Basic Calculus - 1
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Lecture 36 - Part 2
Areas of Surface of Revolution - Part 2

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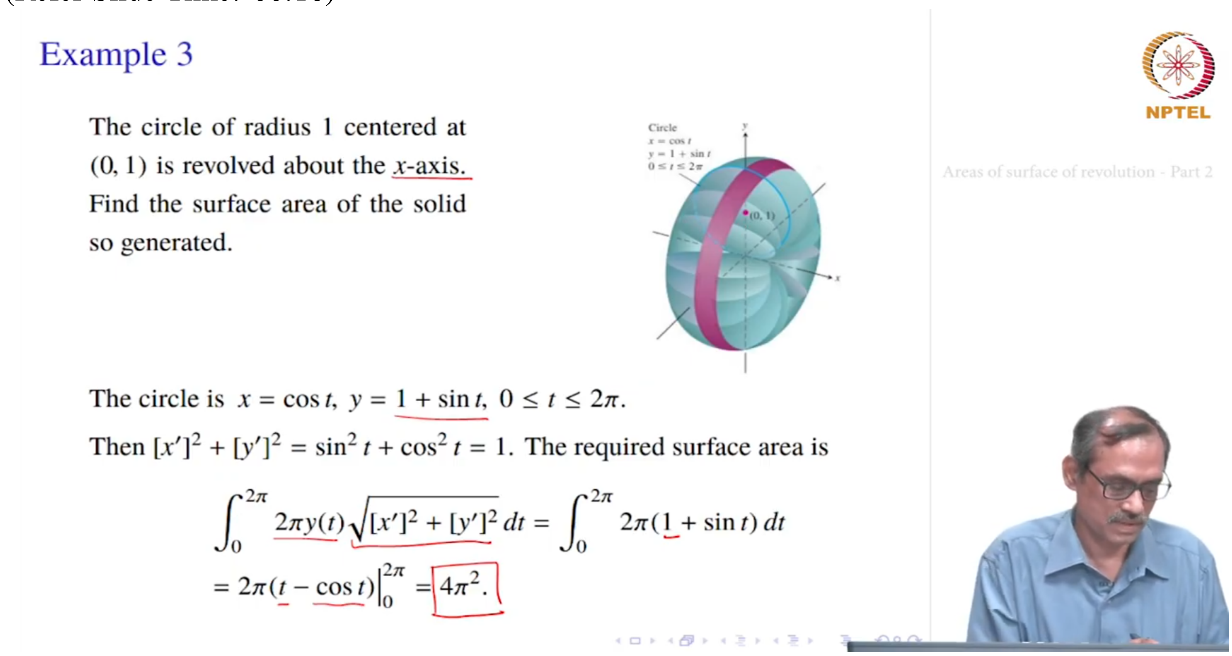

If the curve is given in parametric form, then what do we do? Let us consider this example. We have a circle of radius 1 centered at $(0, 1)$. We have a circle having its center at $(0, 1)$ and it has radius 1. That is revolved about the x-axis. Does it touch the x-axis? Yes, since its center is $(0, 1)$ and radius is 1. When it revolves about the x-axis, what is the surface area of revolution?. Look at the picture this way. You have the circle which is touching the x-axis its revolution about the x-axis would give a picture like this. It is a surface, and we want to find the area of this surface.

Since it is a circle, it will be easier to parameterize. If a circle has center at (a, b) and radius r, then it is written as $x = a + \cos t,\ y = b + \sin t$ for $0 \le t \le 2\pi$. Here then the circle is parameterized by $x = \cos t,\ y = 1 + \sin t$ for $0 \le 2 \le 2\pi$. That is how we get the parameterization of the circle.

Then we should obtain ds, which will be computed as the square root of $(dx/dt)^2 + (dy/dt)^2$ into dt. We now have $dx/dt = d(\cos t)/dt = -\sin t$ and $dy/dt = d(1 + \sin t)/dt = \cos t$. Thus, $ds = \sqrt{(-\sin t)^2 + (\cos t)^2}\,dt = dt$. Therefore, the surface area will be the integral from 0 to 2π, as t varies from 0 to 2π, of 2π times $y(t)$, since the revolution is about the x-axis, times the square root of $[x'(t)]^2 + [y'(t)]^2$. That is, $\int_0^{2\pi} 2\pi y(t)\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$, which is equal to $\int_0^{2\pi} 2\pi(1 + \sin t)\,dt$.

Now, you integrate this. It gives $2\pi(t - \cos t)$ as 1 gives t and $\sin t$ gives $-\cos t$. This is to be evaluated at 0 and 2π and subtracted. That simplifies to $4\pi^2$.

This is how we are going to find the area of the surface of revolution when the curve is given in parametric form. Basing on these three types of possibilities, we go for solving some problems.

Let us take the first exercise. Here, we want to find the lateral surface area of the cone, generated by revolving about the x-axis the line segment $y = x/2$, where x varies between 0 and 4. If revolution is about the x-axis, then we need a function $y = f(x)$. here, it is already given in that form: $y = x/2$. So, we differentiate to get $y'(x) = 1/2$. The surface area will be the integral $\int_0^4 2\pi y\sqrt{1+[y']^2}\,dx$. Now that $\sqrt{1+[y']^2} = \sqrt{1+1/4} = \sqrt{5}/2$, and $y = x/2$, one 2 gets canceled, and we have $\int_0^4 \pi\sqrt{5}2x\,dx$. It gives $\pi\sqrt{5}4x^2$ to be evaluated at 0 and 4, and then subtracted. That simplifies to $4\pi\sqrt{5}$.

The first exercise asks for obtaining the slant surface area of a right circular cone. The cone is now given as a surface obtained by revolving the line segment $y = x/2$ for $0 \le x \le 4$ about the x-axis. Just go through the work-out. you will find that it confirms the old geometrical formula half base into circumference into slant height.

Let us go to the next problem. The problem is to find the area of the surface generated by revolving the portion of the curve $y = \sqrt{2x - x^2}$ given by $1/2 \le x \le 3/2$ about the x-axis.
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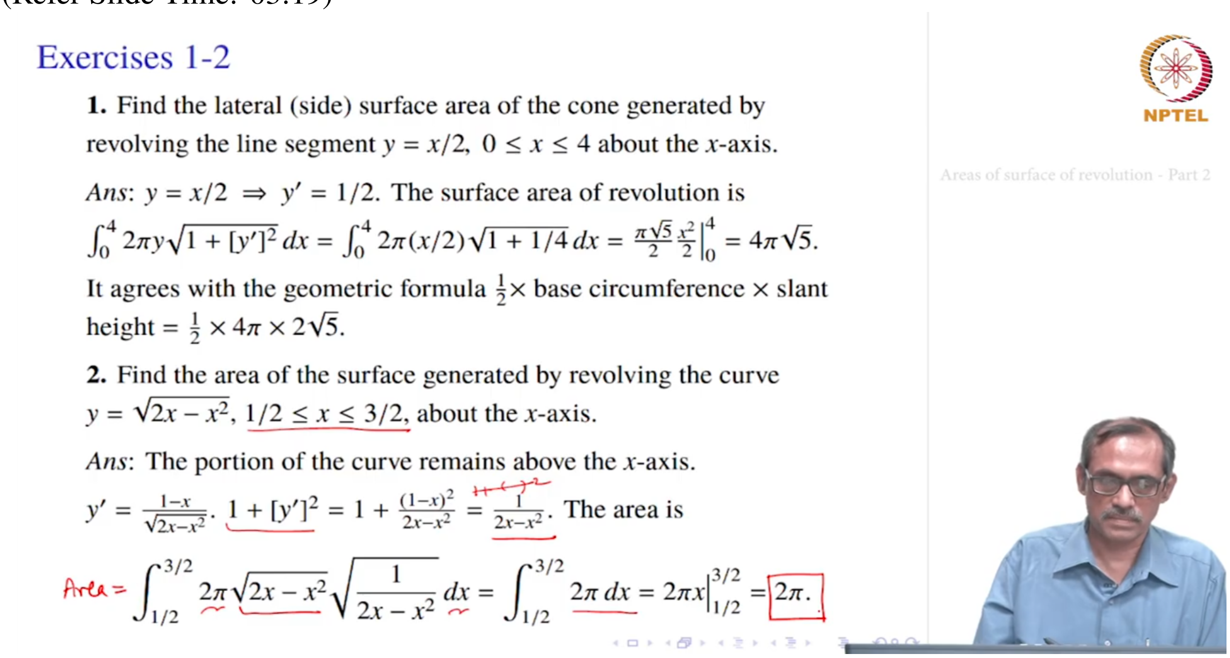

Since the revolution is about the x-axis, we should find the formula with y as a function of x. But that is already given. We should verify that the curve remains above the x-axis. That is also satisfied since it is the square root of $2x - x^2$, it is always greater than equal to 0. We differentiate to get y' is equal to the derivative of $\sqrt{2x - x^2}$ with respect to $2x - x^2$ multiplied by the derivative of $2x - x^2$ with respect to x. That gives $y' = \{1/[2\sqrt{2x - x^2}]\} \times (2 - 2x) = (1 - x)/\sqrt{2x - x^2}$.

So, $1 + [y']^2$ is equal to $[(2x - x^2) + (1 - x)^2]/(2x - x^2)$, which simplifies to $1/(2x - x^2)$. Then, we get the area of the surface as the integral

$$\int_{1/2}^{3/2} 2\pi y\sqrt{1+[y']^2}\,dx = \int_{1/2}^{3/2} 2\pi\sqrt{2x - x^2} \times [1/(2x - x^2)]\,dx = \int_{1/2}^{3/2} 2\pi\,dx.$$

This is $2\pi x\Big|_{1/2}^{3/2} = 2\pi(3/2 - 1/2) = 2\pi$. You should verify these things. If there is some mistake, then you correct it.

Let us take another problem. Find the area of the surface generated by revolving the portion of the curve $x = y^{3/2}/3 - \sqrt{y}$ where y varies from 1 to 3; and the revolution is about the y-axis. When the revolution is about the y-axis, we need to express x as a function of y; and it is already given that x is a function of y.

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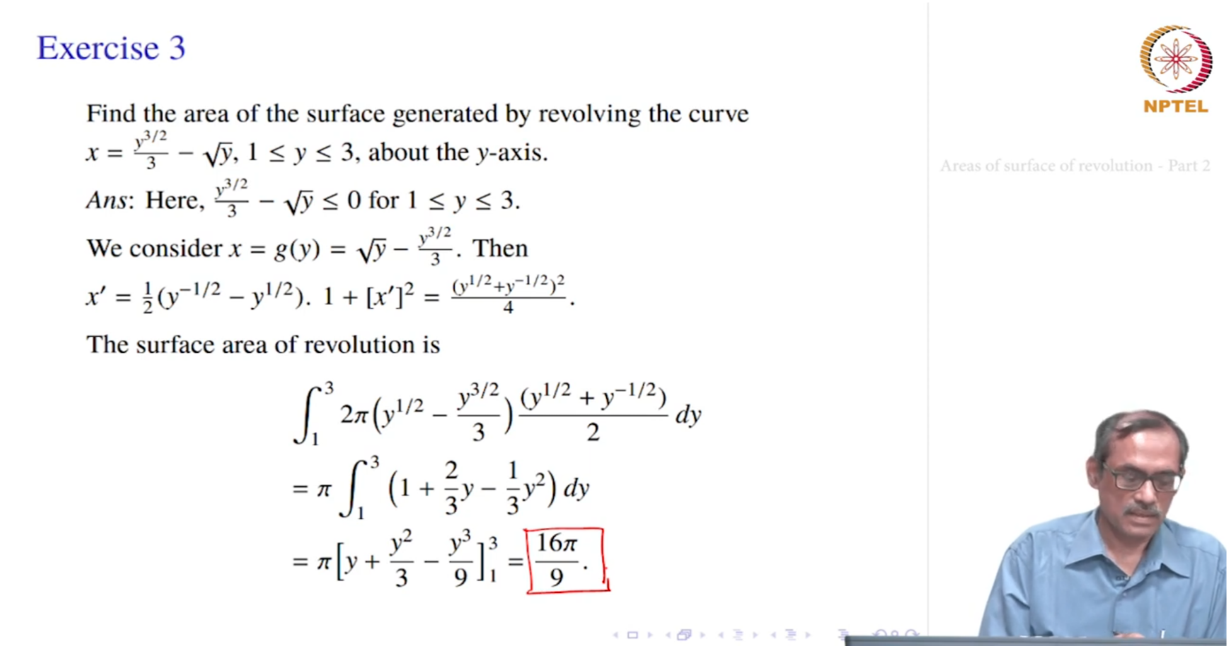

So, it is straight forward. We have to compute the derivatives and so on. Let us compute them. And we should also verify that the curve lies above the x-axis. That means it should be greater than equal to 0. But here, what happens is $y^{3/2}/3$, when y varies from 1 to 3, is less than or equal to $\sqrt{y}$. When $y = 1$, $x = 1^{3/2}/3 - \sqrt{1} = -2/3$, and when $y = 3$, $x = 3^{3/2}/3 - \sqrt{3} < 0$. Of course you can see that throughout $[1, 3]$, $x = y^{3/2}/3 - \sqrt{y}$ remains negative. That means, we should consider its negative to get the correct area. Area will be anyway the integral of the modulus of that thing; so we are assuming the integrand to be greater than or equal to 0. Here, we consider x equal to minus of what is given; that is, $x = \sqrt{y} - y^{3/2}/3$. Had you not taken it, we would have got a negative answer; but it should be same in absolute value.

So, we consider this function $x = \sqrt{y} - y^{3/2}/3$. If you differentiate it, you get $x' = (1/2)y^{-1/2} - (3/2)y^{1/2} \times (1/3) = (1/2)(y^{-1/2} - y^{1/2})$. Then,

$$1 + [x']^2 = 1 + (1/4)(y^{-1} + y - 2) = y^{-1} + y + 2 = [(1/2)(y^{-1/2} + y^{1/2}]^2.$$

Notice that we need the square root of $1 + [x']^2$. Now that we have expressed it as a square, it is easy to get its square root. Now, the surface area will be equal to the integral $\int_1^3 2\pi x\sqrt{1 + [x']^2}\,dy$, which is equal to $\int_1^3 2\pi(y^{1/2} - y^{3/2}/3)(1/2)(y^{-1/2} + y^{1/2})\,dy$. Multiplying out the factors and simplifying, we get it to be equal to $\int_1^3 2\pi(1 + 2y/3 - y^2/3)\,dy$. It gives $2\pi(y + y^2/3 - y^3/9)$, which is to be evaluated at 3 and 1, and then subtracted. Verify that it is equal to $16\pi/9$.

Let us take one more problem. Find the area of the surface generated by revolving the curve $x = y^4/4 + 1/(8y^2)$, when y varies between 1 and 2, about the x-axis. Had it been about y-axis, you would have done it directly. But now this surface is generated by revolving this curve about the x-axis. So, you need to find y in terms of x. But that looks a complicated problem. Also limits of y are only given; from which you may have to find limits for x.

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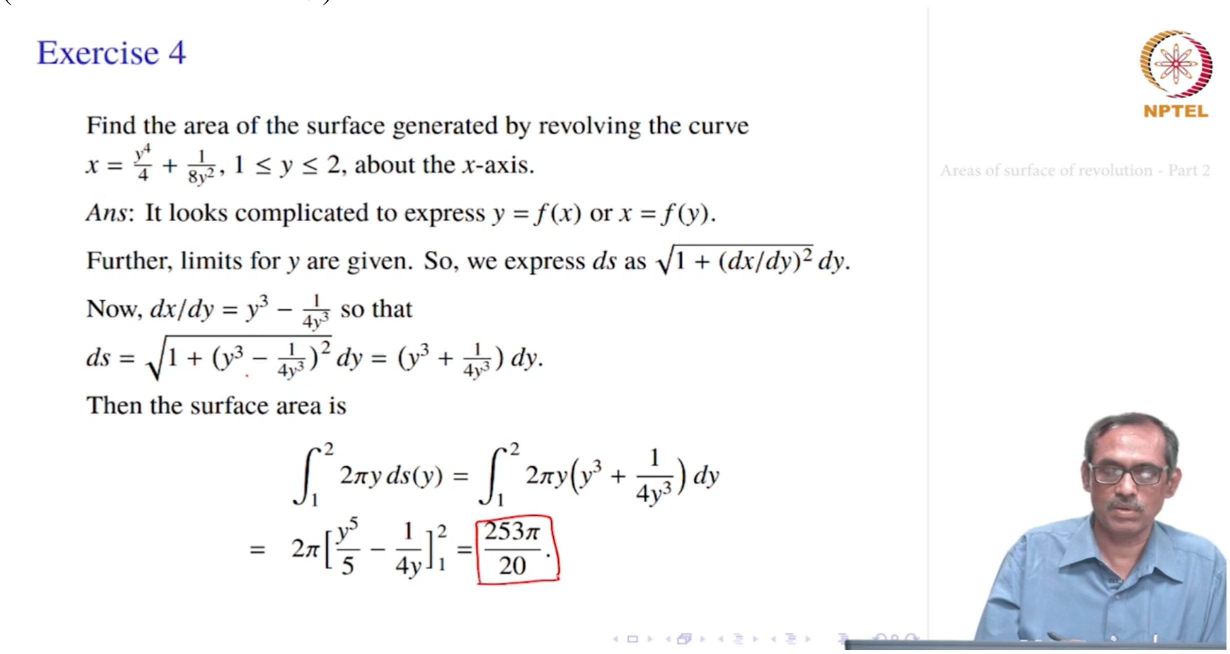

Instead of going directly to that, let us compute ds. It is anyway $\sqrt{1 + (dx/dy)^2} \times dy$. From $x = y^4/4 + 1/(8y^2)$, we get $dx/dy = y^3 + (1/8)(-2)y^{-3} = y^3 - 1/(4y^3)$. Now,

$$1 + (dx/dy)^2 = 1 + [y^3 - 1/(4y^3)]^2 = 4y^3[1/(4y^3)] + [y^3 - 1/(4y^3)]^2 = [y^3 + 1/(4y^3)]^2.$$

Then, $ds = \sqrt{1 + (dx/dy)^2}\, dy = [y^3 + 1/(4y^3)]^2\, dy$. We can use this to get the surface area. But since the revolution is about the x-axis, the surface area will be equal to the integral of $2\pi y ds$. where s is treated as a function of y so that the limits are for y. All that we have to do is apply the unified formula with ds instead of dx or dy. Notice that the limits for y are 1 and 2. So, the surface area is $\int_1^2 2\pi y\, ds$, which becomes $\int_1^2 2\pi y[y^3 + 1/(4y^3)]\, dy$. We integrate this. The first term is y^4 which gives $y^5/5$; next one is $1/(4y^2)$; that gives $-1/(4y)$. So, it is $2\pi[y^5/5 - 1/(4y)]$ evaluated at 1 to 2. Verify that it simplifies to $253\pi/20$.

Here we used is a trick. If we use the unified formula, it becomes easier. If you go directly, it will be a bit complicated.

We go to Exercise 5, where we want to find the area of the surface generated by revolving the given curve. The curve is given in parametric form: $x = t + \sqrt{2}$, $y = t^2/2 + t\sqrt{2}$, where t varies from $-\sqrt{2}$ to $\sqrt{2}$. And this curve is revolved about the x-axis.

The curve is given in parametric form; so you must get the derivatives. Now, $dx/dt = 1$ and $dy/dt = t + \sqrt{2}$. Then, we compute ds, which is equal to $\sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$. That gives $ds = \sqrt{1 + (t + \sqrt{2})^2}\, dt$. By expanding the square, we have $ds = \sqrt{t^2 + 2\sqrt{2}t + 3}\, dt$.

Since the revolution is about the y-axis, we can write the surface area as the integral $\int 2\pi x\, ds$ with limits for y. Remember, when you take the other formula in terms of y; it will be $\int 2\pi x\sqrt{1+[x']^2}\, dy$, where x is a function of y. Here, directly we get $\int 2\pi x ds$. Is that alright? Because you can see that what happens in the earlier problem. You have taken $\int 2\pi y ds$ when the revolution was about the x-axis. It is the same thing we are applying here. Though it is given in terms of parameters; the formula is the same.

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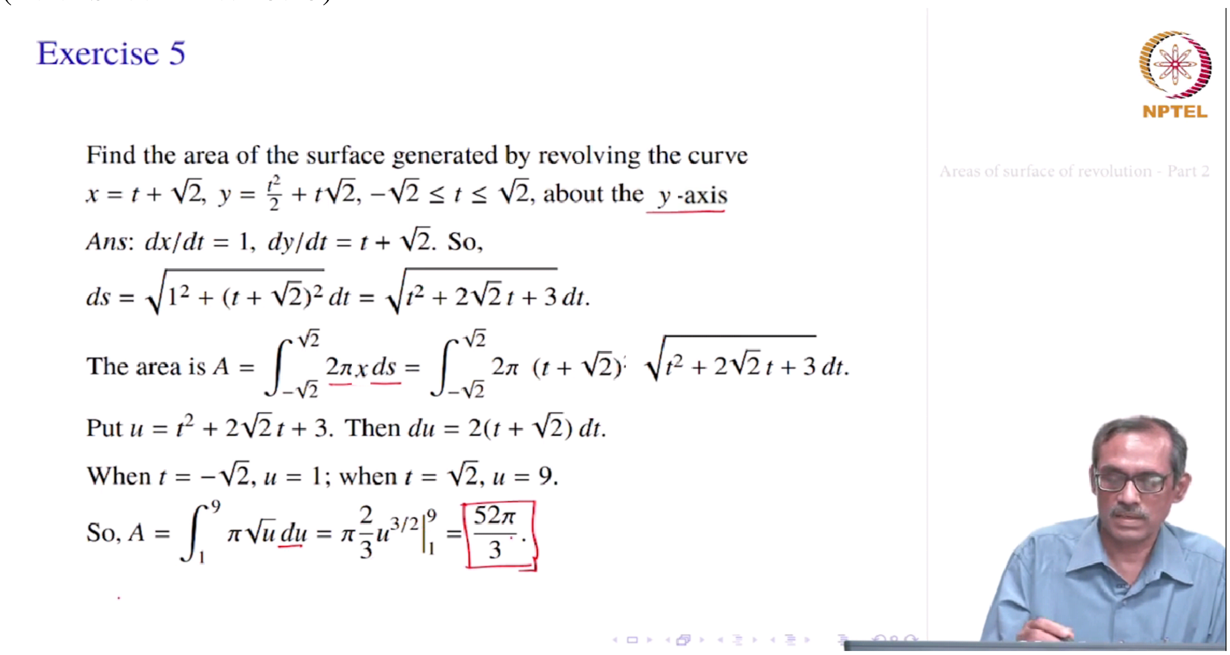

The integral will now be in terms of the parameter t, that is, s is a function of t now; so, the limits of the integration will be from $-\sqrt{2}$ to $\sqrt{2}$. That is, the surface area is $\int_{-\sqrt{2}}^{\sqrt{2}} 2\pi x\, ds(t)$. Now, everything should be express in terms of t. Thus, the integral is equal to

$$\int_{-\sqrt{2}}^{\sqrt{2}} 2\pi(t+\sqrt{2})\sqrt{t^2+2\sqrt{2}t+3}\, dt.$$

How do we integrate it? The problem is $t^2 + 2\sqrt{2}t + 3$. So, let us take that to be u. We substitute $u = t^2 + 2\sqrt{2}t + 3$. Then, $du = (du/dt)\, dt = 2(t+\sqrt{2})\, dt$. For the limits, see that when $t = -\sqrt{2}$, $u = 2 - 2\sqrt{2}(-\sqrt{2}) + 3 = 1$, and when $t = \sqrt{2}$, $u = 2 + 2\sqrt{2}\sqrt{2} + 3 = 9$. Then, you can write the area directly in terms of u. It is $\int_1^9 2\pi\sqrt{u}\, du$. That gives you $\pi(2/3)u^{3/2}$ to be evaluated at 1 and 9; and then subtracted. That simplifies to $52\pi/3$; verify it.

Let us take one more problem, the last one. Find the area of the surface generated by revolving the given portion of the asteroid. You have seen earlier that an asteroid has 4 petals. We have only 2 petals here, because x varies from -1 to 1. The asteroid is $x^{2/3} + y^{2/3} = 1$. The portion of the curve which is to be revolved is is described by limiting x to -1 to 1. And this is revolved about the x-axis. So, we should apply the formula directly.

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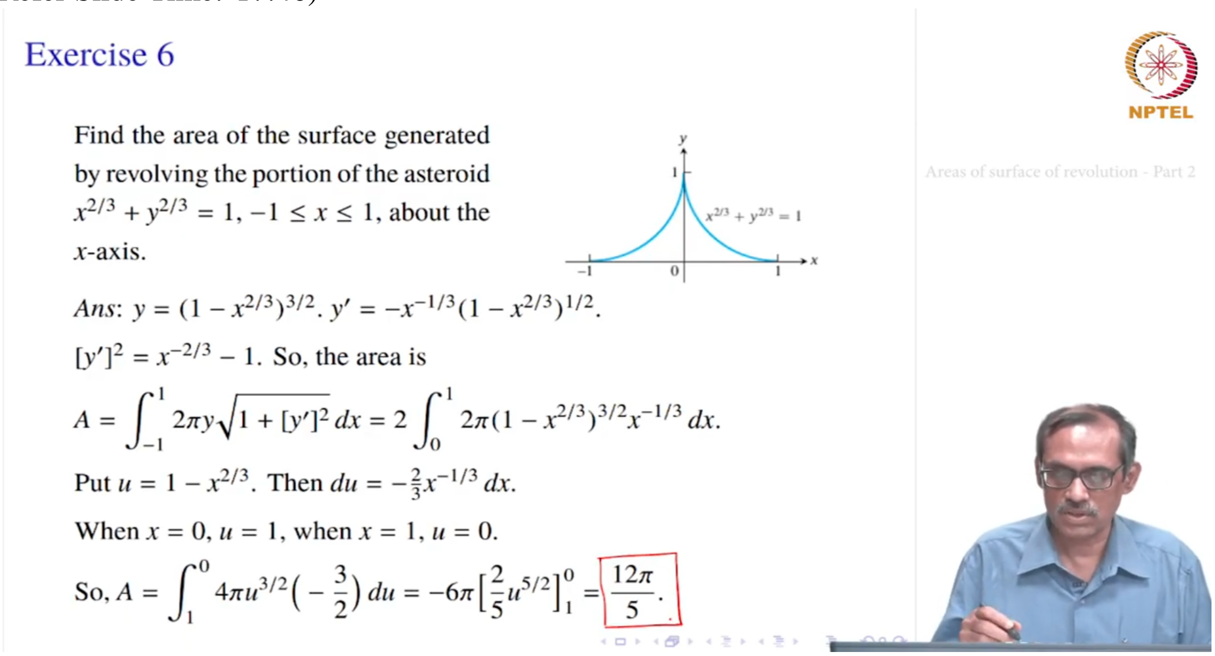

The equation yields $y = (1 - x^{2/3})^{3/2}$, $-1 \le x \le 1$. Then, $y' = -x^{-1/3}(1 - x^{2/3})^{1/2}$. You get $1 + [y']^2 = 1 + x^{-2/3}(1 - x^{2/3}) = 1 + x^{-2/3} - 1 = x^{-2/3}$. The surface area is the integral $\int_{-1}^{1} 2\pi y\sqrt{1 + [y']^2}\, dx = \int_{-1}^{1} 2\pi(1 - x^{2/3})^{3/2}x^{-2/3}\, dx$. However, the integrand is an even function, so the integral is equal to $2\int_0^1 2\pi(1 - x^{2/3})^{3/2}x^{-2/3}\, dx$. This is to be integrated now.

To integrate we take $1 - x^{2/3}$ as u. Then, $du = -(2/3)x^{-1/3}\, dx$. For the limits, we see that when $x = 0$, $u = 1$ and when $x = 1$, $u = 0$. So, the integral is equal to $\int_1^0 4\pi u^{3/2}(-3/2)\, du$. This gives $4\pi(-3/2)(2/5)u^{5/2}$, which is to be evaluated at 0 and 1, and then subtracted. Verify that it simplifies to $12\pi/5$.

We stop and declare that with this lecture the course is over. I hope you have enjoyed this course.