Basic Calculus - 1

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Lecture 4 - Part 2

Transcendental and trigonometric Functions - Part 2

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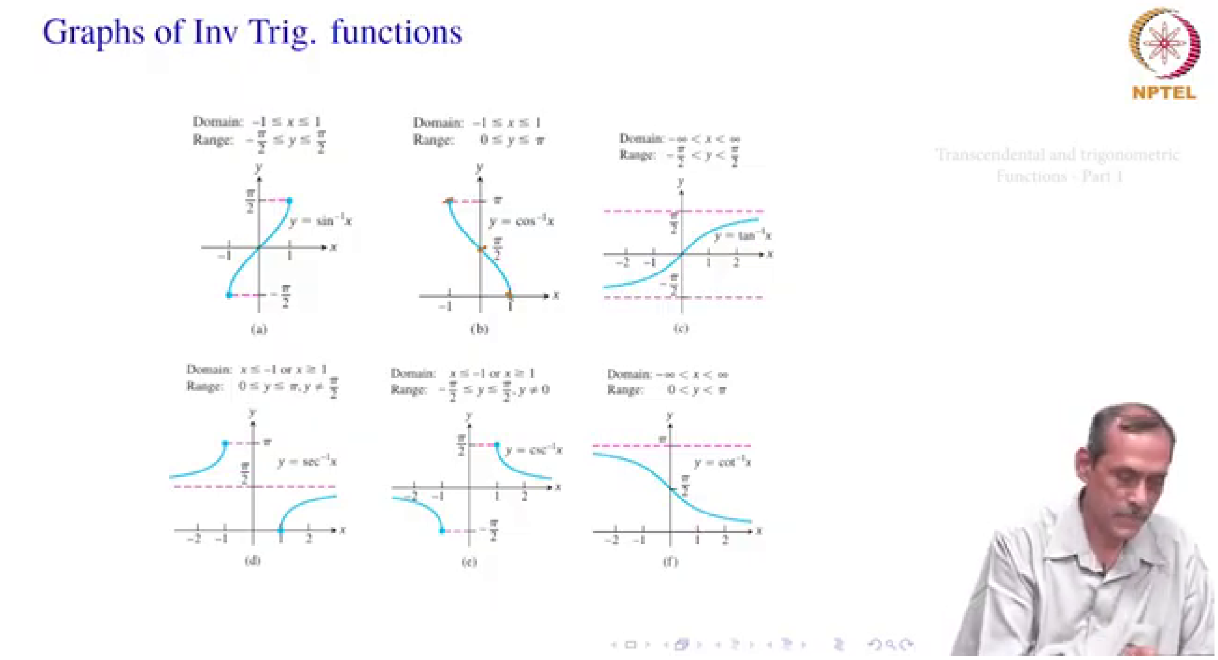

Here are the graphs of those inverse trigonometric functions. Let us look at the first one which is $\sin^{-1} x$. If you look at it taking y-axis as the horizontal one, then it is just your $\sin x$ curve. It is going this way; but that will not be a function, because a vertical line has to intersect at no more than one point. Here there are many points, but repetitions are not allowed. So, we are defining $\sin^{-1}$ from $[-1, 1]$ with the co-domain as $[-\pi/2, \pi/2]$. Of course, that becomes the range here, and there, it is a function. So, the domain of $\sin^{-1} x$ is $[-1, 1]$ and range is $[-\pi/2, \pi/2]$; that is how the curve looks like.

Similarly, we look at $\cos^{-1} x$. It is just a shift of $\sin^{-1} x$; the same pattern is shifted along x-axis by $\pi/2$. You get here this way. So, this is -1, this is π. At -1, it is π and at 0, it is $\pi/2$, and at 1, it is 0. That is how $\cos^{-1} x$ looks like. Its range is $[0, \pi]$ and and domain is $[-1, 1]$.

Now, look $\tan^{-1} x$. Again, you do the same trick of looking at it from the y direction. It would look like this. Its domain becomes the whole of real numbers and its range becomes $(-\pi/2, \pi/2)$.

Look at $\sec^{-1} x$. That comes from the cosine or even from the secant x, if you take its graph. So, at $y = \pi/2$, it will not be defined; and $y = \pi/2$ is never achieved with $\sec^{-1} x$. Its domain is again $\mathbb{R}$ excluding $(-1, 1)$, the open interval; so, $x \le -1$ or $x \ge 1$. Its range will be $[0, \pi] - \{\pi/2\}$, because at $\pi/2$, $\cos(\pi/2) = 0$, and then secant is not defined at $\pi/2$. So, $\sec^{-1} x$ will never achieve that value.

Now, Look at cosec ^{-1}x. Its domain is again same as that is $\sec^{-1} x$: $x \leq -1$ or $x \geq 1$; and its range will be from $-\pi/2$ to $\pi/2$; it is just a shift of $\pi/2$ from that of $\sec^{-1} x$. But $\sec^{-1} x$ is not defined for $\pi/2$; so cosec ^{-1}x will not be defined for $x = 0$. Thus, the range will be $[-\pi/2, \pi/2] - \{0\}$.

Similarly, the domain for $\cot^{-1} x$ is the whole of real numbers, and range is $(0, \pi)$. It just shifts from $(-pi/2, \pi/2)$ by $\pi/2$; so, we get $(0, \pi)$. This is the strip within which the values of $\cot x$ remain. That is how the inverse trigonometric functions would look like.

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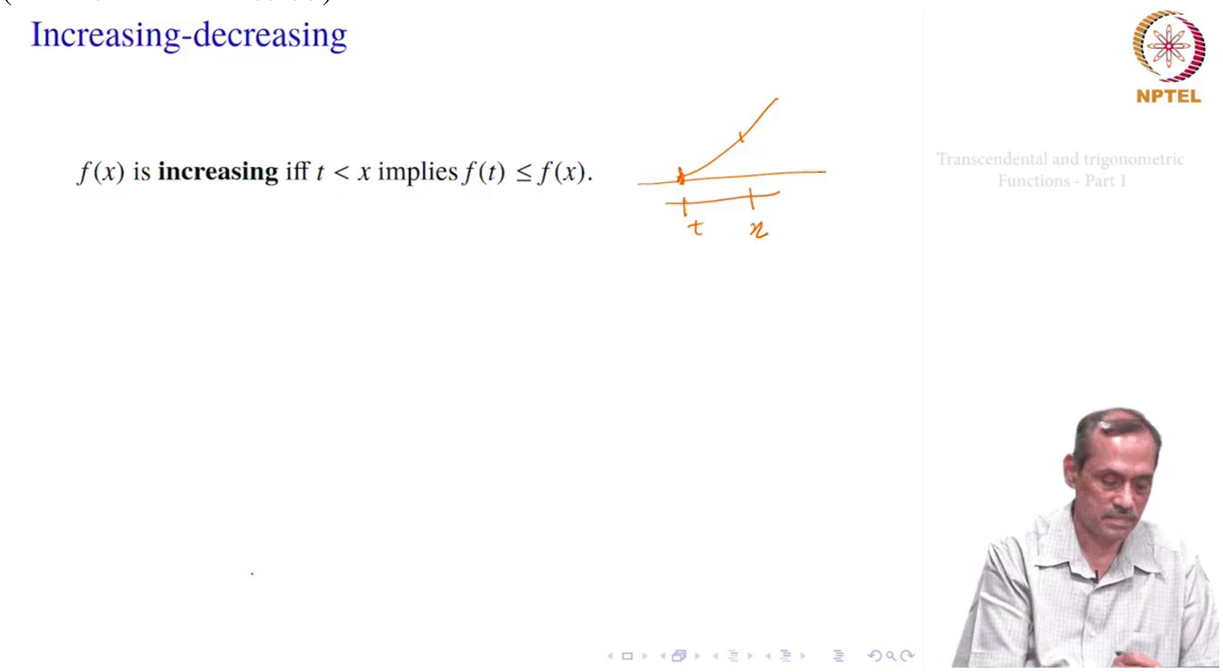

Let us look at what properties these functions do have. We will need something of this kind. When do we say that f is an increasing function? An increasing function means it is really climbing up in the graph. It will be climbing up like this when in its domain, whenever $t < x$, you see that its values, that is, $f(t) \leq f(x)$. That means, we also admit that a constant function is an increasing function; thus the equality sign. We say that $f(x)$ is increasing if and only if $t < x$ implies $f(t) \leq f(x)$, where t and x are any distinct numbers in the domain of f.

Similarly, we would say that $f(x)$ is a decreasing function if the inequality is reversed. That is, it is climbing down. When, instead of climbing it is falling, we will say that it is decreasing. That is, when $t < x$ implies $f(t) \geq f(x)$. Again, constant functions will become decreasing functions. These are both increasing and decreasing. When we write this equality, we allow this.

To eliminate the constant functions we will use the strict inequality and say that it is strictly increasing. That is, we say that $f(x)$ is strictly increasing if $t < x$ implies $f(t) < f(x)$. We thus use the adjective 'strictly'. Similarly, we say that $f(x)$ is strictly decreasing if $t < x$ implies $f(t) > f(x)$; now equality is not allowed.

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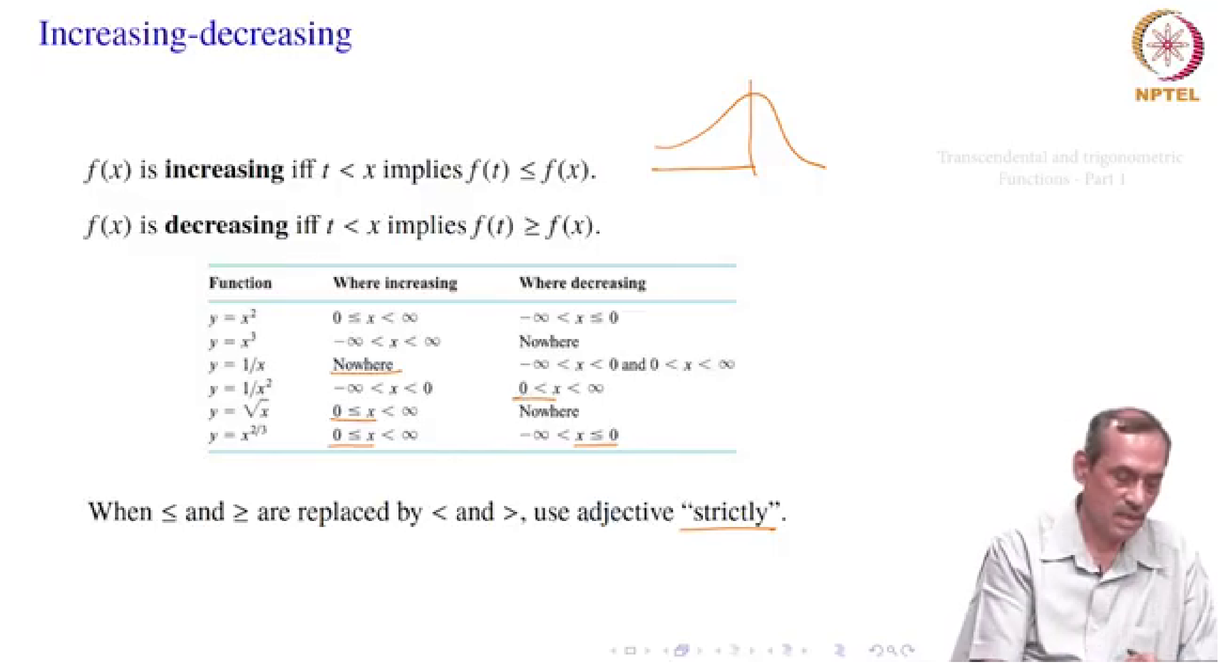

Let us see some examples, of increasing and decreasing functions. However, sometimes these conditions are not satisfied on the whole of its domain. A function can be climb and again fall. So, we will say that, in some part of the domain it is increasing and in some part of the domain it is again decreasing.

So, let us look at $y = x^2$. Where is it increasing? It is increasing for all non-negative x. And it is decreasing for all negative x. At 0, well, we can include it on both the sides. So, we say $y = x^2$ is increasing for $0 \leq x$ and decreasing for $x \geq 0$.

Let us look at $y = x^3$. If you remember its graph, then you find that it is an increasing function. It is never decreasing anywhere. That is also clear algebraically. You can see it very easily. If $t < x$, then $t^3 < x^3$. That is why it is increasing; and it never decreases.

Similarly, $y = x$ is increasing; because $t < x$ implies $t < x$.

If you take $y = 1/x$, the 'one by' will change the inequality: $t < x$ implies, $1/x < 1/t$ or $1/t > 1/x$. So, it will be decreasing; and it is not increasing nowhere. Whereas, $y = x$ is increasing. So, we say that $y = 1/x$ increases nowhere. And where does it decrease? We forget about 0; when $x < 0$, it decreases and when $x > 0$, it also decreases. So, it decreases everywhere. Of course, it is not defined at 0. The point 0 is not in its domain.

Look at $y = 1/x^2$. The function x^2 is increasing for x positive and decreasing for x negative. So, $y = 1/x^2$ is increasing for x negative and decreasing for x positive; at $x = 0$, it is not defined.

Now, look at $y = \sqrt{x}$. It is just like your x. But square $\sqrt{x}$ is only defined for non-negative x. It is increasing everywhere in its domain. It is increasing for $0 \leq x$; x is non-negative, and it decreases nowhere.

Let us take $y = x^{2/3}$. It is similar to the earlier ones, like x^3 0r $x^{1/3}$. It is increasing for non-negative x and decreasing for negative x. Of course, the point $x = 0$ can be included in the 'negative;' side also. These are some of the examples of the notion of increasing and decreasing functions. We will see later why they are useful.

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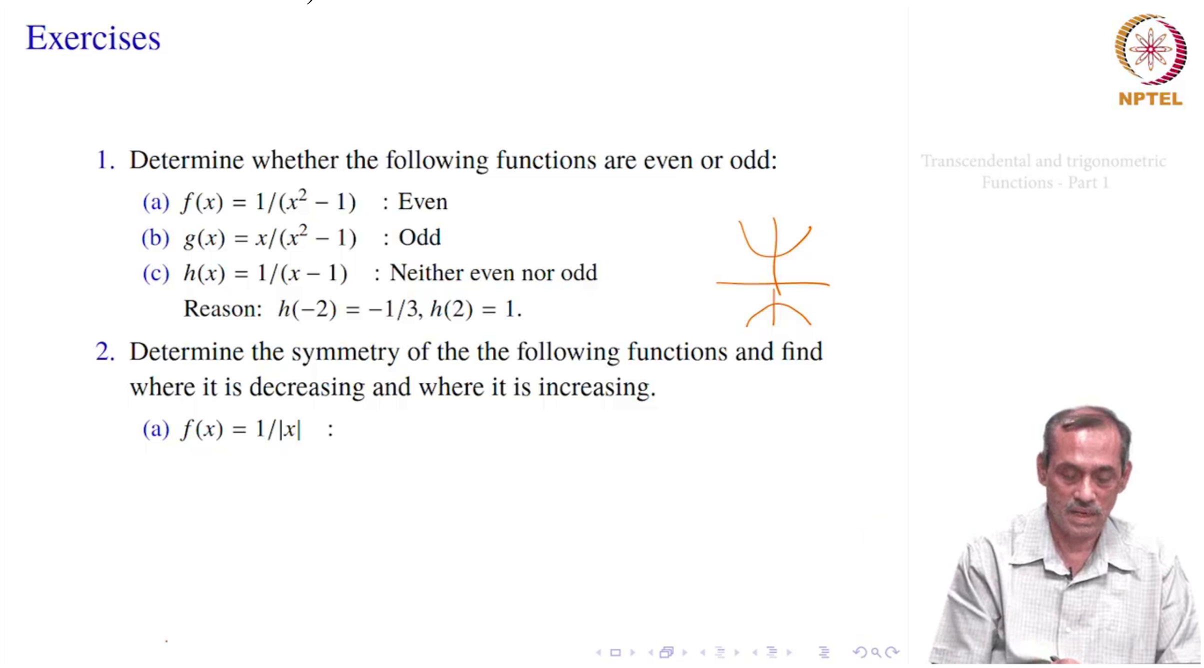

Let us see how to solve some problems basing on these ideas. Say, we are asked to find out whether the following functions are even or odd. All that you have to do is substitute x with $-x$ in the formula and see what happens, if it is given by a formula. Here, it is given that $f(x) = 1/(x^2 - 1)$. If you substitute x with $-x$, you would get the same $f(x)$. So, it is an even function. In fact, $y = x^2$ is even, $1/x^2$ is be even, 1 is even, $x^2 - 1$ is even; and when you divide it, it also remains even.

What about $g(x) = x/(x^2 - 1)$? We substitute x with $-x$ to get $g(-x) = -x/(x^2 - 1)$. So, $g(x)$ is odd.

And, what about $h(x) = 1/(x - 1)$? Of course, it is not defined at $x = 1$ like the earlier ones. We are not considering $x = 1$ for $g(x)$ also. Here, let us take $h(-x)$. It gives $1/(-x - 1)$. It is neither $h(x)$ nor $-h(x)$. So, we cannot say that it is even or odd. In fact, it is neither even nor odd. It is different from point to point. For example, you take $h(-2)$. It is $-1/3$ but $h(2)$ is 1. This gives a counterexample that nothing will happen; $h(x)$ is neither even nor odd.

Look at the second problem. It asks us to find the symmetry of the following functions and find where it is decreasing and where it is increasing.

What about the symmetry? We say that some function f is symmetric about y-axis when "if you reflect along y-axis, the graph remains the same". Say, this is y-axis. You reflect along y-axis; these things come on the right side, right side becomes the left side; so the graph remains the same.

Similarly, symmetry about x-axis means that you reflect something along the x-axis that graph remains same. So, this one is not symmetric about x-axis, this is symmetric about y-axis. What about this one? If you reflect on the x-axis, it becomes the new one. But it is about the graphs of functions. This is not a function. So, we will say that this is symmetry about x-axis and also y-axis.

We also talk of symmetry about the origin. This means, if you rotate the graph about 180 degrees, then you will find that the graph remains the same. We will call this symmetric about the origin when rotating 180 degrees.

Let us see what happens here. We need to to see what is the symmetry and decreasing and increasing nature of the functions. The first one is $f(x) = 1/|x|$. As you remember, $|-x| = |x|$; that is, $f(-x) = f(x)$; so $f(x)$ is even. That is what we expected; it is even.

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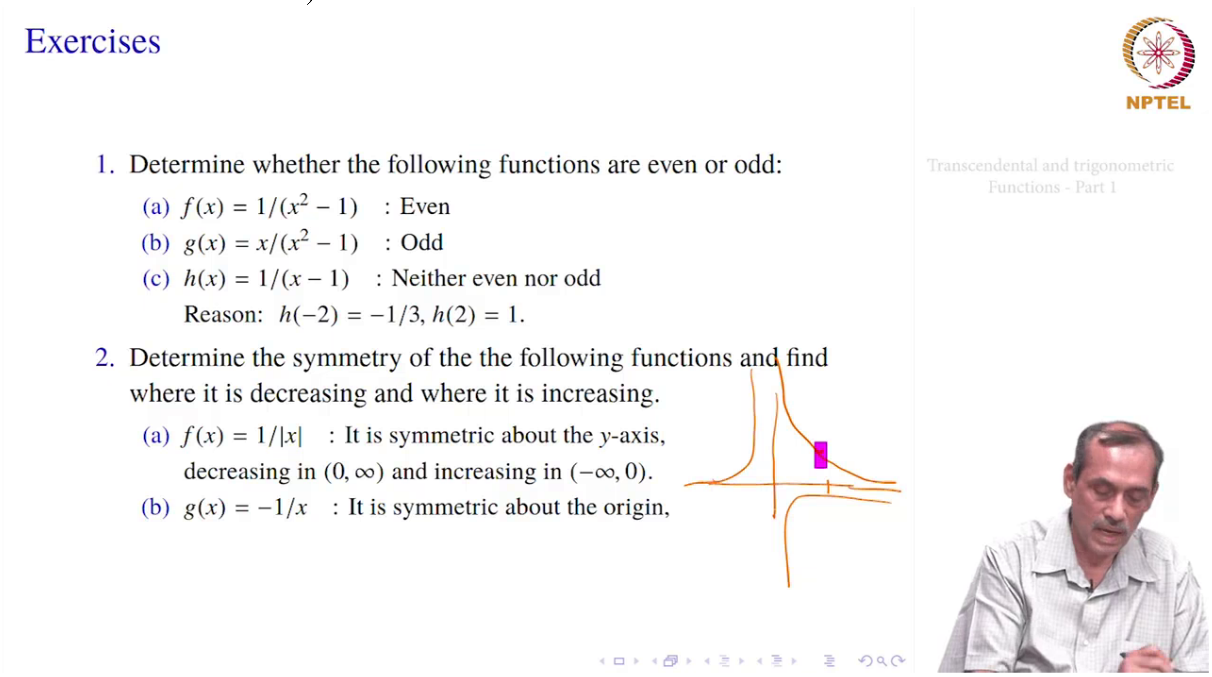

And what about the symmetry? It says that it is symmetric about the y-axis, why? That will simply reflect the graph of $y = |x|$, which looks something like this: $|x|$ and it is 1 divided by those values.

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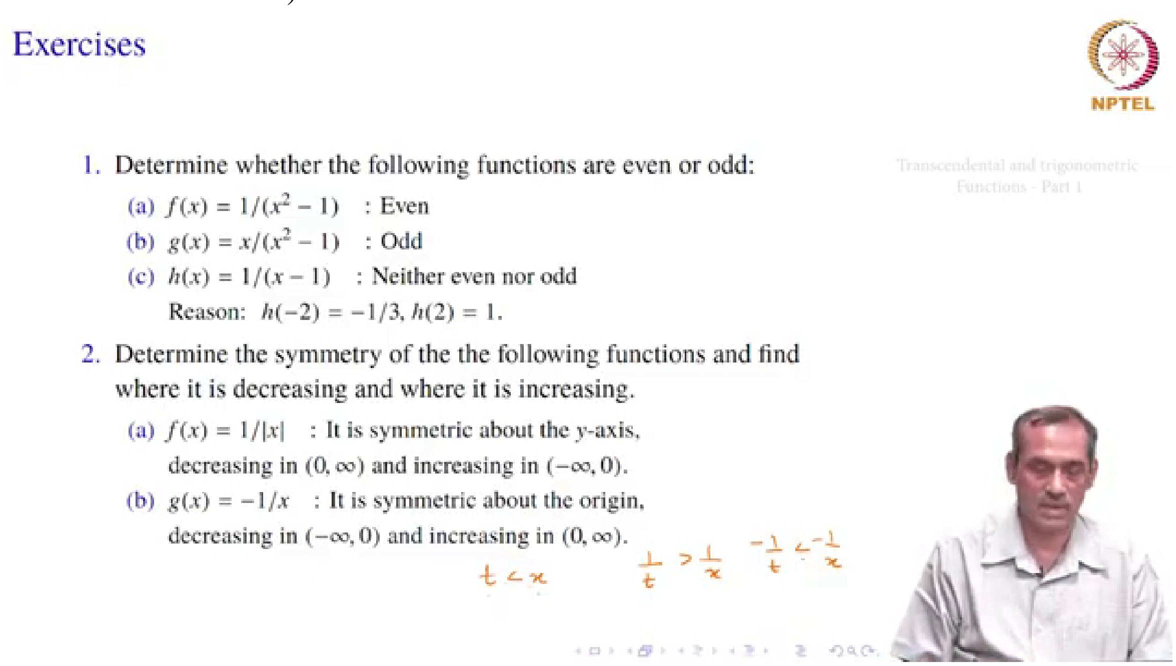

So, that should be looking the same way; it is symmetric about the y-axis. As $|x|$ is increasing there, $y = 1/|x|$ should be decreasing in $(0, \infty)$. $y = |x|$ is increasing in $(0, \infty)$; so $y = 1/|x|$ is decreasing. And, $y = 1/|x|$ is increasing in $(-\infty, 0)$. That is, for the negatives, it will be the other way around.

Now, what about this $g(x) = -1/x$? Here, if you take $1/x$ and $-1/x$, they will be looking different about the symmetry. How does $-1/x$ look like? So, $f(x) = -1/x$. We will be looking at something like, say, 0 to 1. At 1 it is 1 and that near 0 it is going this way. That is how $1/x$ looks like. And here, of course, it continues like this. When you take minus, instead of here, the same thing will be reproduced here. That is how $-1/x$ looks like; and we do not have the other side. This one is not there. So, all that we have is $-1/x$. When x is negative, it will be somewhere here. If you rotated by 180 degrees, this side would come to the side, this side will become this side. So, it remains the same. It is symmetric about the origin.

And what about decreasing and increasing nature? It is decreasing in $(-\infty, 0)$ as you have seen, and increasing in $(0, \infty)$. When x is positive, it is increasing as you can see from this. If $t < x$, then $1/t > 1/x$, and then you take the minus sign. You would say $-1/t < -1/x$. So, it is increasing for positives, and it will be decreasing for negatives. Because here itself it will be decreasing. So, the same thing will go to the last one. Let us stop here.