $f(x)$ should lie between $\ell - \epsilon$ to $\ell + \epsilon$. That is $|f(x) - \ell| < \epsilon$. If this condition is satisfied, we would say that the limit of a $f(x) = \ell$ as x approaches c.

Notice that ℓ is not given with $f(x)$; we have to guess that. Once we guess one ℓ, we can see whether this condition is satisfied for that ℓ or not. If it is satisfied, then ℓ becomes the limit, and if it is not satisfied, then ℓ is not the limit. Possibly there is another guess, or maybe there is no possibility of any guess; whatever ℓ we choose it is not possible, that can also happen.

So, this is the central idea of the limit. Writing $D_\delta(C)$ for the deleted neighborhood:

$(c-\delta, c+\delta) - \{c\}$, it says that each ϵ-neighborhood of ℓ must contain the f-image of the deleted neighborhood $D_\delta(C)$. That is what the notion of limits is. We will take some examples and see how it happens. But we should not forget this intuitive idea that if x is near c, then $f(x)$ is near ℓ. By quantifying the idea of nearness o closeness we get this formalization.
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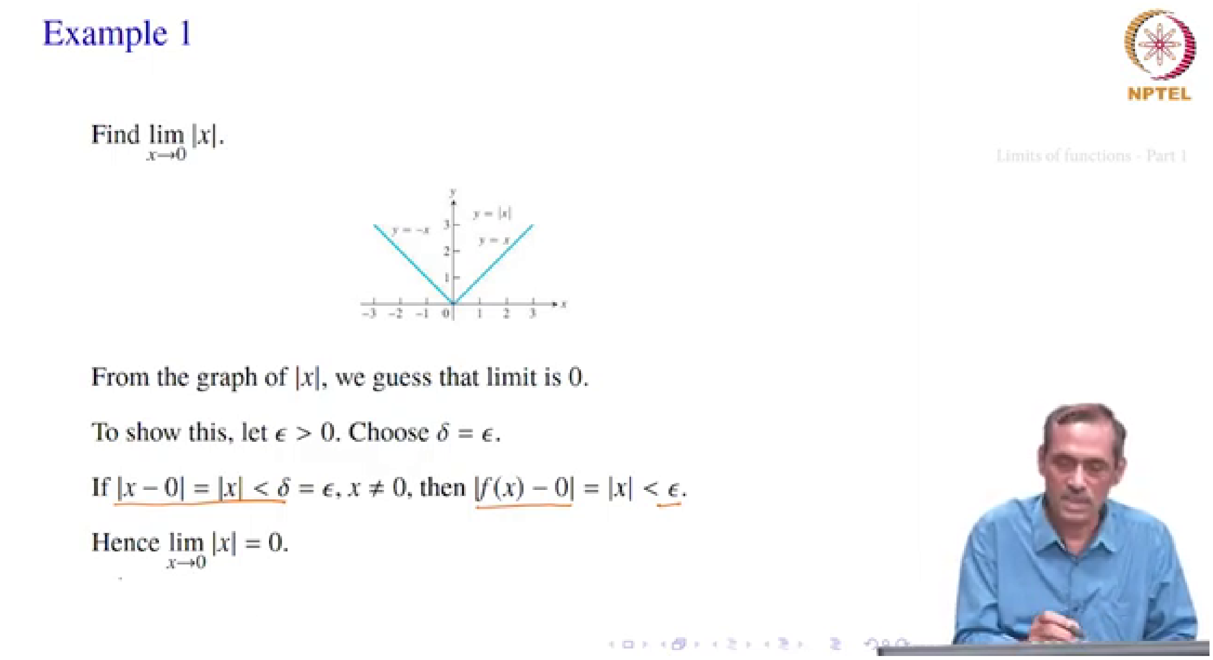

Let us take the function $|x|$. It is defined from the whole of real numbers and its range is non-negative real numbers. We are asked to find out its limit, if possible near 0; that is the limit of $|x|$ as x approaches 0. If you remember its graph, $|x|$ looks something like this. On the left side of 0, it is $y = -x$ and on the right side of 0, it is $y = x$. When x remains near 0, somewhere here, it looks that $f(x)$ is also remaining near 0. So, our guess is that the limit must be 0, if it already exists. We have to really verify whether it happens or not. Well, that means for any ϵ given, we have to get a δ such that the condition is satisfied. It looks that the same delta will do because $|x|$ is roughly equal to x or $-x$. So, if the distance of x from 0 is fixed, then that of $|x|$ is also fixed the same way. So, we are trying to choose $\delta = \epsilon$ and see what happens.

This is the crux of the problem. We have to choose a delta, depending on the epsilon. Here $c = 0$. In general, this δ might depend on both ϵ and c. And how to choose δ needs some experience. But let us see how it goes by looking at this function. It helps to know how the function is behaving, like x or like x^2 and so on. Here, it is roughly like x, that is why you are choosing $\delta = \epsilon$, and trying to see whether it is satisfied or not. If it satisfied, it is fine. If it is not satisfied, then that will not say that it is not the limit; there can be another choice of δ. Let us see what happens.

So, with this $\delta = \epsilon$, suppose I take $|x - 0| < \delta$. For all these x, I look for the difference between $f(x)$ and 0. It is again $|x|$ and $|x|$ is already less than ϵ. So, it says that if $|x - c| < \delta$ with $x \neq c$, then $|f(x) - \ell| < \epsilon$. Therefore, this ℓ must be the limit of $f(x)$. So, $\lim_{x\to 0} |x| = 0$. This is how we are going to solve some problems. In fact, once you get experienced, you do not have to do all these things; you can see through how this intuitive idea of nearness works.

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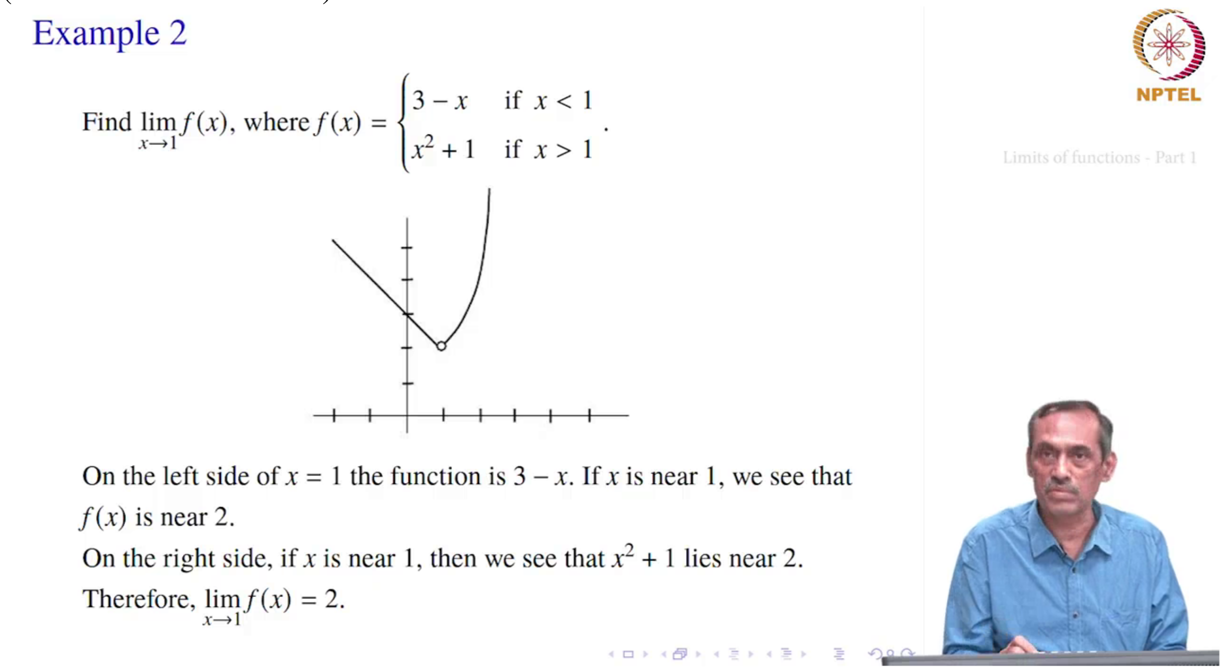

Let us look at this. Say, $f(x) = 3 - x$ if $x < 1$, $f(x) = x^2 + 1$ if $x > 1$, and $f(1)$ is not defined. That means the domain of f is $(-\infty, 1) \cup (1, \infty)$. Except at 1, f is defined everywhere else. We want to find the limit of $f(x)$ as x goes to 1. So, how do we see it? On the left side if I take x is near 1, then this is something smaller than 1, but it is near 1. Then, $3 - 1 = 2$, that is, it is near 2. On the right side, if I take $x > 1$ but near 1, then x^2 is still near 1, and $1 + 1 = 2$. On both sides it is 2. So, my guess is that at $x = 1$ the limit of $f(x)$ should be 2.

If you plot the graph, it looks like this. Here is your 1. On the left side, it is $3-x$. This is $3-x$. On the right side, it is $x^2 + 1$. And this point is omitted. Because at 1, function is not defined. This is how the graph looks like. From the graph it is clear that $f(x)$ is remaining near 2. So, that should be our limit. Now, our guess is that. Let us see what happens.

How do we really justify it? You may think of this. On the left side, you may have choose a δ for an ϵ; and on the right side again, you may have to choose a δ. Sfter choosing, you will check formally.

Suppose, $\epsilon > 0$ is given. Now, we choose $\delta = \min\{\epsilon, \sqrt{1+\epsilon} - 1\}$. It is very unintuitive. How did I choose this? You will see it only during verification process. The idea is roughly this. On the left side, it is $3 - x$, just a shift of x So, δ should be equal to ϵ. So, that comes from the left side. And on the right side, it is $x^2 + 1$. If I substitute $1 + \delta$ there, it is $(1+\delta)^2 + 1$ and then minus 2. That will be $(1+\delta)^2 - 1$. And when I go to x, it will be the square root of that. That is why this one is taken: $\sqrt{1+\epsilon} - 1$. It is put to work with the function $x^2 + 1$ on the right side of 1. But at this moment, there is no need to see how. We just say: choose delta equal to this and see how it is verified.

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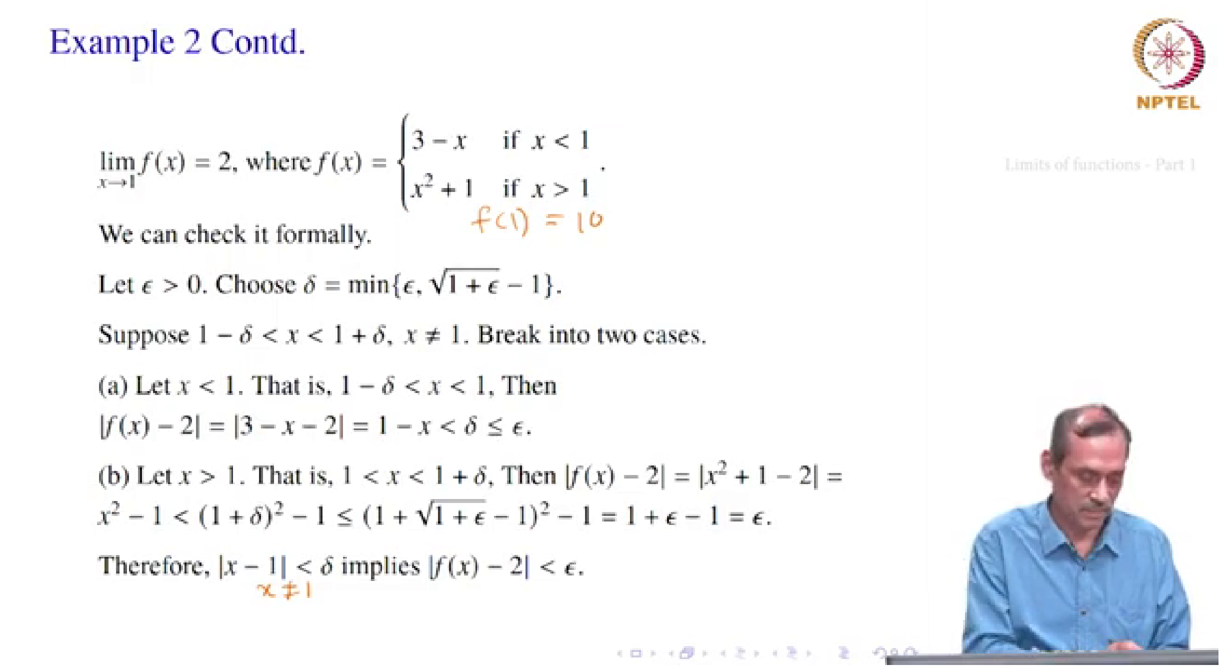

Now you have to start with any x near this 1, which is δ-close; that is, with $1 - \delta < x < 1 + \delta$, but $x \neq 1$. Choose any such x. On the left side, the function is given in a different way than from the the right-side. So, we break it into two cases depending on the value of x: when $x < 1$ or $x > 1$.

Let us consider the first case: $x < 1$. This condition along with $x < 1$ gives rise to $1-\delta < x < 1$; this is our assumption. Now under this assumption, we have to verify whether $f(x)$, which is $3-x$, and then minus 2; and its absolute value, that is, $|3 - x - 2|$ should be less than ϵ. Is it so? $|3 - x - 2| = |1 - x|$ and $1 - \delta < x < 1$ say that $1 - x < \delta$. As delta is always minimum of epsilon or this, it is less than or equal to epsilon. Hence, this condition is satisfied. That is, if we assume that $x < 1$ and $1 - \delta < x < 1 + \delta$, then we are able to verify that $|f(x) - 2| < \epsilon$.

Now you take the second case: $x > 1$. In this case, with the assumption $1 - \delta < x < 1 + \delta$, we see that x must be lying between 1 and $1 + \delta$ So, $1 < x < 1 + \delta$. We should verify our condition for $|f(x) - 2|$. Here, it is $|x^2 + 1| - 2 = x^2 + 1 - 2 = x^2 - 1$. Now, $\delta \leq \sqrt{1+\epsilon} - 1$. So, $(1 + \delta)^2 - 1$ is less than or equal to this. We just substitute that here and that would be simplified. So, 1 cancels and this gives $1 + \epsilon$ and again minus 1, so that is ϵ.

That is why we have chosen this hoping that it might work this way. You have to take care of the algebra; anticipating what is going to happen. And according to that you have to choose delta. Sometimes we do back-calculation. If $f(x) - 2 < \epsilon$ is going to be satisfied, then what would have been our x. We come back slowly going from bottom to top. We will reach there. That is why this delta was chosen this particular way: minimum of ϵ and $\sqrt{1+\epsilon}-1$. You see that the limit condition as required is satisfied. Therefore, we conclude that $|x - 1| < \delta$ with $x \neq 1$ implies $|f(x) - 2| < \epsilon$.

Of course, with $x = 1$ something else might happen. But we are not concerned about that f is undefined at $x = 1$. Also, it does not matter whatever value is assigned to $f(1)$. Suppose, f is defined on the whole of $\mathbb{R}$. Say, along with these two conditions, we have $f(1) = 10$. Then, it does not matter because $f(1)$ is never used anywhere in this competition. Still, the limit of $f(x)$ as x goes to 1 will be equal to 2. It is so even if you define $f(1)$ equal to anything. So, it does not really matter whether it is defined there or not defined.