Basic Calculus - 1
Professor. Arindama Singh
Department of Mathematics
Indian Institute of Technology Madras
Lecture 35 - Part 1
Lengths of Curves - Part 1

This is lecture 35 of Basic Calculus - 1. We were discussing applications of integrals to computing volumes of solids of revolution. Today, we will have another application, which is to find the lengths of curves using integrals. Let us see how to do this.
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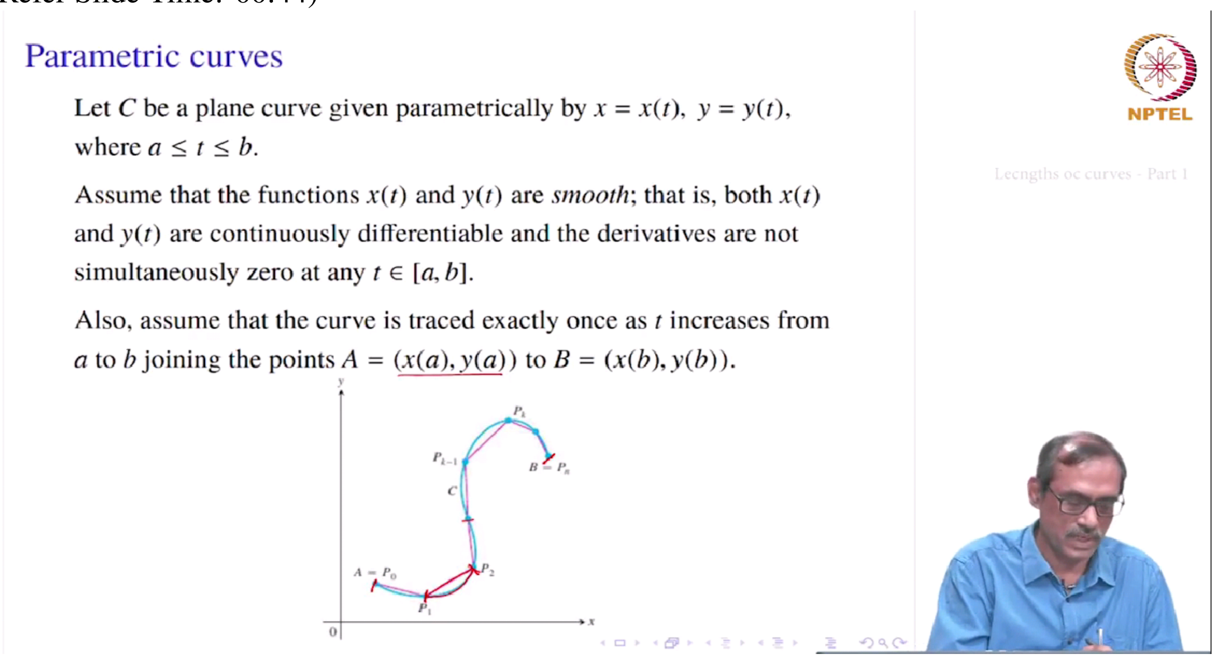

First of all, we have to see what is a curve. What do you mean by a curve in the plane? Imagine that you are drawing a curve in the plane and you are keeping your time; you will look at your watch. Suppose you have started at time $t = 0$ and then go on drawing the curve. As t increases, these points are drawn in the plane. So, any point on the curve can be thought of as having components, one is the x component and the other is the y component. When t varies both these components vary. So, you may think of them as functions, $x(t)$ and $y(t)$. Once a curve is there, we may think of that as if it is given parametrically by $x = x(t)$ and $y = y(t)$.

Basically, it may not be the time t. For instance, x can be taken as time t itself, and y can be a function of x, $y = f(x)$. Here, x itself is taken as a parameter and $y = f(x)$. In general, you can think of some parameter t, and as t varies over some interval, you get these points: the x-coordinate is a function of that parameter t and the y-coordinate is a function of that parameter t so that you get the point (x, y). That is how the curve is drawn.

So, let us assume that C is a plane curve given parametrically by $x = x(t),\ y = y(t)$. Both $x(t)$ and $y(t)$ are functions over the interval $[a, b]$, where t varies over this interval. We also add some nice properties. As time proceeds, it does not stop anywhere, unless t reaches that point b. So, you may say that the derivatives $x'(t)$ and $y'(t)$ at any point t are not simultaneously 0. Of course, we assume that the derivatives of $x(t)$ and $y(t)$ exist with respect to t.

We thus say that the functions $x(t)$ and $y(t)$ are smooth; that is, $x(t)$ and $y(t)$ are continuously differentiable, and the derivatives are not simultaneously 0 at any $t \in [a, b]$. With this assumption, let us look at the curve. We want to find its length. Let us say that when $t = a$, the point $(x(a), y(a))$ on the curve is the beginning point. And at $t = b$, we have the end-point $(x(b), y(b))$. Look at the blue curve in the picture. We want to approximate its length.

What do we do to approximate its length? We choose many points on the curve. And then any small portion of that curve is approximated by the length of the straight line segment joining those two points. Of course, there will be some errors in this. But as you see, when the points get closer and closer, we will be committing less and less error. We are planning to use the idea of Riemann sum; we make a partition of $[a, b]$ and then take the sum of all these lengths and finally, we take the limit of this sum as the norm of the partition goes to 0.

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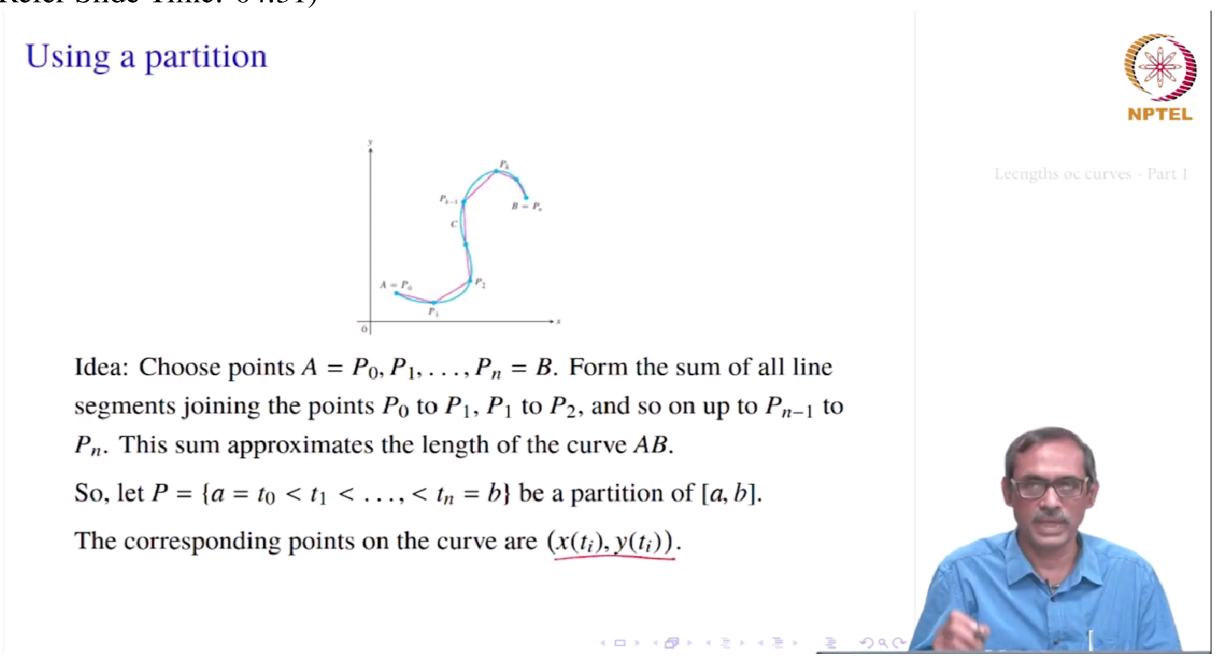

Let us describe the details. We take a partition of the interval $[a, b]$ with points $a = t_0 < t_1 < \cdots < t_n = b$. The partition points $t_0, t_1, \ldots, t_n$ correspond to the points P_0, $P_1, \ldots, P_n$ on the curve. That is, for each i, the point P_i on the curve is simply $(x(t_i), y(t_i))$. The sum of the lengths of these straight line segments will give us an approximation for the length of the curve.

How to get the length of the line segment joining P_{i-1} to P_i? We have the points $P_{i-1} = (x(t_{i-1}), y(t_{i-1}))$ and $P_i = (x(t_i), y(t_i))$. We need to find the length of the line segment joining P_{i-1} to P_i. You know that this length is the length of the hypotenuse of this right angled triangle. It is the square root of $[x(t_i) - x(t_{i-1})]^2 + [y(t_i) - y(t_{i-1})]^2$. This is the length of the line segment.

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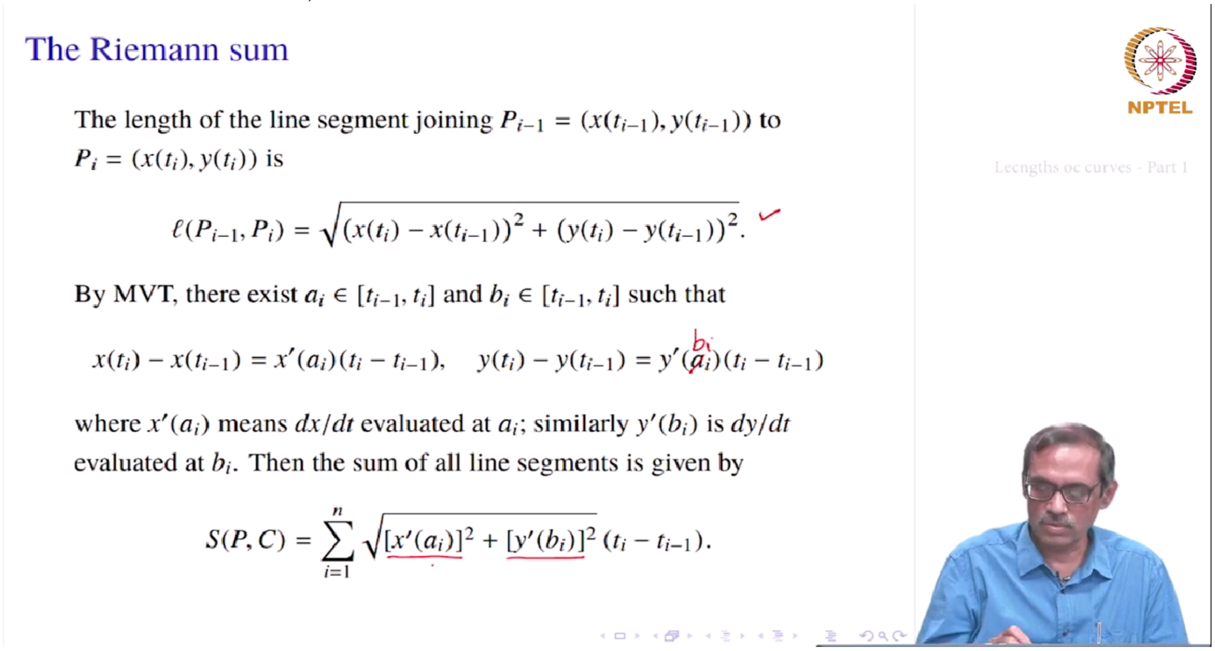

Before we take the sum, let us look at a convenient way of expressing this length. We know that $x(t_i) - x(t_{i-1})$ is equal to $x'(a_i)(t_i - t_{i-1})$ for some point a_i between t_{i-1} and t_i. That is what our mean value theorem for the differentials say. Since $x(t)$ is continuously differentiable, we can write like this for some a_i between t_{i-1} and t_i. Same thing happens for the function $y(t)$. By mean value theorem, we say that $y(t_i) - y(t_{i-1})$ is equal to $y'(b_i)(t_i - t_{i-1})$ for some b_i between t_{i-1} and t_i.

If instead of the mean value theorem, you use Cauchy mean value theorem, then you can take some point c_i instead of possibly different points a_i and b_i. That allows taking the same point for both the functions simultaneously. However, we can manage with the usual mean value theorem. We take possibly different points such as a_i and b_i. Notice that $x'(a_i)$ means the derivative of $x(t)$ with respect to t evaluated at a_i. Similarly, $y'(b_i)$ is the derivative of $y(t)$ with respect to t evaluated at b_i.

Then what happens, the length of any line segment joining P_{i-1} and P_i, is the square root of $[x'(a_i)]^2 + [y'(b_i)]^2 \times [t_i - t_{i-1}]^2$. Simplifying, we get the length as $\sqrt{[x'(a_i)]^2 + [y'(b_i)]^2}(t_i - t_{i-1})$. We then take the sum of all these lengths to get this expression. This approximates the length of the curve. Now, this sum looks like a Riemann sum. When the norm of the partition goes to 0, the maximum of the lengths $t_i - t_{i-1}$ goes to 0. In that case, the sum will be the integral $\int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$. This is how we are going to define the length of a curve. We say that this integral is the length up the curve.

Let us summarize what we have done. Suppose the curve is given in the form $x = x(t)$, $y = y(t)$, where the parameter t varies from a to b. Then its length can be expressed as $\int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$. Now, we take a particular case, where the curve is given as $y = f(x)$. Once y is a function of x, the parameter t is really x. That means $x = t$ and $y(t) = y(x) = f(x)$. Then, $x'(t) = 1$ and $y'(t) = f'(x)$. So, the integral can be written as $\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$.

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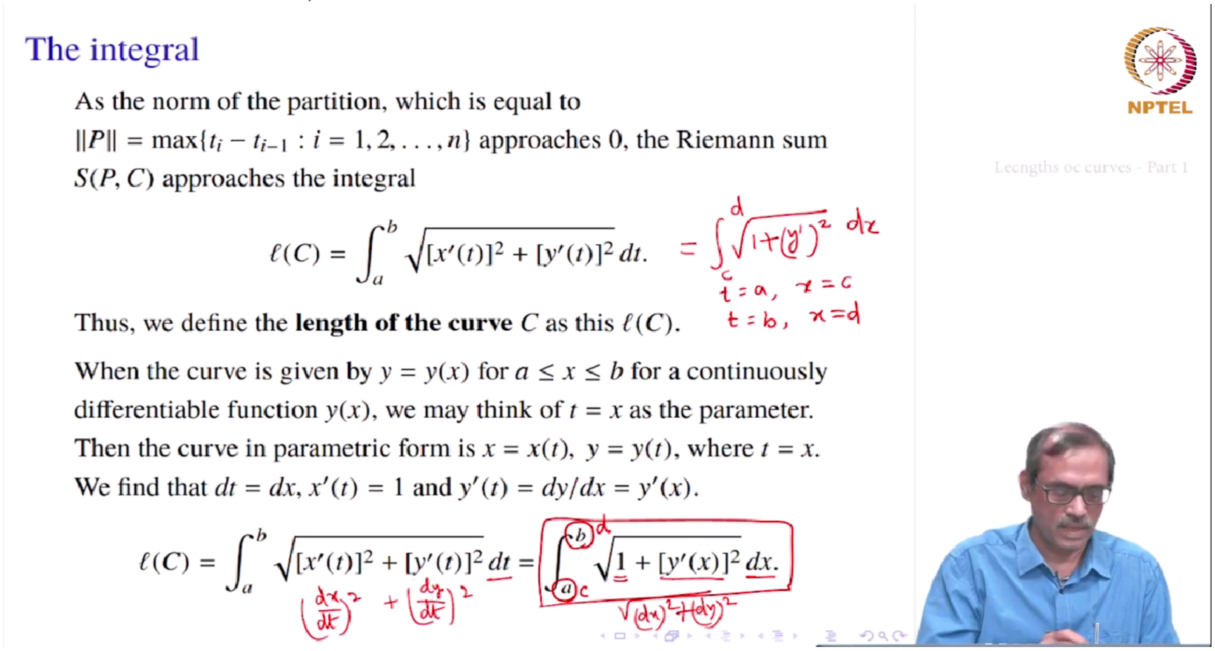

If you think of using the differentials, then $\sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$ is really $\sqrt{[dx/dt]^2 + [dy/dt]^2}\, dt$, where dt cancels and you get $\sqrt{(dx)^2 + (dy)^2}$. When y is a function of x, say, $y = f(x)$, as earlier this differential gives $\sqrt{1 + [f'(x)]^2}\, dx$. To remind ourselves that the variable is now x and not t, we may write that x varies from c to d. Then, the length of the curve will be written as $\int_c^d \sqrt{1 + [f'(x)]^2}\, dx$.

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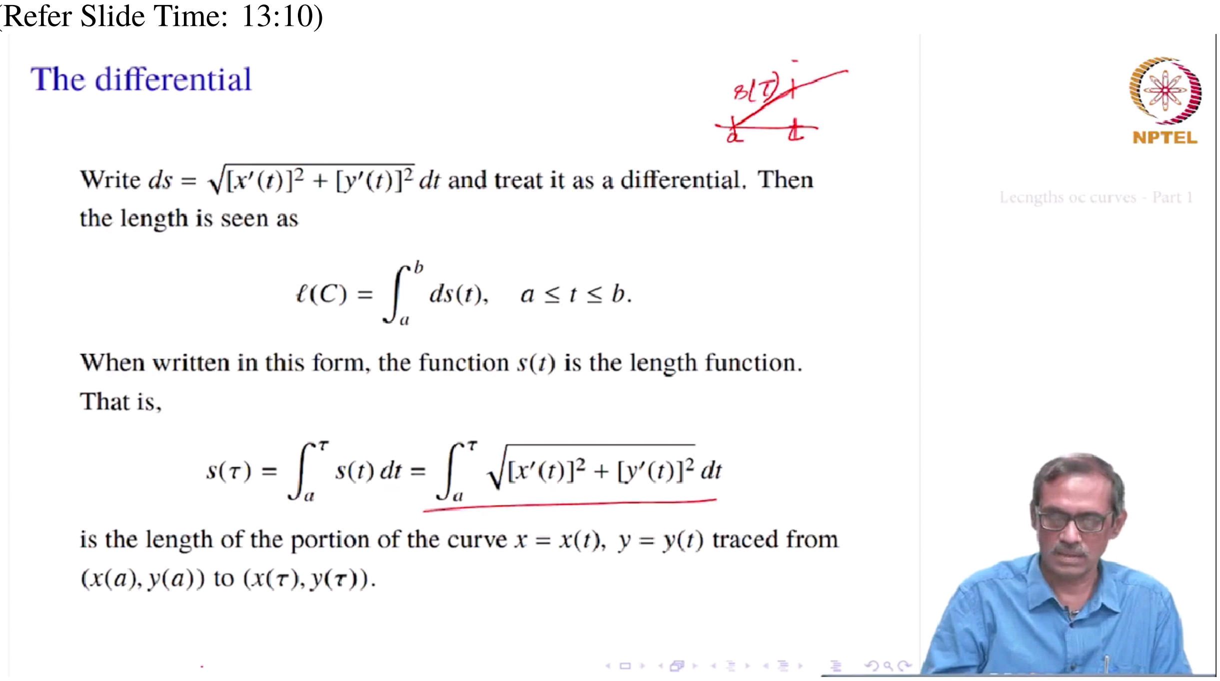

Let us use the differential notion again and see how does it look. The integrand, which is integrated here can be thought of as a differential. We usually write $\sqrt{(dx)^2 + (dy)^2}$ as the differential ds, where s represents the length of the curve measured from the start point to the current point along the curve. Then, it makes sense to write the total length as the integral of

ds. Then, you can write the length of the curve in a simpler way. It is $\int_a^b ds(t)$. Notice that by the fundamental theorem, $\int_a^b ds(t) = s(t)\big|_a^b$ is simply the length of the curve. That is how the differential form takes care of the length. To have a proper perspective, you may think of an arbitrary point τ. When $t = \tau$, $s(\tau)$ is the length of the curve from $t = a$ to $t = \tau$. That is, $s(\tau)$ is the length measured along the curve from the point $(x(a), y(a))$ to $(x(\tau), y(\tau))$. The differential notation is useful when you want to express that length as a function of t. At any point $t = \tau$, $s(\tau)$ will be the length starting from the beginning of the curve, which corresponds to $t = a$ to the current point $t = \tau$. That is how this formula can be seen.

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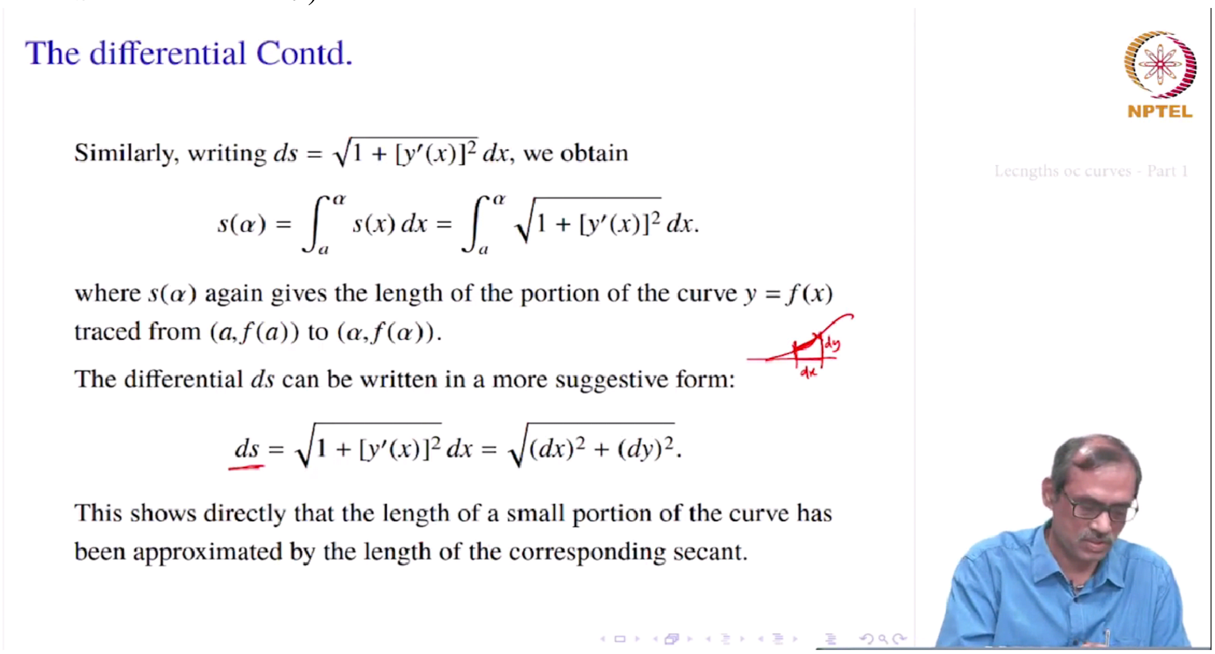

Again, if you look at the differential way of writing the length, we may write $ds = \sqrt{1 + [y'(x)]^2}\, dx = \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$. Suppose a point on the curve corresponds to $t = \alpha$. Then, the length of the curve from the start point to this point is written as $s(\alpha)$, and $s(\alpha)$ is equal to the integral $\int_a^\alpha s(t)\, dt$. Now, if you want to express it in terms of x, then, writing t as x in the integral you would get the same length as $\int_a^\alpha s(x)\, dx$. This is again equal to $\int_a^\alpha \sqrt{1 + [y'(x)]^2}\, dx$ provided the curve is expressed by y as a function of x. We can express the same length of the curve in various ways. Similarly, if the curve is given by x as a function of y, then you can write a corresponding formula, but we will come to it later.

So, how to interpret the integral with ds? We can think of ds as a small length on the curve, say, this portion of the curve. Then, the total length is simply the integral of ds. Now that $ds = \sqrt{(dx)^2 + (dy)^2}$, the same length can be written as the integral of $\sqrt{(dx)^2 + (dy)^2}$. To emphasize, this is really dx and this is dy; so you may think of this hypotenuse as $\sqrt{(dx)^2 + (dy)^2}$. That shows clearly that $ds = \sqrt{(dx)^2 + (dy)^2}$. That means the length of the curve has been taken as the limit of the sum of approximate lengths of the secants of the curve joining the close by points.

Anyway, these are just different ways of looking at the same formula. All that we have is the direct formula. If the curve is given in terms of t and t varies from a to b, then you can express its length as $\int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$, where this prime means differentiation with respect to t. If the curve is given by y as a function of x, where x varies from a to b, then the length of the curve is the integral $\int_a^b \sqrt{1 + [y'(x)]^2}\, dx$, where the prime denotes differentiation with respect to x.

Let us apply this to one of the examples. First of all, let us verify that the way we have formulated te length of a curve abstractly is really matching with that of some known curves. Suppose C is the circle of radius r centered at the point (a, b). We know that the length of C is $2\pi r$. So, let us see whether the abstract definition of length conforms to this or not.

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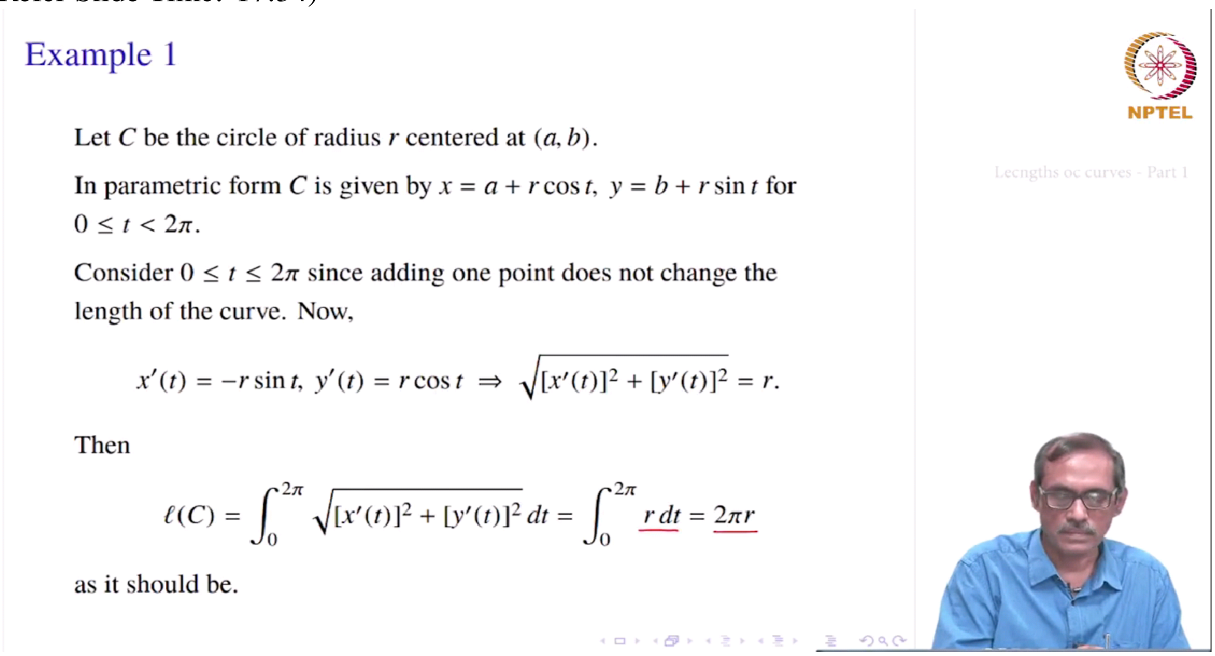

To apply our formula, we have to express that circle in parametric form. Since r is the radius and the x-coordinate of the center is a, b is the y-coordinate of the center, we can write C in parametric form by $x = a + r\cos t,\ y = b + r\sin t$, where $t \in [0, 2\pi]$.

We can now verify directly. With x and y as parametrized above, we see that $x^2 + y^2 = a^2 + b^2$. This is the square of the length joining any point on C to the center (a, b), and that turns out to be r^2. So, the parametrization is correct. Alternatively, the parametrization of the circle with center at the origin and radius r is $x = r\cos t,\ y = r\sin t$. Then shifting the origin to (a, b) gives the parametrization $x = a + r\cos t,\ y = b + r\sin t$. Here, t varies from 0 to 2π. So, this is the parametric form of this given circle.

Then, we compute the length of the curve. For that, we need $x'(t)$ and $y'(t)$. As $x = x(t) = a + r\cos t$, $x'(t) = -r\sin t$. Here, the derivative of the constant a is 0, of $\cos t$ is $-\sin t$. Similarly, $y'(t) = r\cos t$. So, $[x'(t)]^2 + [y'(t)]^2$. is equal to $r^2(\cos^2 t + \sin^2 t)$. It simplifies to r^2 so that the integral becomes $\int_0^{2\pi} \sqrt{r^2}\, dt = \int_0^{2\pi} 2\pi r\, dt = 2\pi r$. So, it really confirms with whatever formula we are having for the case of the circle. Let us take some more examples.

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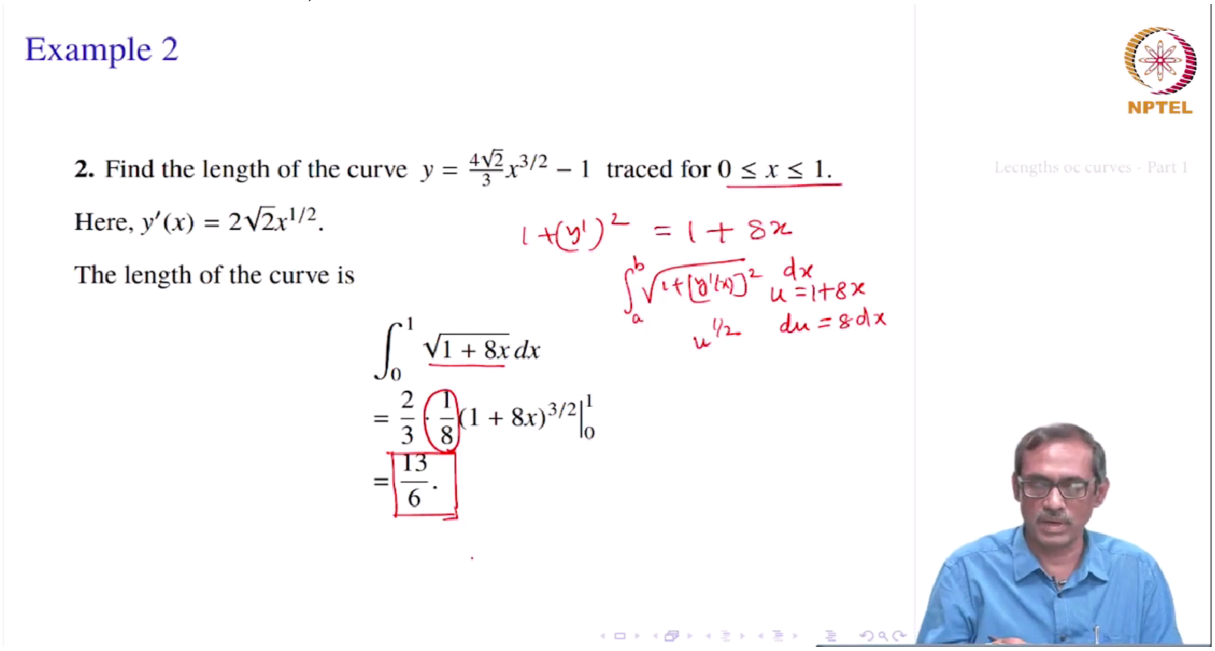

Here is another example. Find the length of the curve $y = 4\sqrt{2}/3\,x^{3/2} - 1$. Here, y is given as a function of x. The curve is traced when x varies from 0 to 1. If you want to convert it to parametric one, it will be $x = t$ and $y = y = 4\sqrt{2}/3\,t^{3/2} - 1$. So, our parameter t is x itself, when y is a function of x. Therefore, $x'(t) = 1$, and $y'(t) = 4\sqrt{2}/3\,(3/2)t^{1/2}$. With $x = t$, we have $[x'(t)]^2 + [y'(t)]^2 = 1 + [dy/dx]^2$.

So, let us apply that directly. We need $y'(x)$. Differentiating y with respect to x, we get $y'(x) = (4\sqrt{2}/3)(3/2)x^{1/2}$. This 3 gets canceled and this becomes $2\sqrt{2}x^{1/2}$. Then, the length of the curve is the integral $\int_0^1 \sqrt{1 + [y'(x)]^2}\,dx$. We have to take $[y'(x)]^2$; that is equal to $8x$. So, the length of the curve is $\int_0^1 \sqrt{1+8x}\,dx$. Now, it is a matter of integration. In this integration, you substitute $u = 1 + 8x$ so that the integrand is $u^{1/2}$ and $du = 8\,dx$. Thus, $\int \sqrt{1+8x}\,dx = \int (1/8)u^{1/2}\,du$. Its integration gives $(1/8)(2/3)u^{3/2} = (2/3)(1/8)(1+8x)^{3/2}$. And this is to be evaluated at 0 and 1 and subtracted out. It simplifies to $13/6$.

So, that is how we are going to evaluate the length of curves.