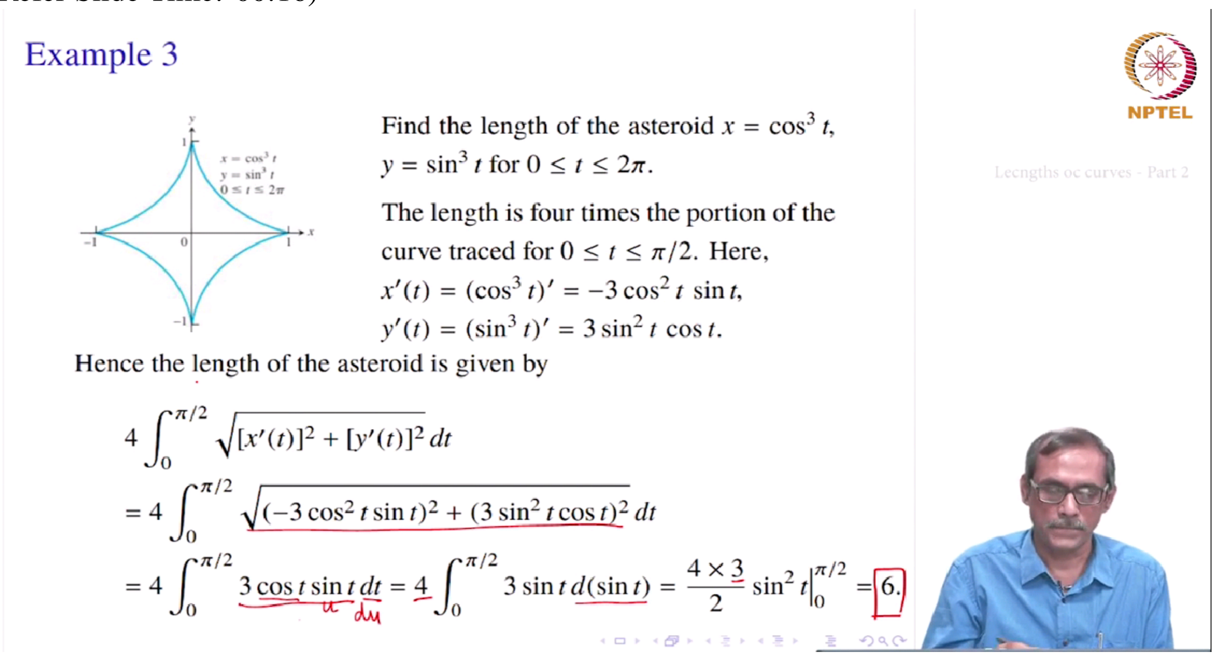
Basic Calculus - 1
Professor. Arindama Singh
Department of Mathematics
Indian Institute of Technology Madras
Lecture 35 - Part 2
Lengths of Curves - Part 2

(Refer Slide Time: 00:16)



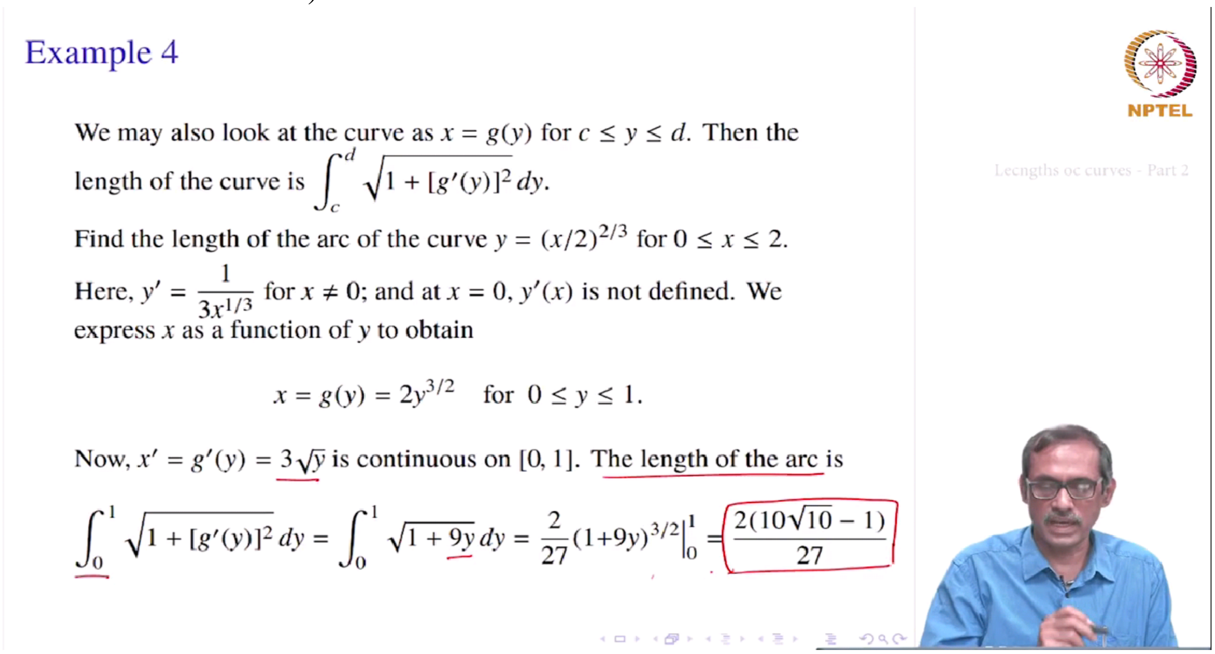
Let us take one more example. Here, we want to find the length of the asteroid, which is given in parametric form $x = \cos^3 t$, $y = \sin^3 t$, where t varies from 0 to 2π. If you plot it, it looks something like this, having four petals, one in each quadrant. Since it is symmetric, its length will be 4 times the length of any one petal. So, it is enough to compute the length of one petal. In that case, x will vary from 0to $\pi/2$ instead of from 0 to 2π. So, that is the first thing we can see. The length of the asteroid is four time the curve traced for $x \in [0, \pi/2]$, which is this portion.

Let us go to computing the length. We will compute this length and multiply it by 4. Here, $x'(t) = (d/dt)(\cos^3 t)$. That gives $3\cos 2t$ into the derivative of $\cos t$, which is $-\sin t$. That is, $x'(t) = -3\cos^2 t \sin t$. Similarly, $y'(t) = 3\sin^2 t \cos t$ because the derivative of $\sin t$ is $\cos t$, and there is no minus sign. We plug in the formula. The length of the asteroid is 4 times $\int_0^{\pi/2} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$.

That gives $4 \times \int_0^{\pi/2} \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2}\, dt$. If you simplify this, that turns out to be $3\cos t \sin t$, using of course, the formula $\cos^2 t + \sin^2 t = 1$. That gives rise to $3\cos t \sin t$. Then we take its integral. For this, we substitute $u = \sin t$. Now, if $u = \sin t$, then $du = \cos t\, dt$. Or, we may directly write $4 \times 3 \int_0^{\pi/2} \sin t\, d(\sin t)$. So, we get $(4 \times 3/2)\sin^2 t$, which is to be evaluated at 0 and $\pi/2$, and subtracted out. Verify that it simplifies to 6.

Of course, you could have done directly instead of multiplying 4, but this becomes easier. Once you know the shape of the curve, where the curve is exactly lying, that makes the computations easy.

(Refer Slide Time: 03:25)



Let us take one more example to fix the idea. Suppose the curve is given by x as a function of y instead of y as a function of x. That means, we can think of y as the parameter t. So, our t is equal to y. In that case, $y'(t) = 1$ and $x'(t)$ will be equal to dx/dy. Then, the formula for the length of the curve will be $\int_c^d \sqrt{1 + (dx/dy)^2}\, dy$, where y varies from c to d. If the curve is given by $x = g(y)$, then its length would be written as $\int_c^d \sqrt{1 + (g'(y))^2}\, dy$.

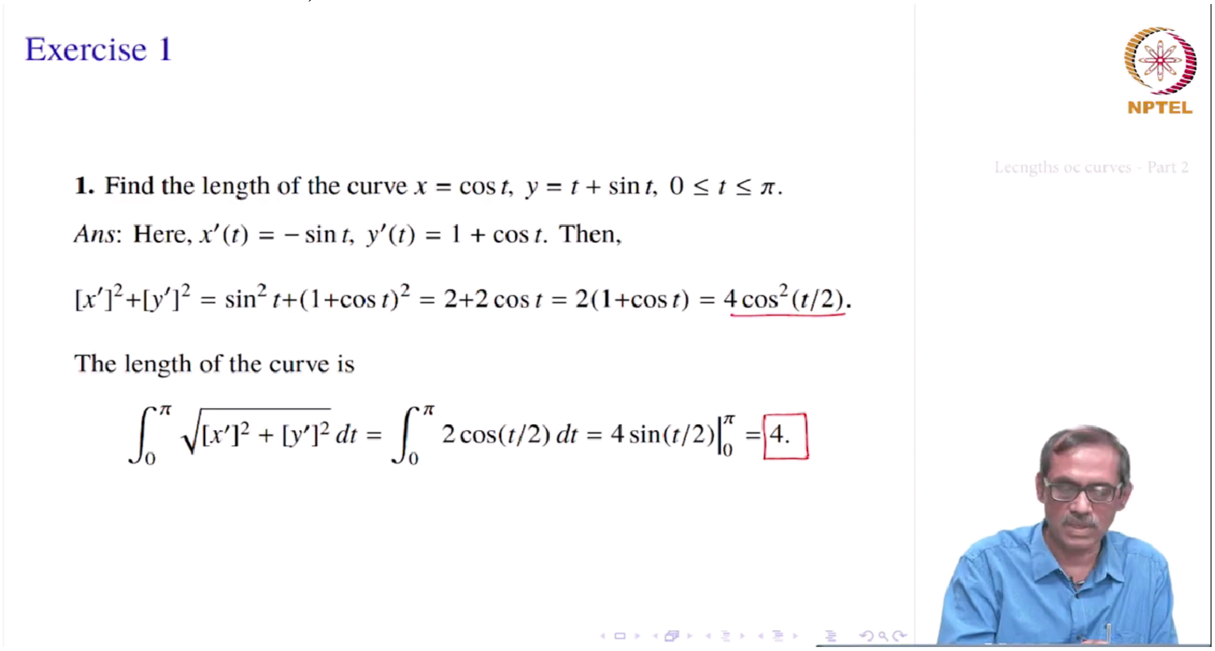
Let us use this formula in solving the given problem. Here, we are asked to find the length of the arc of the curve $y = (x/2)^{2/3}$ for $0 \le x \le 2$. The arc is defined when x varies from 0 to 2. That is how it is given. You can do of course directly, but let us see it via y; that will be a bit easier. We think of x as a function of y. And there is an advantage here because at $x = 0$, this function is not differentiable, but we want it to be differentiable. Of course, we can omit that one point, because it is an endpoint so that it will not affect the integral. However, if you look at it by x as a function of y, then that will be differentiable. Expressed this way, the same curve is written as $x = 2y^{3/2}$. You can verify: $x = 2y^{3/2} = 2[(x/2)^{2/3}]^{3/2} = 2(x/2) = x$. So, it is correct. When $x = 0$, $y = 0$ and when $x = 2$, $y = 1$. That is, when x varies from 0 to2 2, y varies from 0 to 1. Hence, the given curve is expressed as , $x = g(y) = 2y^{3/2}$ for $0 \le y \le 1$.

Of course, you could have done earlier way, but let us see how this variation looks like. In this case, our length of the curve will be computed where you will be getting $c = 0$ and $d = 1$, plug in $1 + [g'(y)]^2$ etc. We need to compute $g'(y)$; that is easier. Since $g(y) = 2y^{3/2}$, $g'(y) = 2(3/2)y^{1/2} = 3y^{1/2}$. Notice that this also continuous on the close interval $[0, 1]$. That is required for getting our formula so that the Riemann sum would give us the integral.

Now, we take the length of the arc, which is equal to the integral $\int_0^1 \sqrt{1 + [g'(y)]^2}\, dy$. Now, $1 = [g'(y)]^2 = 1 + 9y$ so that the integral is $\int_0^1 \sqrt{1 + 9y}\, dy$. To integrate, we substitute $u = 1 + 9y$.

Then, $du = 9dy$ and the integral of $\sqrt{u}$ will give $(2/3)u^{3/2}(1/9) = (2/27)(1+9y)^{3/2}$. Thus, the length of the curve is this evaluated at 1 and 0, and then subtracted. That simplifies to this number: $2\big(10\sqrt{10}-1\big)/27$.

(Refer Slide Time: 08:04)



Let us take some more problems. Find the length of the curve $x = \cos t$, $y = t + \sin t$, where t varies from 0 to π. This is the portion of curve since t varies from 0 to π. It is given in parametric form already and the limits for t are also given. We then need to plug in the formula, find out the derivatives of x and y with respect to t and then compute the length.

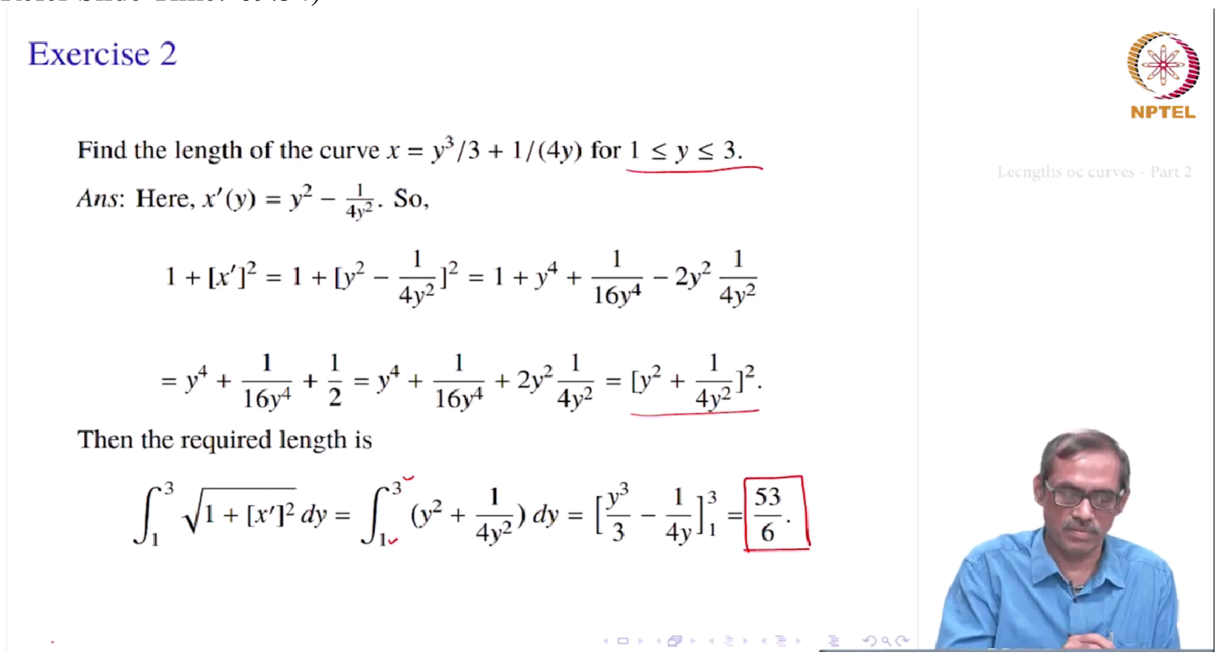
Now, $x'(t) = -\sin t$ and $y'(t) = 1 + \cos t$. Therefore, $[x'(t)]^2 + [y'(t)]^2 = \sin^2 t + (1+\cos t)^2$. It simplifies to $2 + 2\cos t$, which we write as $4\cos^2(t/2)$. Now, the length of the curve is the integral $\int_0^{\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt = \int_0^{\pi} 2\cos(t/2)\,dt$. This can be integrated easily. Integration gives $4\sin(t/2)\big|_0^{\pi}$, which simplifies to 4. As you see, it is quite straight forward to compute the lengths of curves.

Let us take another problem. Find the length of the curve $x = y^3/3 + 1/(4y)$ for $1 \le y \le 3$. Here, x is given as a function of y; then we can continue the same way instead of rewriting it. We have a formula that for this when x is a function of y and the limits for y are known. To apply the formula, we should find out the derivative of x with respect to y. The derivative of x with respect to y is the derivative of $y^3/3$, that gives y^2 plus the derivative of $1/(4y)$ which gives -1 times $(4y)^{-2}$ times 4. So, $x'(y) = y^2 - 1/(4y^2)$. We require $1 + [x'(y)]^2$. With this $x'(y)$ we have

$$1 + [x'(y)]^2 = 1 + \left(y^2 - \frac{1}{4y^2}\right)^2 = 4(y^2)\,\frac{1}{4y^2} + \left(y^2 - \frac{1}{4y^2}\right)^2 = \left(y^2 + \frac{1}{4y^2}\right)^2.$$

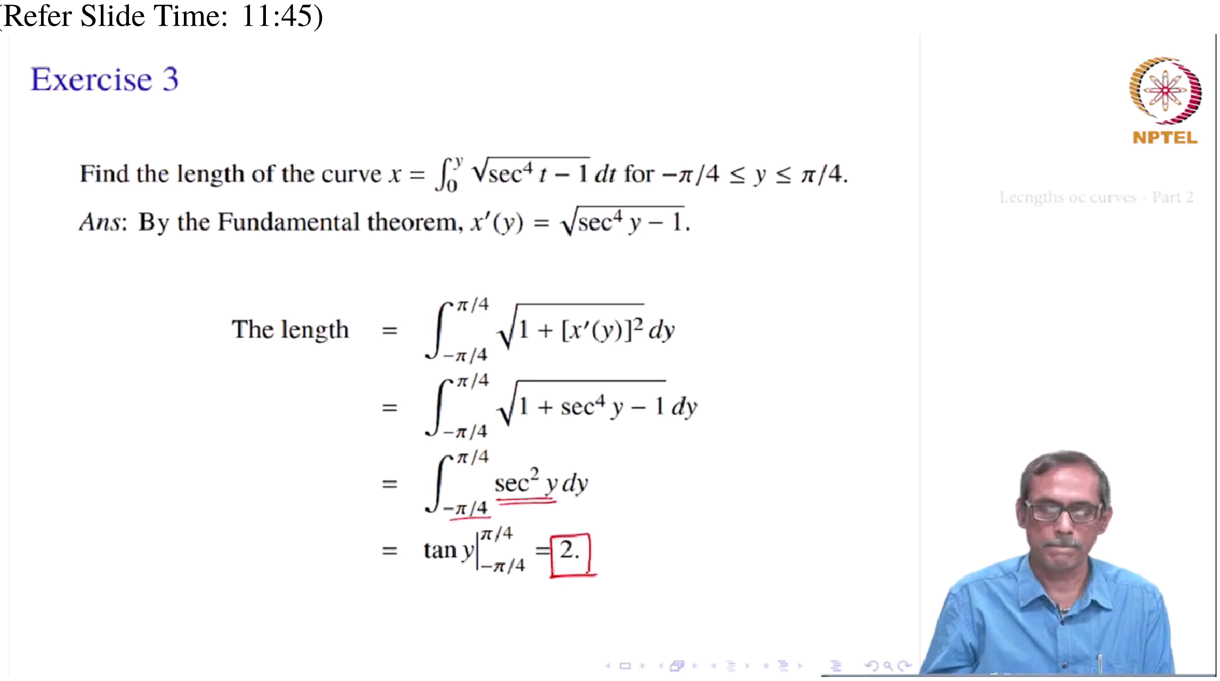
Then, the length of the curve is the integral $\int_1^3 \sqrt{1 + [x'(y)]^2}\,dy = \int_1^3 [y^2 + 1/(4y^2)]\,dy$. This gives $y^3/3 - 1/(4y)$. This is to be evaluated at 1 and 3, and subtracted. Verify that it simplifies to $53/6$.

(Refer Slide Time: 09:34)



Now you see whether x is a function of y or y is a function of x whichever form is given, we can easily compute the length; we have one corresponding formula.

(Refer Slide Time: 11:45)



Let us take another example. Here, we want to find the length of the curve, which is given as x is a function of y. The function is itself an integral. That is, the curve is given by $x = \int_0^y \sqrt{\sec^4 t - 1}\,dt$ for $-\pi/4 \le y \le \pi/4$. You may be thinking that "it is given as an integral, do we need to compute this integral?" But you do not know how to compute this integral. It is $\sec^4 t$. It may be very complicated to evaluate this integral. However, we need the derivatives of that function of y. Now, you can use the Fundamental Theorem of Calculus to get the derivative of this integral, right?

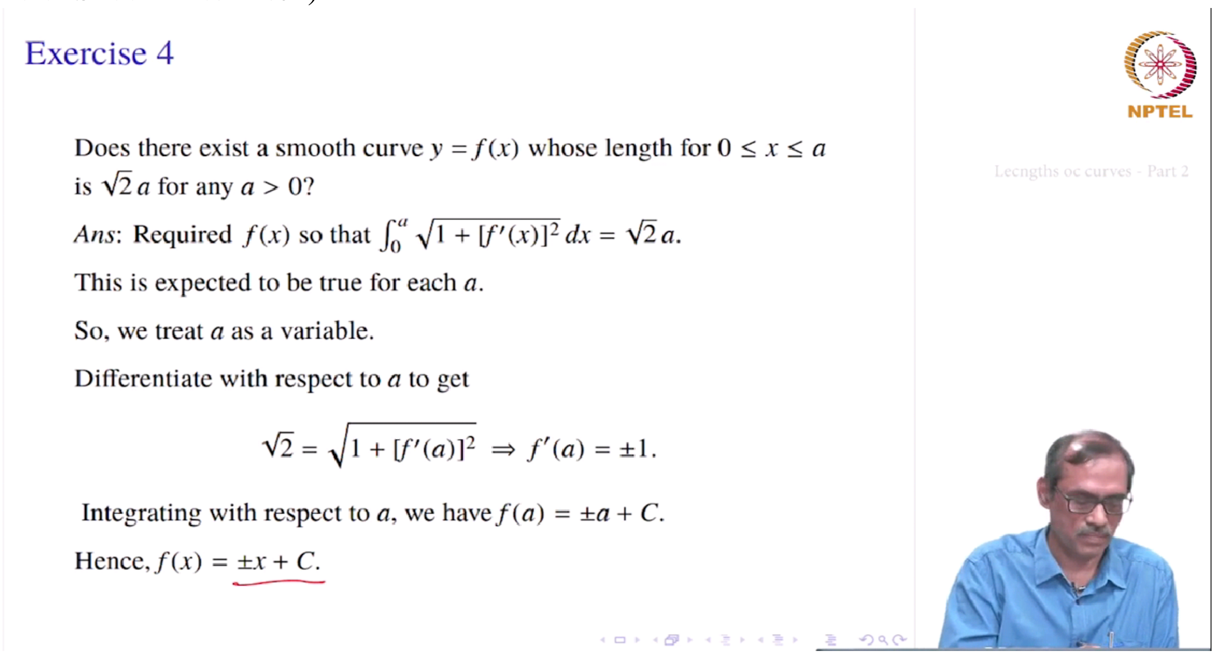
Since it is an integral from 0 to y, the derivative of the integral will be $\sqrt{\sec^4 y - 1}$. Now that

$x'(y) = \sqrt{\sec^4 y - 1}$, we can compute the length of the curve. The length will be $\int_{-\pi/4}^{\pi/4} \sqrt{1 + [x'(y)]^2}\, dy$. We know $x'(y)$, which is the square root of $\sec^4 y - 1$. The square of $x'(y)$ is $\sec^4 y - 1$ so that $\sqrt{1 + [x'(y)]^2} = \sqrt{1 + \sec^4 y - 1} = \sec^2 y$.That makes it simpler for us. The length is

$$\int_{-\pi/4}^{\pi/4} \sqrt{1 + [x'(y)]^2}\, dy = \int_{-\pi/4}^{\pi/4} \sec^2 y\, dy = \tan y\Big|_{-\pi/4}^{\pi/4} = 2.$$

That is how we use earlier results instead of directly computing the integral here.

(Refer Slide Time: 14:04)

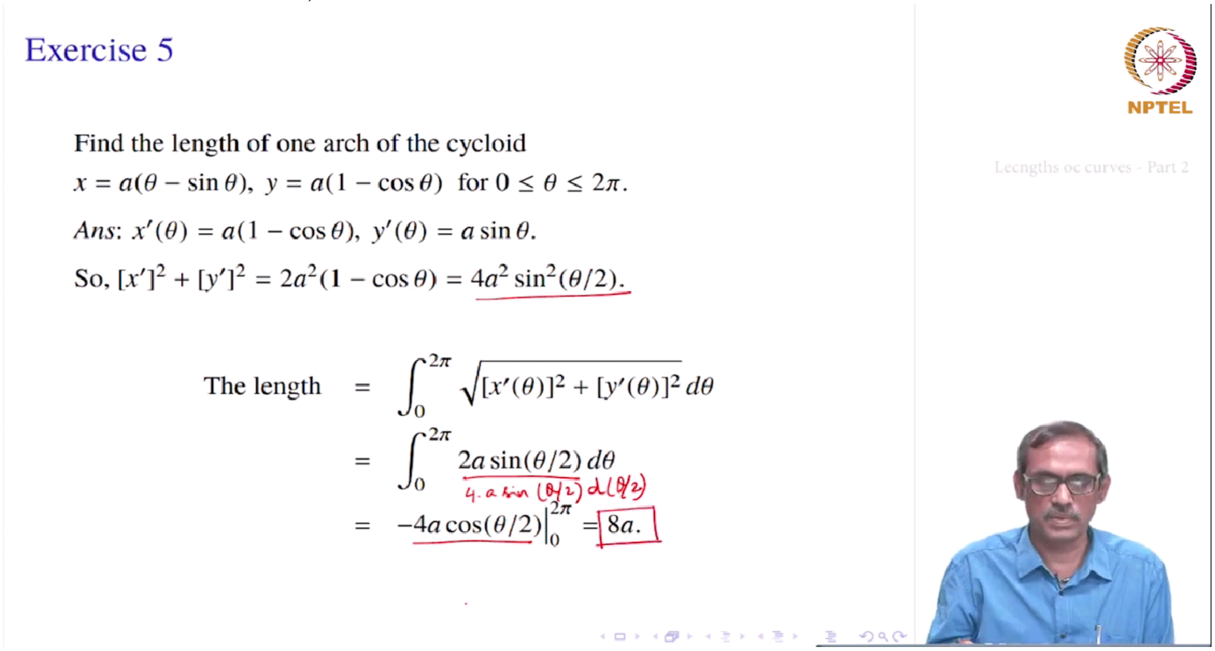


Let us go to the next problem. It is a different kind of problem. It is asking whether there exists a smooth curve $y = f(x)$ whose length from 0 to a for any $a > 0$ is $\sqrt{2}a$? We are given the length of the curve, and we want to find out what the curve is. Of course, the curve is assumed to be in the form y as a function of x. And this is true for any a greater than 0. That means we can treat a as a variable. Now, the length of this curve $y = f(x)$ is the integral from 0 to a of the square root of $1 + [f'(x)]^2$. And this value is given as $\sqrt{2}a$. We want to find this $f(x)$.

Again, we may have to use the fundamental theorem, because it is not known what this integrand is, we cannot integrate as such. We can really differentiate with respect to a, because it is true for any a, and we can treat a as a variable. We Differentiate $\int_0^a \sqrt{1 + [f'(x)]^2}\, dx = \sqrt{2}a$. We get $\sqrt{2}$ on the right side. On the left side, we use the fundamental theorem to differentiate with respect to a. That means this one, the integrand is evaluated at a, that is the answer. So, $\sqrt{2} = \sqrt{1 + [f'(a)]^2}$.

From this, we should get $f'(a)$, and then we go back to $f(x)$; that is our plan. Now, the above equation gives $2 = 1 + [f'(a)]^2$, or $f'(a) = \pm 1$. We integrate it to with respect to a to get $f(a) = \pm a$. However,it is an indefinite integral, so we have $f'(a) = \pm a + C$ for an arbitrary constant C. And, that is true for any a. As a is a variable, we may re-write it using the symbol x, that is, $y = f(x) = \pm x + C$. This is the curve. It is simple. The curve is a straight line; of course, it is not one line, it represents infinitely many straight lines for which the length will be equal to root $\sqrt{2}a$. Is that right?

(Refer Slide Time: 16:44)



Let us take another problem. Here, we want to find the length of one arch of the cycloid. The name of this curve is cycloid. There can be many arch in the cycloid; we want one arch. That arch corresponds to the portion of the curve when θ varies from 0 to 2π. And this curve, the cycloid is given in terms of the parameter θ; it is $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$. We need the derivatives $x'(\theta)$ and $y'(\theta)$ of x and y with respect to θ to get the length of the curve.

Now, $x'(\theta) = a(1 - \cos\theta)$ and $y'(\theta) = a\sin\theta$. as the derivative of θ is 1, of $\cos\theta$ is $-\sin\theta$, of $\sin\theta$ is $\cos\theta$. We need $[x'(\theta)]^2 + [y'(\theta)]^2$. It is $[a(1 - \cos\theta)]^2 + [a\sin\theta]^2$. Expanding and using $\cos^2\theta + \sin^2\theta = 1$, we obtain $2a^2(1 - \cos\theta)$. Since we will have to take the square root of it, we express it as a square. It is same as $2a^2 \times 2\sin^2(\theta/2) = 4a^2\sin^2(\theta/2)$.

Therefore, the length will be equal to $\int_0^{2\pi} \sqrt{[x'(\theta)]^2 + [y'(\theta)]^2}\, d\theta$. We already know the integrand which is equal to $\sqrt{4a^2\sin^2(\theta/2)} = 2a\sin(\theta/2)$. Then, we have the length as $\int_0^{2\pi} 2a\sin(\theta/2)\, d\theta$. We multiply 2 and and divide 2 with $d\theta$ to get $\int_0^{2\pi} 4a\sin(\theta/2)d(\theta/2)$. It gives $-4a\cos(\theta/2)$ to be evalaueted at 0 and 2π, and then subtracted. At 2π, it is $-4a\cos(\pi) = 4a$ and at 0, it is $-4a\cos(0) = -4a$. When subtracted, we get the anser as $8a$.

That is how we will compute the length of curves. Let us stop here.