Basic Calculus - 1
Professor. Arindama Singh
Department of Mathematics
Indian Institute of Technology Madras
Lecture 4 - Part 1
Transcendental and trigonometric Functions - Part 1

Well, this is lecture 4 of Basic Calculus 1. In the last lecture, we had introduced the notion of function and then gave some examples. The examples were limited to the so called algebraic functions; which means they were coming from the power function. And then, we discussed rational functions, polynomial functions and so on.

Today we will be discussing more examples. They will include transcendental functions; of course, that includes trigonometric functions, which are not explicitly algebraic. Once we have discussed trigonometric functions, we will also discuss inverse of trigonometric functions. So, the notion of inverse of a function also will be introduced. And then, we will discuss some properties like increasing nature or decreasing nature of functions, and so on. So, let us start with our examples.

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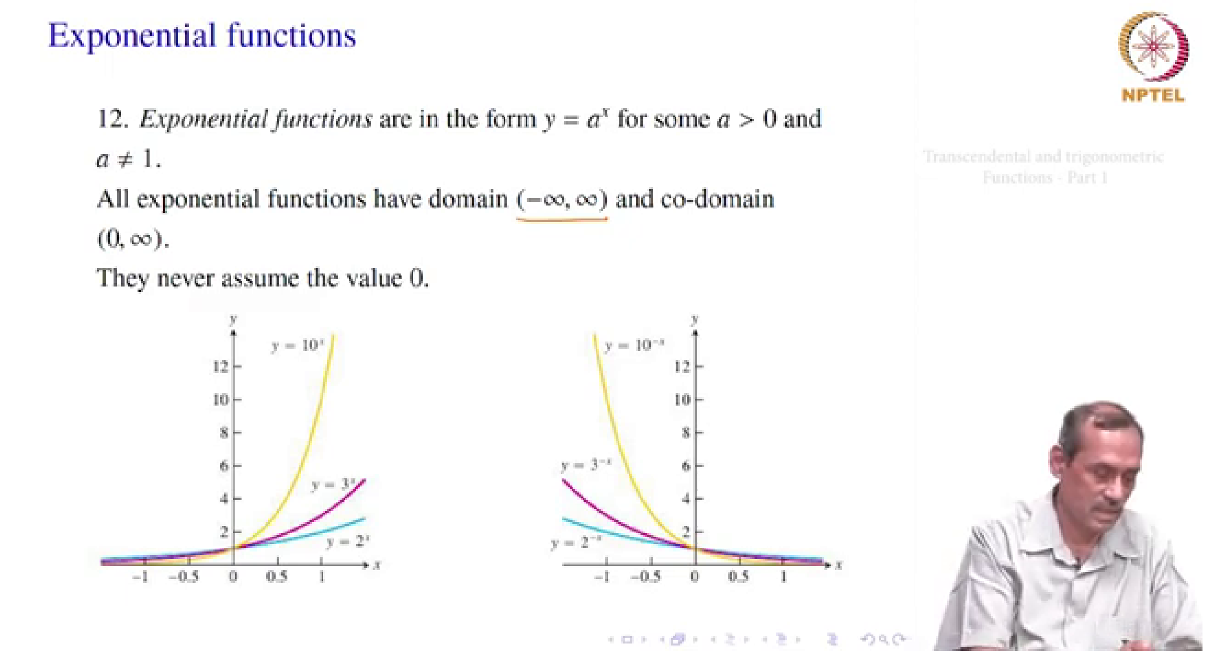
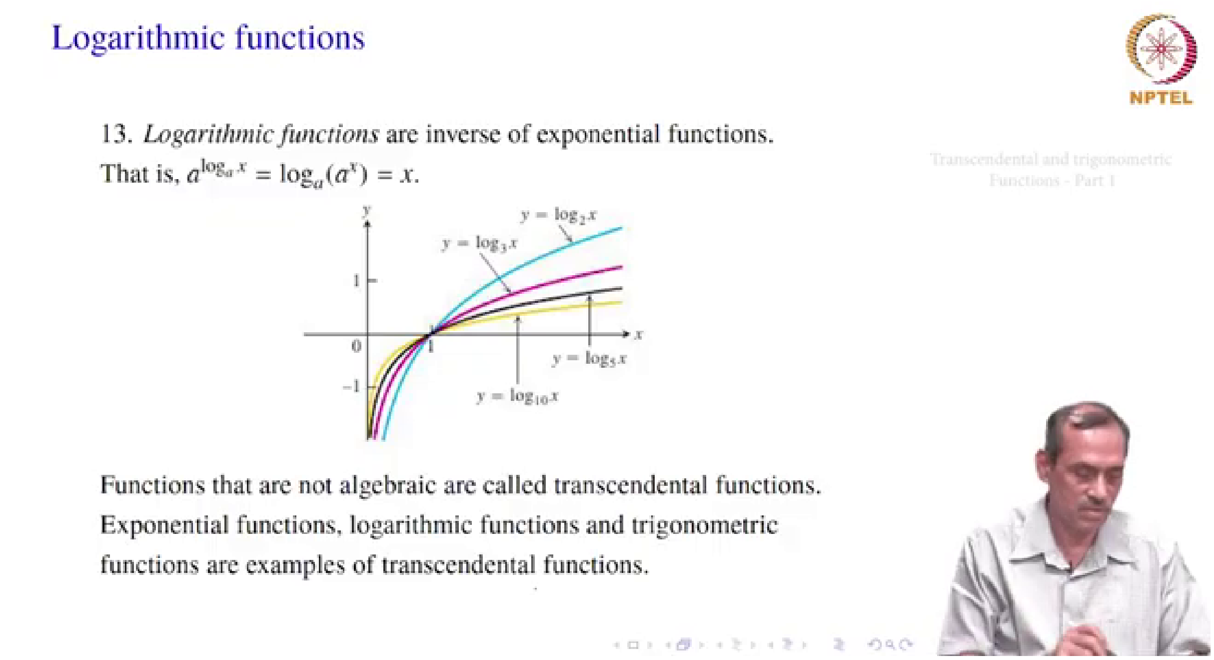


First, we consider the exponential function. It is given as $f(x) = a^x$ for some positive a, a fixed positive number a. Recall that when you take the power function, it is of the form x^a, where a is fixed and x is the variable. Now, this is in the form a^x, where a is fixed and x is the variable. We will assume that $a \neq 1$ because it will be the constant function.

These are called the exponential functions, which are in the form a^x. In a^x, the variable x can vary over the whole of set of real numbers. So, its domain is $(-\infty, \infty)$, the whole of $\mathbb{R}$, and its co-domain is positive reals. a^x can never be 0, of course.

Some examples are here. In $y = a^x$, when you take $a = 10$, you get the function $y = 10^x$, which is this yellow line. When you take $a = 3$, you would get $y = 3^x$, which is your magenta line. And when you take $a = 2$, the function is $y = 2^x$ shown in the blue one. If you take its reciprocal, which is $1/a^x$. And this is same as a^{-x}, where x is replaced by $-x$. When $a = 10$, you get $y = 10^{-x}$, which is the yellow one on the second figure. Similarly, $= 3^{-x}$ gives you the magenta line in the second figure. And, $y = 2^{-x}$ gives the blue one. That is how the curves look like when it is an exponential function.

Look at their nature here. When x decreases, or it is going up in 10^{-x} On the other side, when x increases, 10^x is also increasing. We will come back to increasing and decreasing properties later. (Refer Slide Time: 03:40)



Now, let us take the inverse of this exponential function. We will write that as $\log x$. Here, $y = \log_2 x$. What is the meaning of this to the base 2? It says that when you take a to the power $\log_a x$, that will give you x. Similarly, when you take $\log_a a^x$, that also gives you x. That is how it is the inverse of a^x. Their composition will give you the identity function. That is what it means. In the expression $a^{\log_a x}$, in order that log to be difined, x must be positive. So, the domain of $a^{\log_a x}$ should be positive reals.

Of course, you get back any x from this. Let us see how this is behaving when x is less than 1 and when x is bigger than 1. If $x > 1$, you would get $\log_a x$ to be positive. When $x < 1$, you get $\log_a x$ to be negative. It looks like the yellow one, where we have taken $\log_5 x$. Notice that if you take 5 to the power x instead of x there, then you would get back x.

Similarly, everything has to be interpreted. The yellow one is really $\log_{10}$ and the black one is $\log_5$, the magenta one is $\log_3$, and the blue one is $\log_2$. That is, how the curve should look like. Look at them from the side of y-axis by just rotating by 90 degrees. That is, think of y-axis as the x-axis, as the horizontal one. It will just look like your a^x curve.

These are called the transcendental functions. They are not algebraic. And that includes all exponential functions a^x, the logarithmic functions, which are inverses of power functions, and of course, the trigonometric functions, which will come soon.
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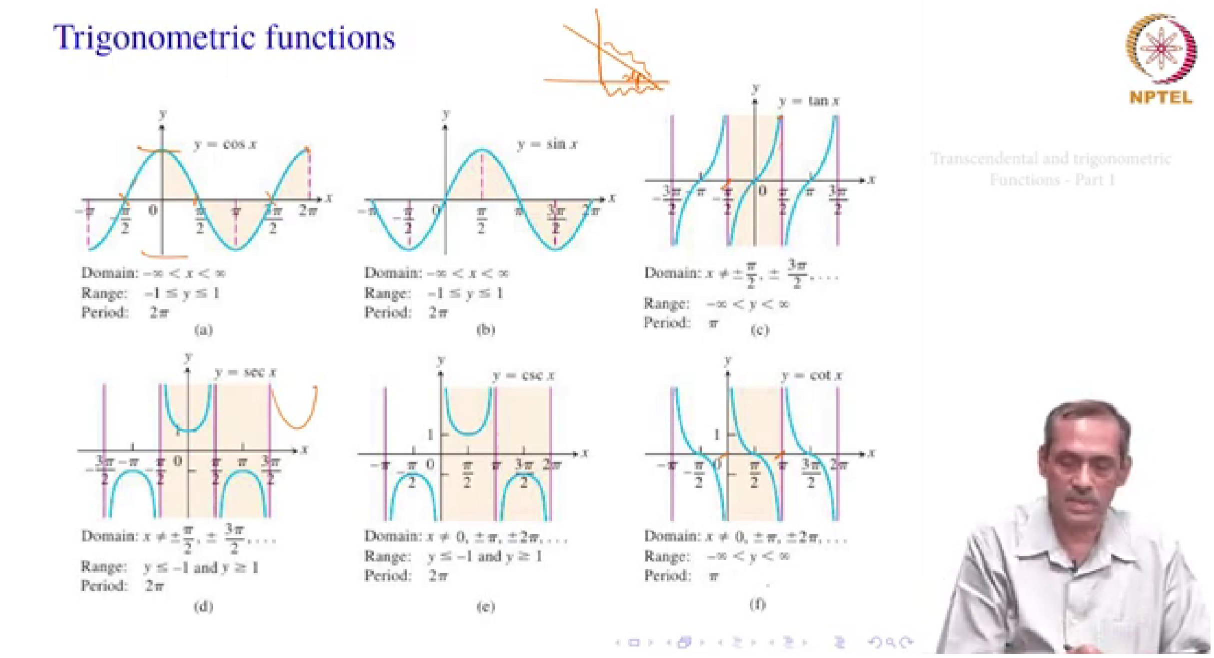

Let us take the trigonometric functions. As you know, we have the six trigonometric functions such as $\cos x$, $\sin x$, $\tan x$, $\sec x$, $\operatorname{cosec} x$ and $\cot x$. In the triangle, if you take this angle as your x, then $\cos x$ is this length: base divided by hypotenuse. Here, this x is really measured in radians. It is a real number, and unit is the radians now. With that $\cos x$, you would get a curve like this. Its plot looks like this. It decreases from 0 to $\pi/2$ to π, increases from π to $3\pi/2$ to 2π and so on.

Similarly, $\sin x$ from 0 to π will be looking like this. It slightly looks like a parabola, but it is not a parabole. It is not up to scale; in y-axis we have a scale different that in x-axis. If you take both the scales same, then it will be looking a bit flatter, something like this. So, that is $y = \sin x$.

If you take $y = \tan x$, it is graph, say, from $-\pi/2$ to $\pi/2$, it will look this way. And the same thing is repeated everywhere. But at $-\pi/2$, tan is not defined, it is not defined at $\pi/2$ also. Because $\tan x = \sin x/\cos x$ shows that $\cos x$ becomes 0 at $-\pi/2$ and also at $\pi/2$. So, $\tan x$ is not defined at these points $-\pi/2$ and $\pi/2$; and also at $\pm 3\pi/2$, and so on. That is, when x is in the form $(2n+1)\pi/2$, $\tan x$ is not defined.

Now, look at $y = \sec x$. There again, as $\sec x = 1/\cos x$, it is not defined at $-\pi/2$, at $\pi/2$, and so on. Like $\tan x$, it is repeating, though in a different way. So, this is repeating here, from $3\pi/2$ to $5\pi/2$. Similarly, if we look at $y = \operatorname{cosec} x$, there is a shift from $\sec x$; the shift is by $\pi/2$. You get this way, just like there is a shift in sin and cos. Also, $\operatorname{cosec} x = 1/\sin x$. It is not defined whenever $\sin x$ becomes 0. You can see that at 0 and π, $\sin x$ is 0. So, at 0 and at π, $\operatorname{cosec} x$ is not defined. Similar thing happens for $y = \cot x = 1/\tan x$. The curve $y = \cot x$ looks like this.

Look at the domains. $\cos x$ has the domain as the whole of real number. $\sin x$ also has the whole of real number as its domain. But the domain of $\tan x$ has all real numbers except $\pm\pi/2$, $\pm 3\pi/2$, and so on. Since $\sec x = 1/\cos x$, its domain is the whole of $\mathbb{R}$ except $\pm\pi/2$, $\pm 3\pi/2$, and so on. Except these points, everywhere else it is defined. Look at $\operatorname{cosec} x = 1/\sin x$. Except at 0, $\pm\pi$, $\pm 2\pi$, and so on, it is defined everywhere else. Similarly, $\cot x$ is defined everywhere, except the interger multiples of π.

Now, look at their ranges. $\cos x$ is always lying between -1 to 1. Its range is $[-1, 1]$, the closed interval minus 1 to 1, and its period is 2π. The pattern is the pattern is repeated in every interval of length 2π. And, in $\sin x$, you get the range similarly. It is $[-1, 1]$, now the curve is shifted by $\pi/2$, and the period is also 2π. Look at $\tan x$. Its range is $(-\infty, \infty)$. That is, every value is achieved, and its period is π. Similarly, what is the range of $\sec x$? It is $1/\cos x$ and the range of $\cos x$ is $[-1, 11]$. When you take 1 by, it will exclude the numbers between -1 and 1. That means the open interval $(-1, 1)$ is excluded from $\mathbb{R}$; that is its range. And the period of $\sec x$ is, of course, 2π. Now, when you look at $\operatorname{cosec} x$ similarly, you get the same range excluding minus the open interval $(-1, 1)$, everything else is there. And in $\cot x$, the range will be whole of real numbers, just like your $\tan x$, and its period is π. So, this is how the trigonometric functions look like.

These trigonometric functions provide nice examples of odd and even functions. We say that a function $f(x)$ is even, if $f(-x) = f(x)$ for each x in the domain of $f(x)$. The domain is the whole of real numbers in case of $\cos x$. There, $f(x) = \cos x$ satisfies $\cos(-x) = \cos x$; that is why it is an even function. And, we say that the function $f(x)$ is odd, if $f(-x) = -f(x)$. You see that $\sin x$ satisfies this condition; that is, $\sin(-x) = -\sin x$. So, $\sin x$ is an odd function.

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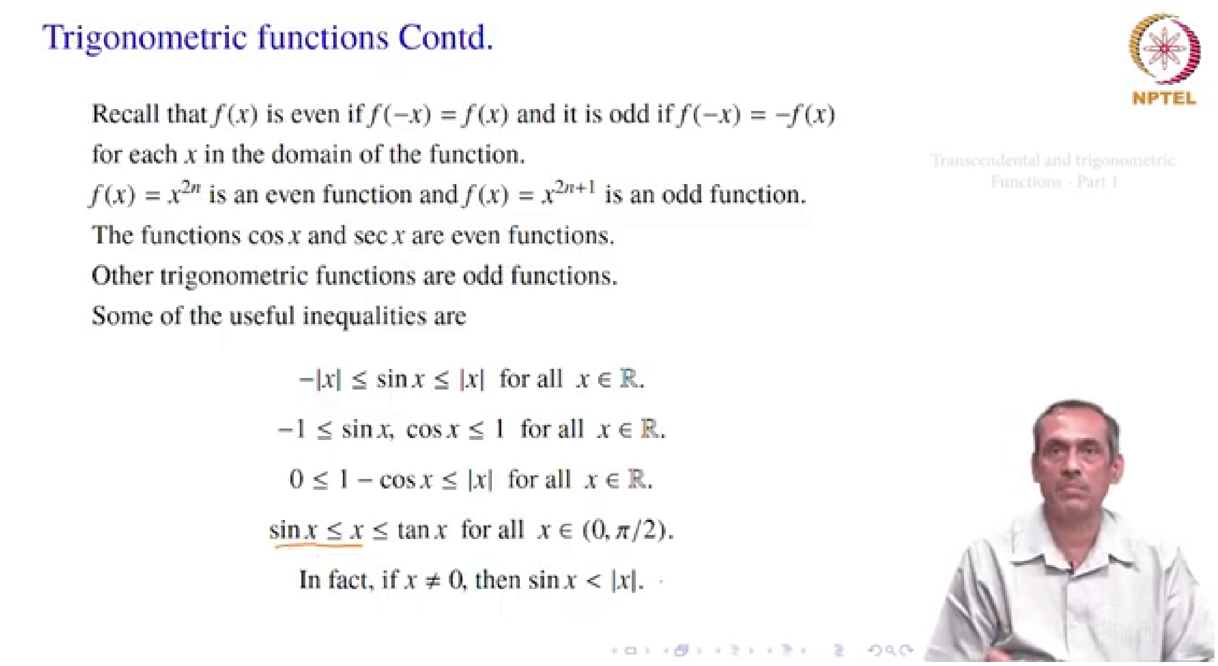

Now, we look at the power functions or algebraic functions such as x^n where n is an integer. As we discussed earlier, x^2 becomes an even function. Because, $x^2 = (-x)^2$. Similarly, x^{2n} is an odd function. But $(-x)^{2n+1} = -x^{2n+1}$. So, x^{2n+1} is an odd function. Similarly, $\sec x$ will become an even function. And all the other trigonometric functions are odd functions.

We will also have some inequalities, which you could have seen from the graph itself. First, $\sin x$ always lies between $-|x|$ to $|x|$ for every x in its domain, which is real number. That is, $-|x| \leq \sin x \leq |x|$. Of course, both $\sin x$ and $\cos x$ have range as $[-1, 1]$, the closed interval. So, we get $-1 \leq \sin x, \cos x \leq 1$ for every $x \in \mathbb{R}$.

We have another inequality which also becomes very helpful. That is, $1 - \cos x$ lies between 0 and $|x|$. It is always positive because $\cos x$ is less than or equal to 1. So, 0 less than or equal to this. And this of course needs something more, which we will see later. But this will be useful: $0 \leq 1 - \cos x \leq |x|$.

Then another useful inequality is: $\sin x \leq x \leq \tan x$ for all $x \in [0, \pi/2)$. Of course equality happens only for $x = 0$; at each other point, it is less than. In general, $\sin x < |x|$ for any $x \neq 0$.

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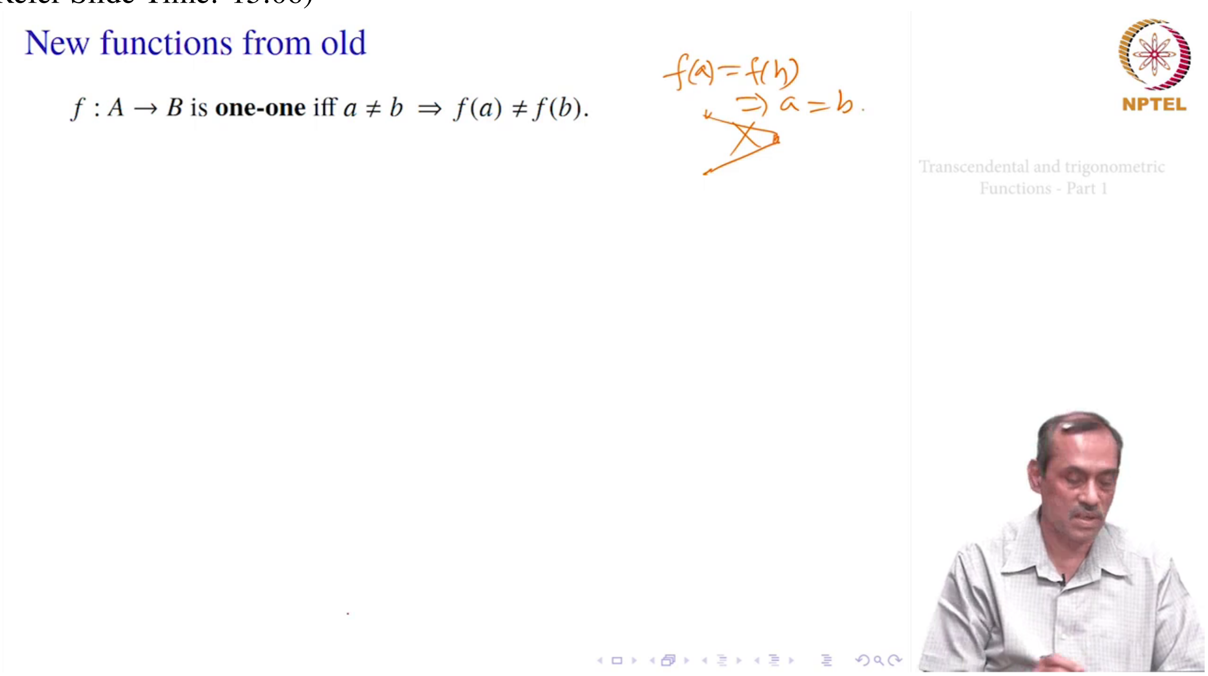

Look at the idea of a function. While defining a function f, we said that for each element in the domain, there is a unique value, in the co-domain, to which it associates. That mean whenever $a = b$, $f(a) = f(b)$, which we also expressed in the other way that $f(a) \neq f(b)$ implies $a \neq b$. We will say that such a function one to one, if and only if, the reverse condition is satisfied; that is, if $a \neq b$, then $f(a) \neq f(b)$. It means two different elements a and b cannot be mapped to the same element on the other side. So, this kind of thing will never occur; two elements a and b will never be mapped to the same element here. This condition when expressed as the other side implication becomes more helpful in applications. An equivalent condition is: $f(a) = f(b)$ implies that $a = b$. This is the condition for f to be one to one; $f(x)$ becomes one to one if this condition is satisfied. And of course, f is onto, if its range coincides with its co-domain; that is, every element in the co-domain has been mapped. That is, the range of f must be equal to B, the co-domain.

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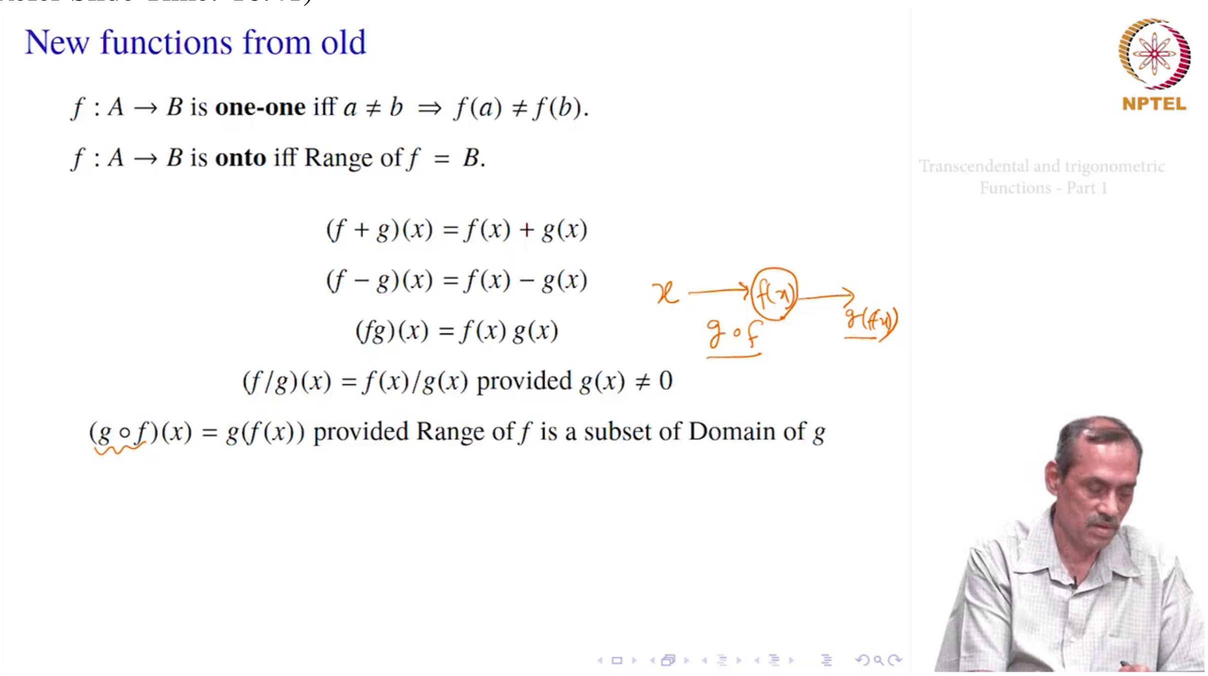

We also find many more functions from the old ones. As we obtained the polynomials from the power functions. We have taken the polynomials like $f(x)$ is a constant, say 1; $f(x) = x$, the identity function. From these you get another function such as $1 + x$. Now we are giving a notation for it. We will say that this is addition of two functions. If f is a function and g is a function with same domain as that of f and the same co-domain as that of f, then you say that $f + g$ is another function, which at takes the value $f(x) + g(x)$ at x. It looks very trivial. It has to be read this way: $f + g$ is a new function, whose value at x is equal to the old value of f at x plus the old value of g at x. So, that gives you the value of $f + g$ at x, and that is how we define $f + g$. When you take $f - g$, it is again the $-g(x)$ on the other side.

Similarly, consider the product. You can have another function which is f times g, also written as fg. Here, $(fg)(x) = f(x) \times g(x)$ For example, take $f(x) = x$, the identity function. Now, $(ff)(x) = f(x)f(x) = x^2$. That is why that is an example of this. We can also define their division. We say $(f/g)(x) = f(x)/g(x)$ provided $g(x) \neq 0$ for any x, because we have to take care of division by 0.

There is another way for obtaining new functions, which is called the composition. If f is a function, g is also a function, then from x, we can first go to $f(x)$. Then, g acts and we get $g(f(x))$ here. That is how the composition looks like: first $f(x)$ next $g(x)$. So, this is written as $g \circ f$. As you come to $g(f(x))$ from x, we preserve this notation $g \circ f$ and not $f \circ g$. So, $g \circ f$ is a new function with $(g \circ f)(x) = g(f(x))$. But its definition needs some condition to be satisfied. Since x goes to $f(x)$, this value $f(x)$ must be in the domain of g so that g can carry it to the value $g(f(x))$. That means the range of f should be contained in the domain of g; then only you can define this. Thus we say: “provided the range of f is a subset of the domain of g”.

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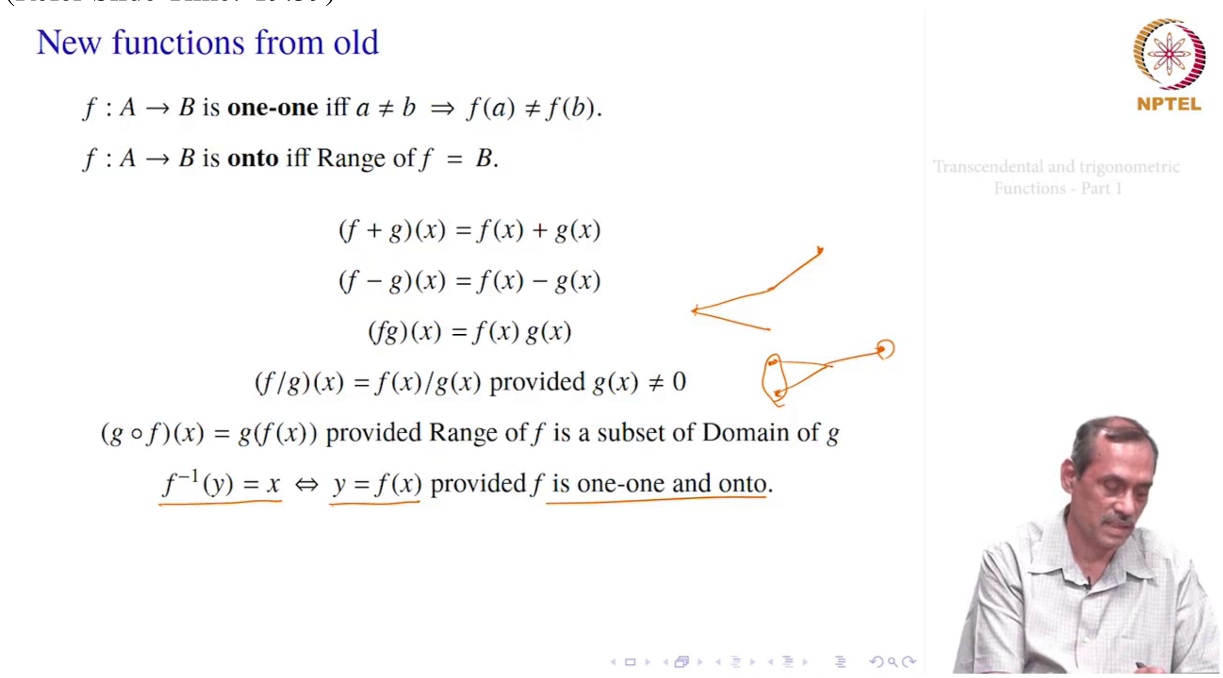

Your inverse function will come from this composition. How? You take a function, say g such that $g \circ f$ is equal to the identity function and $f \circ g$ also should be the identity function. Notice that $f \circ g$ is defined the other way: first g would work then f would work. So, in case, both $g \circ f$ and $f \circ g$ are equal to the identity function, we would say that g is the inverse of f.

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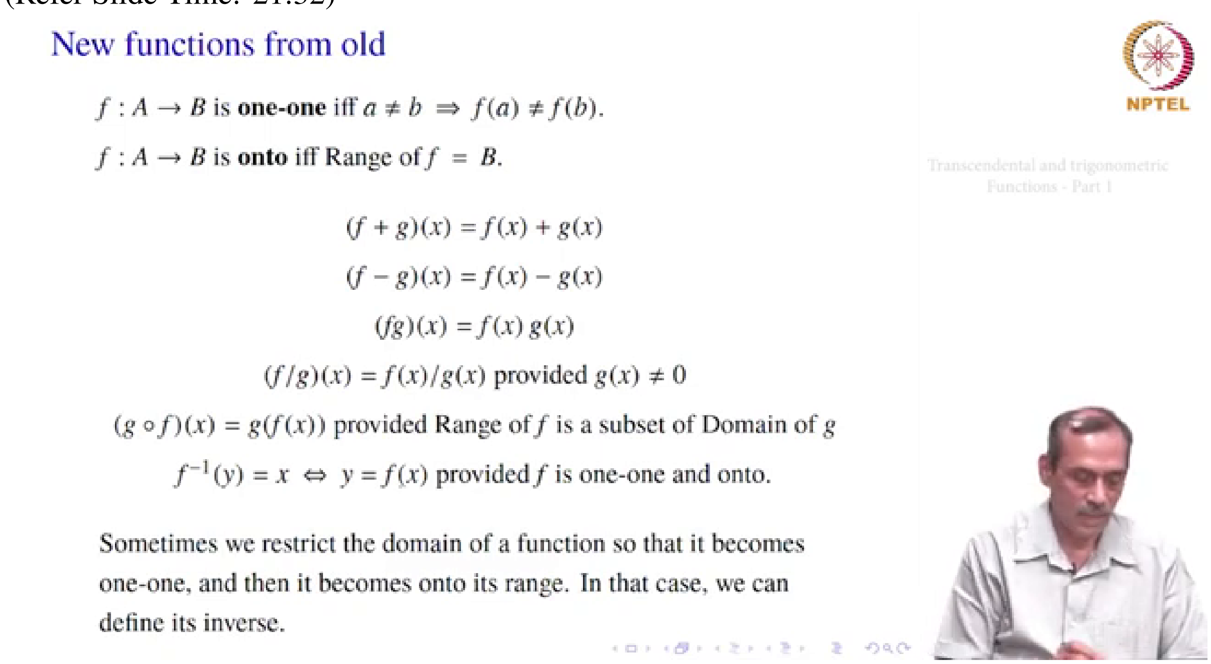

Then, it is required that f is one-one and f is onto. Because, if you go from x to $f(x)$, then g acts and it takes back to x; so it becomes the identity function. These two are the same. When you come back from the other side, that should be defined as a function. If it is not one to one, then there are two points to which it goes. Now that g acts and that becomes identity, that means these two have to be identified as same. Similarly, these are to be identified as same. That means, all these are same, so that f has to be one-one.

Similarly, 'on to' also will come out. That means f^{-1} will be defined provided f is both one to one and onto. If f is a function from A to B, then f^{-1} is defined from B to A; it is one-one and onto. So, you just read the reverse arrows; it brings you back to the same x. We say that $f^{-1}(y) = x$ if and only if $y = f(x)$; that is the definition of f^{-1}. But this can only be defined whenever f is both one-one and onto.

Once f is one-one and onto, f^{-1} also becomes one-one and onto. The inverse function is defined only for those functions and their inverses are also of similar type so that f becomes the inverse of f^{-1}. This is the way the logarithm $\log_a x$ and the exponential a^x were working. It is this composition to which the inverse refers.

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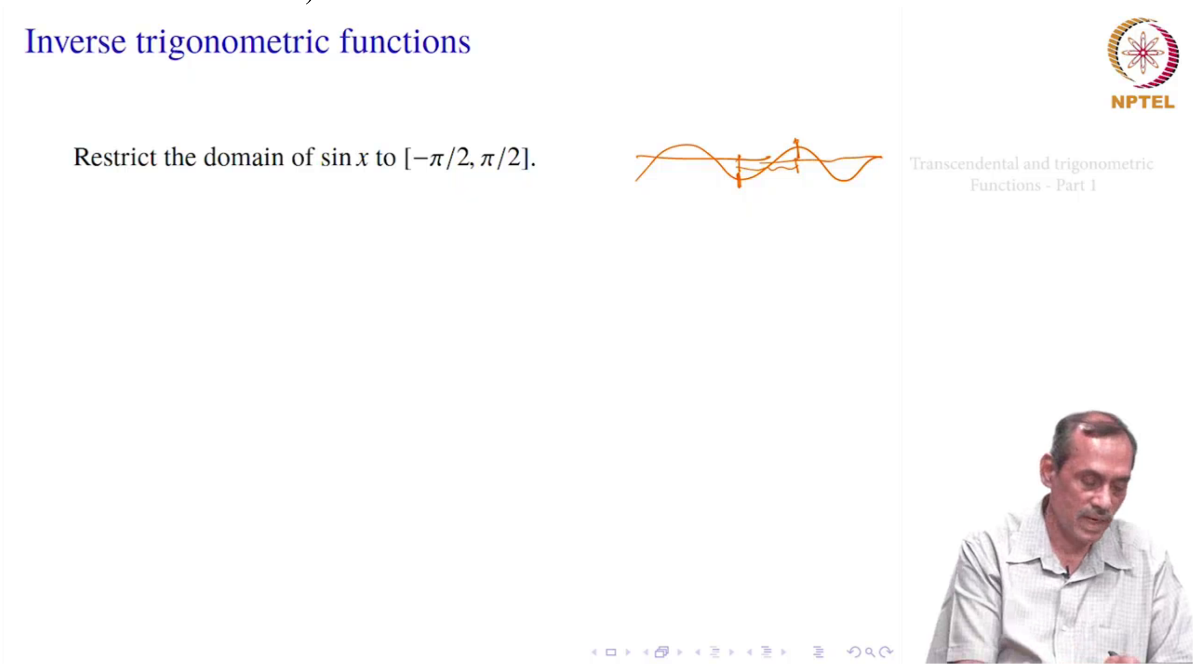

Suppose you take the trigonometric functions. Look at $\sin x$ for example. It has the domain as the whole of real numbers, its range is minus $[-1, 1]$, and it is periodic. It goes something like this; the pattern is repeated. And then you look at the points $-\pi/2$ and $\pi/2$. At these points, $\sin x$ achieves the values -1 and 1, respectively. That means the restriction of $\sin x$ to this domain $[-\pi/2, \pi/2]$ is a one-one and onto function with range as $[-1, 1]$. So, you can define its inverse.

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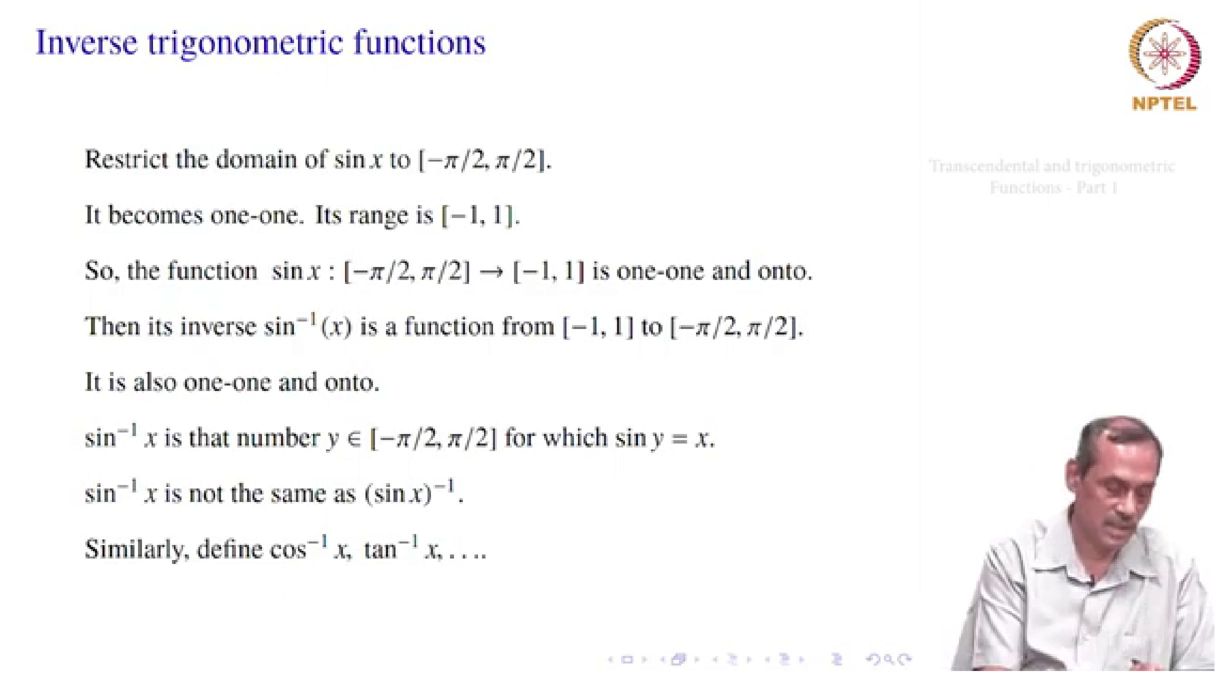

Its inverse is a function from $[-1, 1]$ to $[-\pi/2, \pi/2]$; we write it as $\sin^{-1} x$. So we say that $\sin^{-1} x$ is a function from $[-1, 1]$ onto $[-\pi/2, \pi/2]$. But how does it act? If you take y here, then it goes to $\sin^{-1} y$ here in such a way that if this is your x, this becomes $\sin x$, so y becomes $\sin x$. So, we say that $\sin^{-1} x$ is that number y in $[-\pi/2, \pi/2]$ for which $\sin y$ becomes equal to x.

When you take sine of sine inverse, or sine inverse of sine, they become identity functions on their respective domains. But, remember that sine minus 1, which we write as a superscript, as $\sin^{-1} x$ is not same as $\sin x$ to the power -1, though we write $3^{-1} = 1/3$. It is looking the same way, something like $\sin^{-1}$ but it is different from $(\sin x)^{-1}$; $(\sin x)^{-1}$ is $\operatorname{cosec} x$, but $\sin^{-1} x$ is not $\operatorname{cosec} x$; they are different. The inverse of $\sin x$ is inverse with respect to composition. Similarly, you can define $\cos^{-1} x$, $\tan^{-1} x$, $\sec^{-1} x$, $\cot^{-1} x$ and $\operatorname{cosec}^{-1} x$.