Basic Calculus - 1
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Lecture 33 - Part 2
The Washer Method - Part 2

Let us take another problem. It is asked to find the volume of the solid generated by revolving the region, in the second quadrant, which is bounded above by the curve $y = -x^3$, below by the x-axis and on the left by $= -1$, about the line $x = -2$. You must plot the curve $y = -x^3$ in the second quadrant. This is the second quadrant, and this is the x-axis. The left border of the region is on the line $x = -1$. that is, the region is lying to the right of the line $x = -1$. We want to revolve this region about the line $x = -2$.

That is how $y = x^3$ in the second quadrant looks. This is origin; we have -1 and -2 here on the x-axis. The region is to be revolved about the line $x = -2$. And, what is the region? It is to the left of $y = -x^3$ in the second quadrant, above the x-axis, and to the right of $x = -1$. So, this is the region, the shaded portion here. This is revolving about the line $x = -2$, which is parallel to the y-axis.

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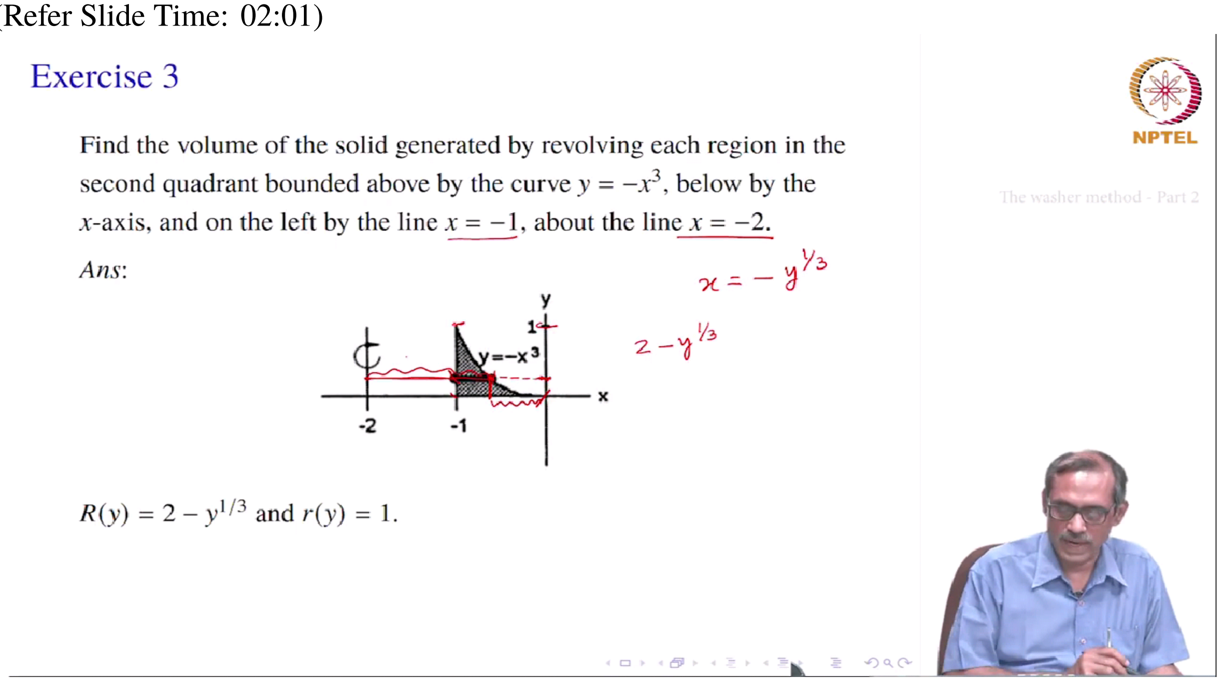

Since the region is revolved about the $x = -2$, which is parallel to the y-axis, we need to express everything in terms of y. Choose any point inside this region, and let y be its y-coordinate. Then, the cross-sectional disk corresponding to that y have an outer radius and an inner radius, which are to be measured from the axis of revolution. The axis of revolution is $x = -2$. Therefore, the outer radius will be this length and the inner radius is this length. The inner radius has the length 1 as it spans from -2 to -1. The outer radius comes from this point which is on the curve $y = -x^3$. Once you take one y, the corresponding x will be equal to $iy^{1/3}$; and that is the x-coordinate here. This is $-y^{1/3}$. Therefore, the outer radius is, this span plus $-y^{1/3}$, which is $2 - y^{1/3}$.

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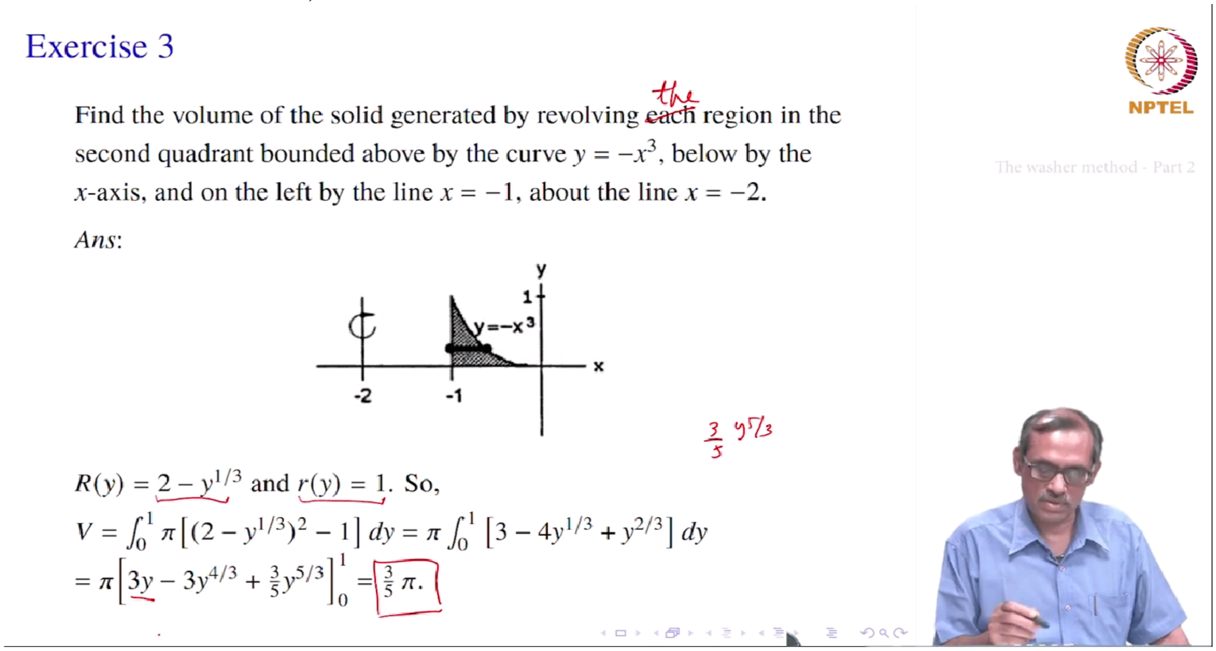

Now, we have the outer radius as $R(y) = 2 - y^{1/3}$ and the inner radius as $r(y) = 1$. The limits of integration for y, are 0 and 1, as y varies betwen these two to describe the region. The below limit $y = 0$ comes from the point of intersection of $y = -x^3$ with the x-axis; and the the top limit comes from the intersection of $y = -x^3$ with $x = -1$.

Therefore, we have the volume as $\int_0^1 \pi[R^2(y) - r^2(y)]\, dy$, which is equal to $\int 0^1 \pi[(2 - y^{1/3})^2 - 1]\, dy$, Or, $\int_0^1 \pi[3 - 4y^{1/3} + y^{2/3}]\, dy$. Integration of 3 gives $3y$, of $y^{1/3}$ gives $(3/4)y^{4/3}$ and of $y^{2/3}$ gives $(3/5)y^{5/3}$. Then, the integral is $\pi[3y - 3y^{4/3} + (3/5)y^{5/3}]$. This is to be evaluated at 1 and 0, and subtracted. That simplifies to $3\pi/5$.

We had to plot the region for understanding what is being asked in the story. Then it became easier for finding the outer radius and the inner radius of the washer, and of course, the limits.

Let us take another problem. Here, we want to find the volume of the solid generated by revolving about the line $x = -1$ the region in the first quadrant bounded above by the curve $y = x^2$, which is a parabola, below by the x-axis, and on the right by the line $x = 1$.

First, let us plot the region. This is the parabola $y = x^2$ in the first quadrant; it passes through the origin. Here is the x-axis on which lies the bottom border of the region. We see that $x = 0$ is already the intersection point. On the right of the region is this line $x = 1$. This crosses the curve, and it is the other point which is $x = 1$. The point of intersection is $(1, 1)$.

This region is to be revolved around the line $x = -1$. This line is parallel to y-axis. Then we have to express everything in terms of y. The limits for y are clear as y varies from 0 to 1 for any point that lies in the region. These values $y = 0$ and $y = 1$ are the limits of integration.

What about the outer and inner radius of the washer? The washer will be something like this; it is a circular disk, an annulus; it is this white portion. The outer radius is the total, which is equal to the length from −1 to 1, that is, 2; the length is 2. So, $R(y) = 2$. What is $r(y)$? It is this length measured from this side to $y = x^2$. And, we want to express that in terms of y. The curve is $x = \sqrt{y}$ in the first quadrant. Therefore, the inner radius is $r(y) = 1 + \sqrt{y}$.

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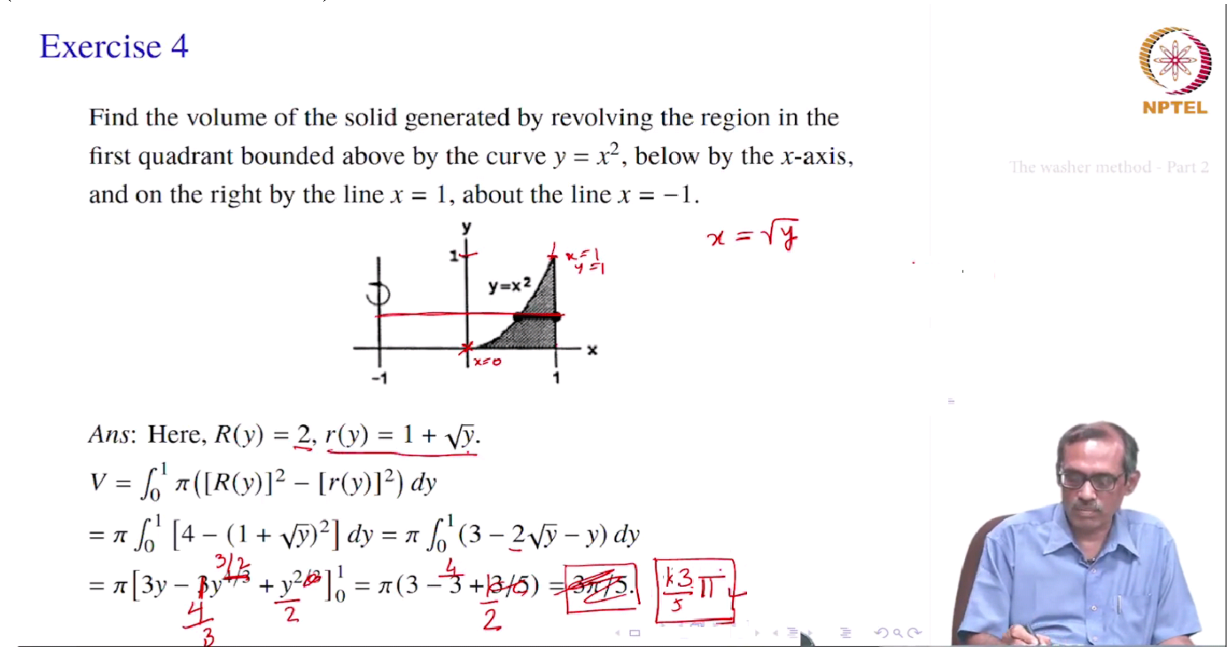

Now, you plug it in the formula. The volume of the solid so generated is the integral $\int_0^1 \pi[R^2(y) - r^2(y)]\, dy$, which is equal to $\int_0^1 [2^2 - (1 + \sqrt{y})^2]\, dy$. We expand the integrand; it is $4 - 1 = 3$ minus $2\sqrt{y} + y$. And we integrate. So, 3 gives $3y$, $2\sqrt{y}$ gives $2(2/3)y^{3/2} = (4/3)y^{3/2}$, and y gives $y^2/2$. The integral is $\pi[3y - (4/3)y^{3/2} + y^2/2]$. This is to be evaluated at 1 and 0, and then subtracted. That simplifies to $13\pi/6$.

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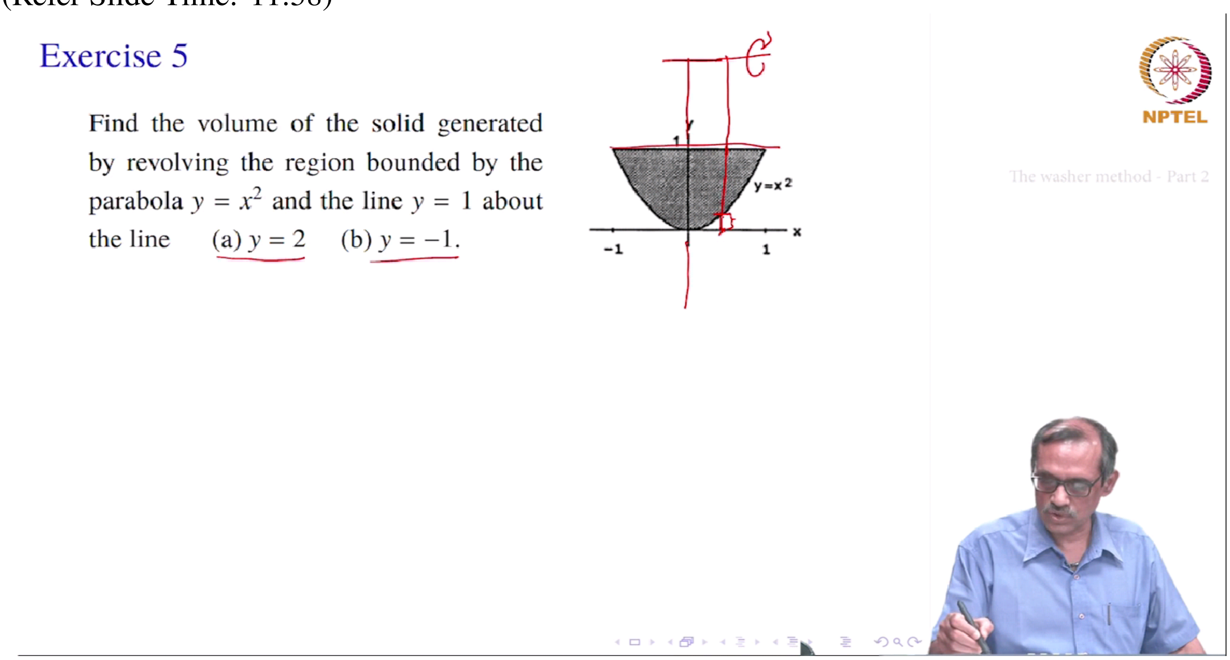

Let us take another problem. We want to find the volume of the solid generated by revolving the region bounded by the parabola $y = x^2$ and the line $y = 1$. There are two problems here. In Part (a) the region revolves about the line $y = 2$, and in Part (b), the same region is revolved about the line $y = -1$.

Let us plot the region first. The parabola is $y = x^2$. It is plotted here along with the line $y = 1$. That gives us the region. We consider Part (a). This region is to be revolved about the line $y = 2$. The line $y = 2$ is somewhere here, which is parallel to the x-axis. So, we must find out the x-coordinates, for the limits of integration and express express the washer in terms of $R(x)$ and $r(x)$. So, take any x here. This length is to be measured from $y = 2$. Then, $R(x)$ is up to this and $r(x)$ is only here. These lengths are to be computed. This length shows $R(x) = 2 - x^2$. From here, we have $r(x) = 1$. Hence, $R^2(x) - r^2(x) = (2 - x^2)^2 - 1$.

The volume of the generated solid is $\int_{-1}^{2} [(2 - x^2)^2 - 1]\, dx$. We expand the integrand to obtain $\pi(3 - 4x^2 + x^4)$. Its integral is $\pi(3x^2 - (4/3)x^3 + x^5/5$. This is to be evaluated at 1 and -1, and then subtracted. That gives the answer as $56\pi/15$.

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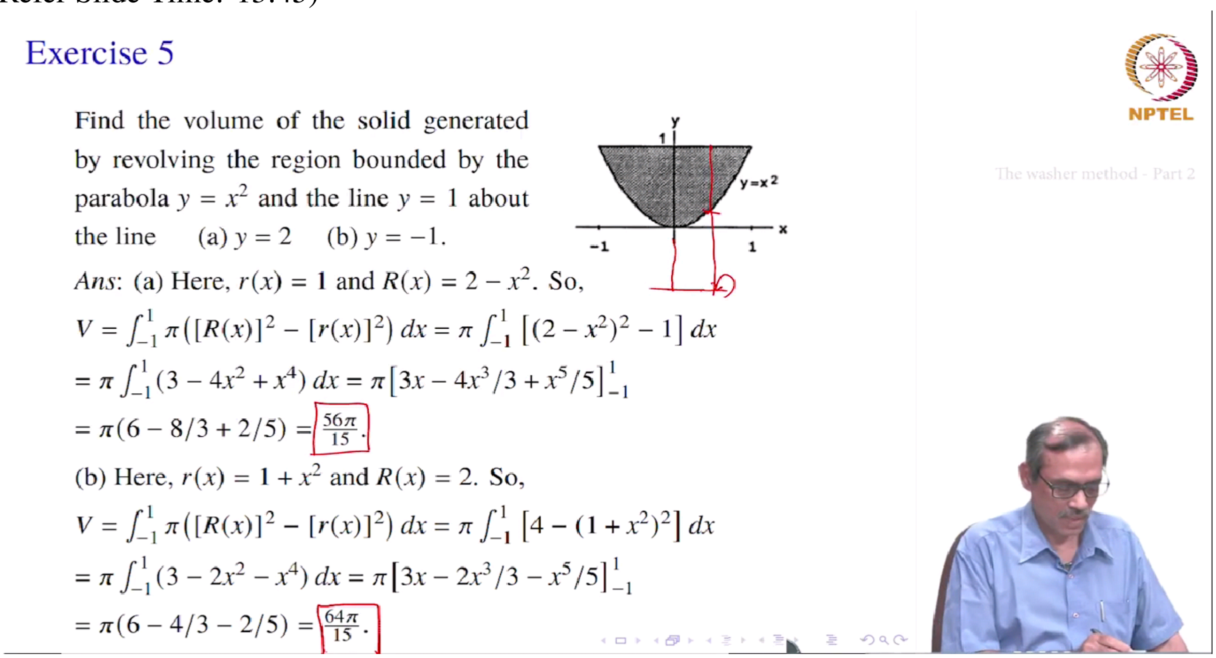

Let us take up Part (b). Here, the same region is revolved about the line $y = -1$. Again, with one particular x we have to find out the outer radius and the inner radius. Now, it need to be measured from $y = -1$. Again, it will be the total minus this distance, which gives 1 plus x^2 and 2; the total is 2. That is, $R(x) = 2$. For any x it is $y = x^2$ and here the length is 1. So it gives $r(x) = 1 + x^2$.

And then we plug these things into the formula. The volume is then equal to $\int_{-1}^{1} \pi[R^2(x) - r^2(x)]\, dx$. This gives $\int_{-1}^{1} \pi[4 - (1 + x^2)^2]\, dx$. And we simplify that to obtain $64\pi/15$ as the answer.