Basic Calculus - 1
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Lecture 33 - Part 1
The Washer Method - Part 1

This is lecture 33 of Basic Calculus - 1. As you remember, we were discussing about how to compute volumes of solids of revolution. Basically, we have discussed only one method, which is the slice method, and then its modification for a solid of revolution that we named as the disk method. In the disk method, our condition is that the axis of revolution should be bordering the region which is being revolved to get this solid. Suppose, the line which is the axis of revolution is not bordering the region, or even not crossing it, then what do we do? Today we will be discussing that, and that is called washer method.

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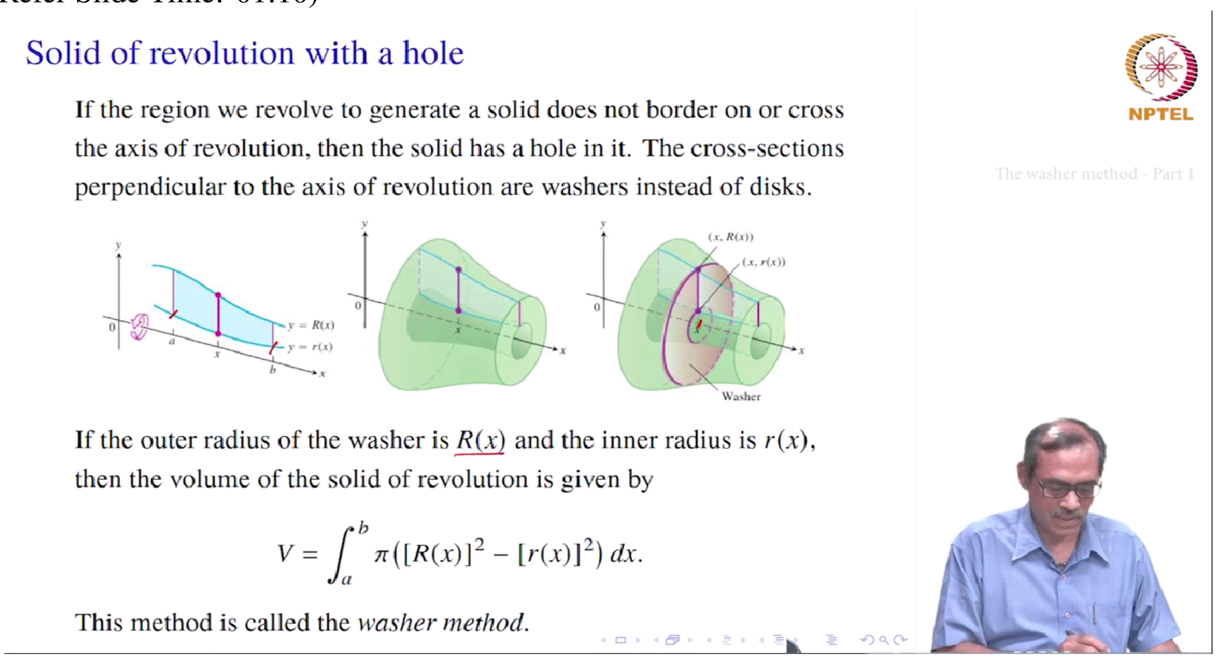

Let us take an example. We have this region which is painted blue in the first picture; this is being revolved about the x-axis as this symbol says. When it revolves, you will not get a solid without any holes, right? There will be a solid with holes as you see in the second picture. When you take a slice, that will not be a disk; it will be a punctured disk, an annulus. So we say that if the region we revolve to generate a solid does not border or does not cross the axis of revolution, then the solid has a hole in it, and the cross-sections perpendicular to the axis of revolution will be annular. So, there will be washer instead of a disk.

Then, you can compute the volume of the total along with the hole and subtract the volume of the hole from it. It amounts to computing the integral $\int_a^b A(x)\,dx$, where $A(x)$ is the area of the washer instead of the disk. That area $A(x)$ will be the area of the disk minus the area of the inside disk. Once you see that the cross-section is an annulus, you take the total first. The total area has the radius from the axis of revolution to that point. Let us denote it b $R(x)$. Similarly, let $r(x)$ be the radius of the inside disk. Then this area $A(x)$ will be π times $R^2(x) - r^2(x)$. That is really the area of this slice. Then it will be integrated to get the volume. That is what we do.

If the outer radius of the washer is $R(x)$, which is measured from the axis and the inner radius is $r(x)$, then the volume of the solid of revolution with that hole will be the integral $\int_a^b \pi[R^2(x) - r^2(x)]\,dx$. Here, a and b are the limits for x describing that region. You may have also a solid of revolution when the region is revolved around y-axis instead of the x-axis. In that case you have to express this region as a function of y, let us denote the area as $A(y)$. Then, the volume will be the integral of $A(y)$ with respect to y with suitable limits. We will come to it shortly.
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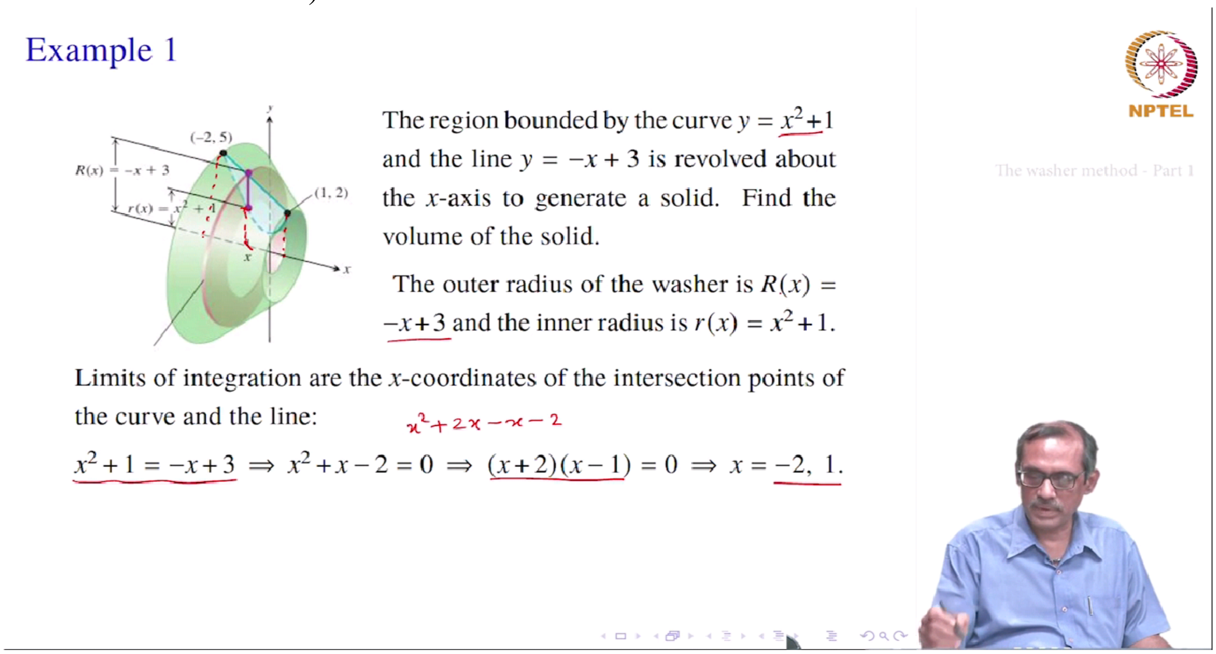

Let us apply this washer method on an example. Consider the region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$. The curve $y = x^2 + 1$ is a parabola and this is the line $y = -x + 3$. So, it is this region, the blue one. Notice that these two curves really make a region. That means, it is assumed that they intersect somewhere in at least two points; and then, the region is formed. This region is revolved about the x-axis. So, by revolving the region about the x-axis, you get the solid. Now you want to find the volume of this solid.

As you see, we must find out first the outer and inner radii $R(x)$ and $r(x)$. Then we should find the limits of the integration, which will come from the intersection points. Let us look around the region. The outer radius of the washer is $R(x) = -x + 3$, because this line is $y = -x + 3$. The inner radius is the one which is not given here. That will be measured from the x-axis to that point on the curve. That is a point on the curve $y = x^2 + 1$. So, $r(x) = x^2 + 1$.

Next, to find the limits of integration, we must find the region. The region is bounded by these two curves. That is, the x-coordinates of their points of intersection would give us the limits of integration. We have $y = x^2 + 1$ and $y = -x + 3$. Eliminating y, we have $x^2 + 1 = -x + 3$, which implies $x^2 + x - 2 = 0$. We can factorize it. That gives you $(x + 2)(x - 1) = 0$, or $x = -2,\ 1$. Thus, the points of intersection have x-coordinates as -2 and 1. This point which is $x = -2$ and this point which is equal to $x = 1$ give the points of intersection.

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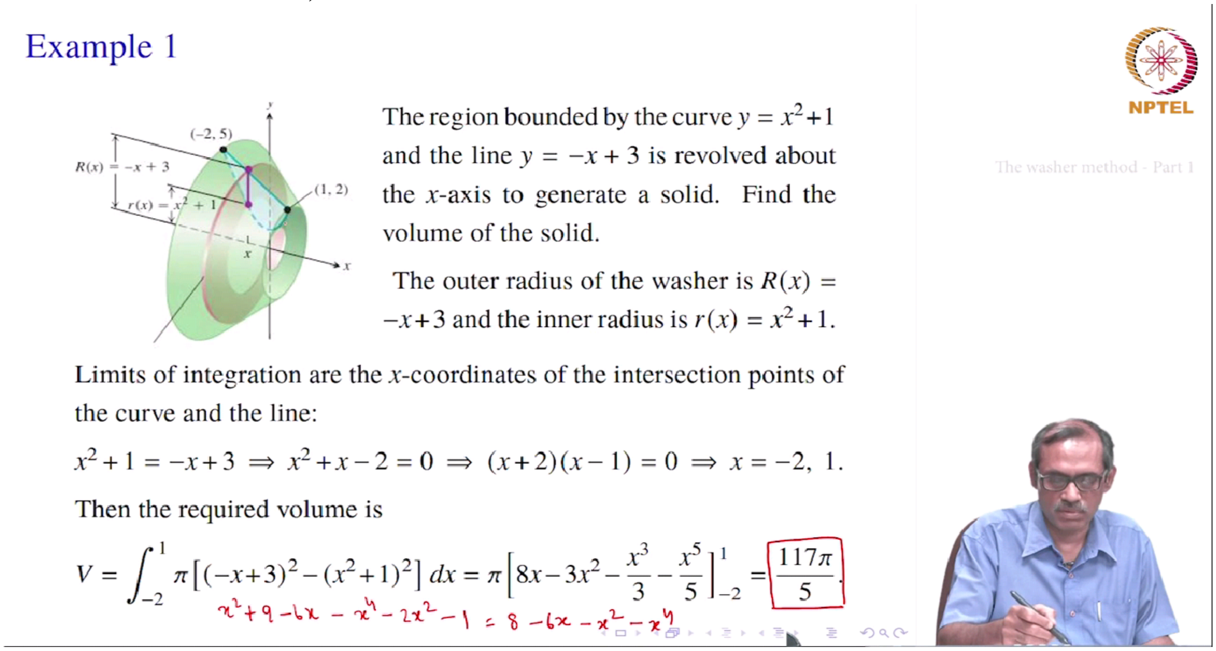

So we have got the limits of integration as -2 and 1. We have already got $R(x)$ and $r(x)$. Thus, the volume is the integral from -2 to 1 of the area of the washer. The area of the washer is $\pi(R^2(x) - r^2(x)) = \pi[(-x+3)^2 - (x^2+1)^2]$. You expand it and integrate. Let us see what is its expansion. It is $x^2 + 9 - 6x$ here, and $-x^2 - 2x^2 - 1$ here. That gives us $9 - 1 = 8$, then $-6x$, $-x^2$ and $-x^4$. When you integrate you get $8x$, $-6x$ gives $-3x^2$, $-x^2$ gives $-x^3/3$, and $-x^4$ gives $-x^5/5$. This multiplied by π and is evaluated at -2 and 1 and subtracted out. That simplifies to $117\pi/5$.

This is how we are going to solve problems applying the washer method. But this is applicable only when you will have a hole in the solid of revolution, and that happens when the axis of revolution is not bordering the region.

Let us take another example. Here, the region is bounded by the parabola $y = x^2$, which is this one, and the line $y = 2x$, which is this one, in the first quadrant. This region is revolved about the y-axis to generate a solid. The solid is painted green here. Find the volume of the solid.

What do we have to do? We should find out the outer radius, inner radius and the limits of integration. Since this is revolved about the y-axis, we should express the area of the region as a function of y. That is, we should find out the outer radius in terms of y, inner radius in terms of y and then the limits of integration should be limits for y.

As you see, we have the curve as $y = x^2$ in the first quadrant. We can write it as $x = \sqrt{y}$. Take any point x here; but it is revolved about y. So, we start with any point y on the y-axis. Corresponding to that, we have the outer radius as this full red line, and the inner radius as this curvy line. This shows that y varies between the line $y = 2x$ and the curve $y = x^2$. The outer radius is the perpendicular distance of a point on the line $y = x^2$ or $x = \sqrt{y}$, from the y-axis, which is equal to $\sqrt{y}$. The inner radius is the perpendicular distance of the same point on the line $y = 2x$ or

$x = y/2$ from the y-axis, which is $y/2$.
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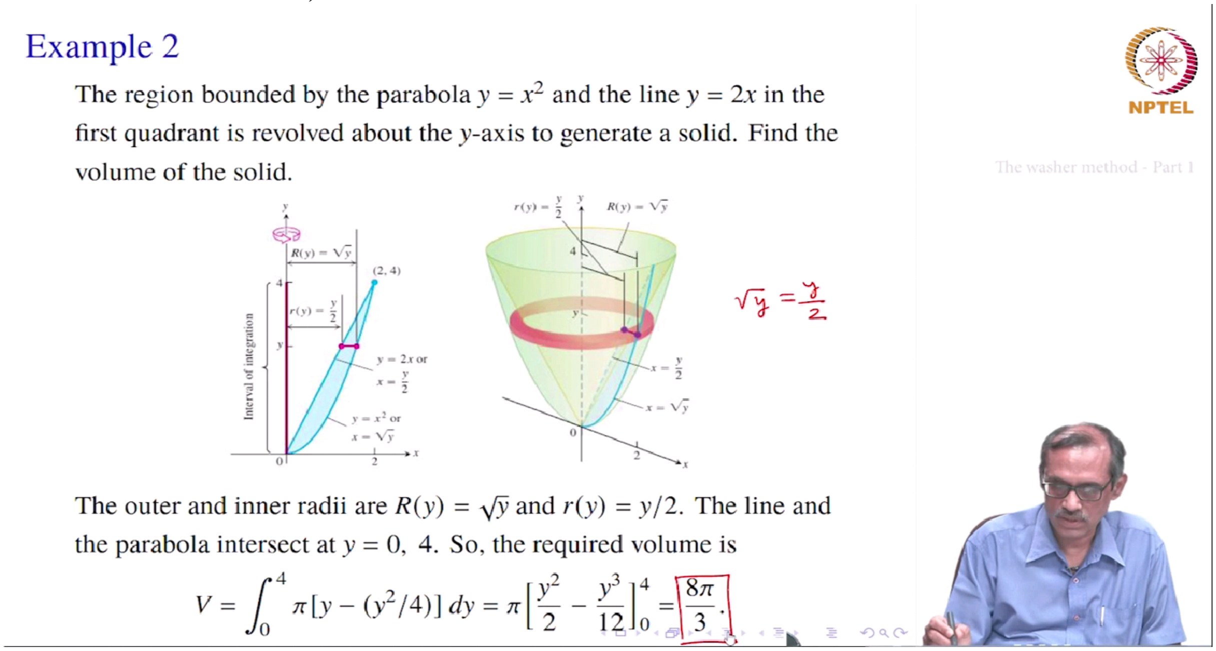

So, we have get $\sqrt{y}$ as the outer radius and $y/2$ as the inner radius. Then we should get the y-coordinates of the points of intersection. For this, we eliminate x from $x = \sqrt{y}$ and $x = y/2$. That gives us $y = 0$ or $y = 4$. So, we have the limits as 0 and 4. Then, the volume is the integral from 0 to 4 of π times square of the outer radius $\sqrt{y}$ minus square of the inner radius $y/2$. That is, $V = \int_0^4 \pi[y - (y/2)^2]\,dy$. It simplifies to $8\pi/3$. This is pretty straightforward.
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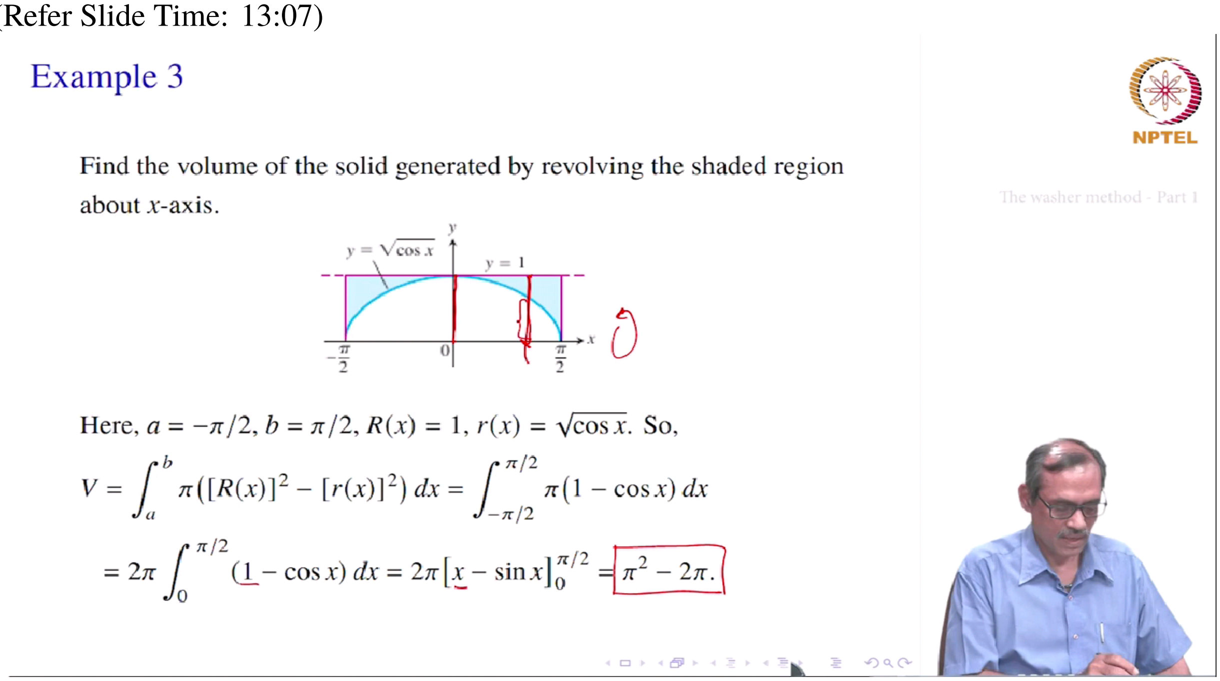

Let us take one more example. We want to find the volume of the solid generated by revolving the shaded region about the x-axis. It is rotated this way, about the x-axis. The shaded region is given via the functions $y = \sqrt{\cos x}$, which is this blue curve. The region lies between $y = \sqrt{\cos x}$

and $y = 1$. This is the shaded blue region. Since the revolution is about the x-axis, we need the limits for x. Those are obtained by solving these two equations for x. So, eliminating y from $y = \sqrt{\cos x}$ and $y = 1$, we get $\cos x = 1$. Hence, the limits for x are $-\pi/2$ to $\pi/2$. We confirm that these are the limits already given.

Then, it is straightforward to write the volume, where the limits will be $-\pi/2$ to $\pi/2$, We have to find the outer radius and the inner radius. Here, the outer radius is really 1, the whole thing. For the inner radius, you take any x where the outer radius is 1; then, the inner radius is this length, which is $\sqrt{\cos x}$. Therefore, the volume is $\int_{-\pi/2}^{\pi/2}[1 - (\sqrt{\cos x})^2]\, dx = \int_{-\pi/2}^{\pi/2}(1 - \cos x)\, dx$.

And now, we integrate. Since it is an even function, instead of $-\pi/2$ to $\pi/2$, we can write the integral as 2 times from 0 to $\pi/2$. Integration of 1 is x, and of $\cos x$ is $\sin x$. So, it is twice $x - \sin x$ evaluated at 0 and $\pi/2$, and subtracted out. That simplifies to $\pi^2 - 2\pi$.

Once the things are given this way instead of a story, it becomes easier, because the limits of integration are known. But this could have been reformulated some other way and you would then have to find out what exactly the limits of integration and what these $R(x)$ and $r(x)$ are.

Let us take a problem. We want to find the volume of the solid generated by revolving the region bounded by the line $y = 2 - x$ and the curve $y = 4 - x^2$ about the x-axis.
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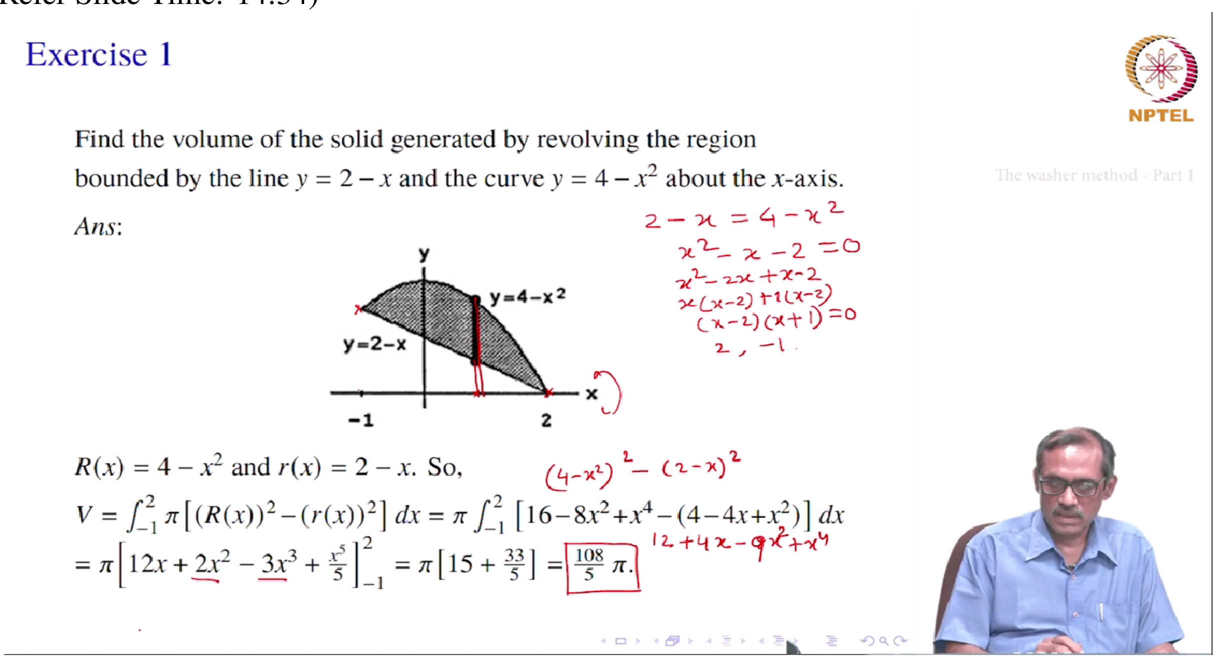

Notice that the region is bounded by $y = 2 - x$ and $y = 4 - x^2$. The limits of integration will come from the intersection of those two curves. The revolution is about the x-axis. So, for the point of intersection, we eliminate y so that we obtain the limits for x. Now, eliminating y from $y = 2 - x$ and $y = 4 - x^2$, we get $x^2 - x - 2 = 0$. Which we rewrite as $x^2 - 2x + x - 2 = 0$ and factorize to obtain $(x - 2)(x + 1) = 0$. That gives us the points as $x = 2$ and $x = -1$.

See the picture. The region is drawn here, where the first point corresponds to $x = -1$ and the second one here on the x-axis corresponds to $x = 2$. Once $x = 2$, of course, y is 0. That is why the point is on the x-axis. Now this region is revolved about the x-axis. Then, the outer radius at any point x will be the one which comes from $y = 4 - x^2$. The inner radius comes from the line $y = 2 - x$.

Then, the required volume will be the integral $\int_{-1}^{2} \pi[R^2(x) - r^2(x)]\,dx$, which is equal to $\int_{-1}^{2} \pi[(4-x^2)^2 - (2-x)^2]\,dx$. We write the integrand as this as $16 - 8x^2 + x^4 - (4 - 4x + x^2)$. This simplifies to $12 - 4x - 9x^2 + x4$. After integration, you get $12x$ from 12, $2x^2$ from $4x$, $3x^3$ from $9x^2$, and $x^5/5$ from x^4. We obtain the integral as $12x + 2x^2 - 3x^3 + x^5$. This is to be evaluated at -1 and 2, then subtracted. That simplifies to $108\pi/5$.

Sometimes it is crucial to see how the picture looks like so that the hole can be subtracted out. It also shows the points of intersection, which of course, can be obtained algebraically. But the plotting of that gives a better feeling and minimizes mistakes.

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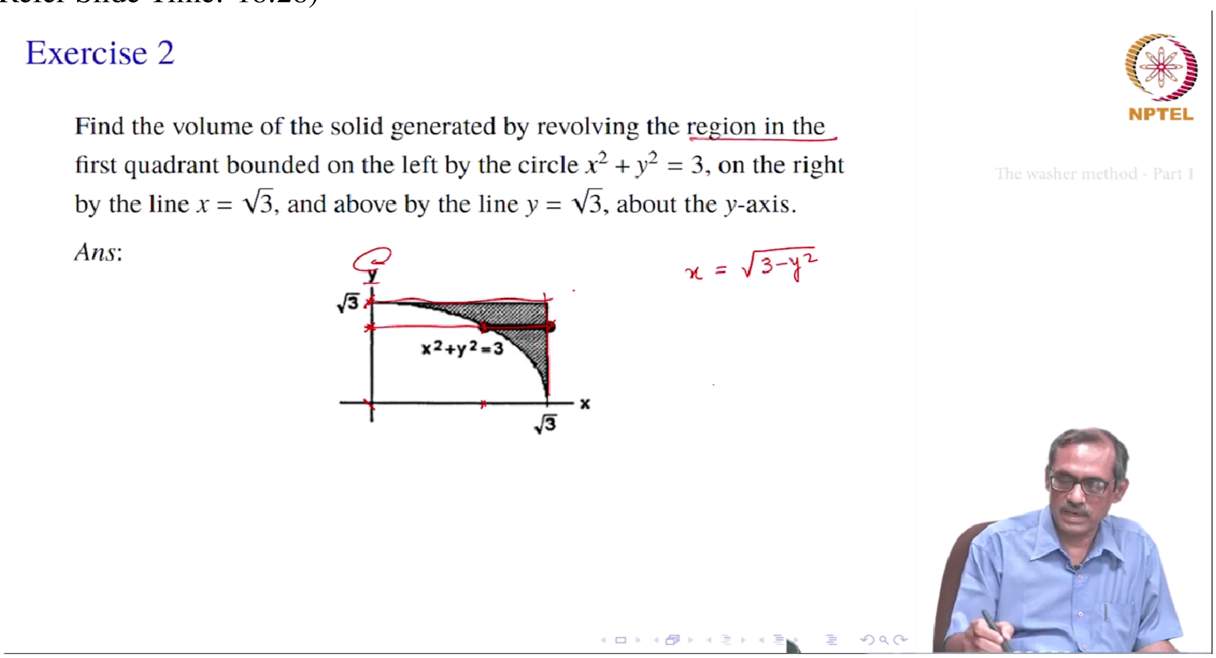

Let us take the next problem. Here, we want to find the volume of the solid generated by revolving about the y-axis the region in the first quadrant, which is bounded on the left by the circle $x^2 + y^2 = 3$, on the right by the line $x = \sqrt{3}$, and above by the line $y = \sqrt{3}$.

First, we have to see what is the region, then think about revolving it about the y-axis. We will need getting the limits for y from the intersections and then find $R(x)$, $r(x)$, and plug it in the formula. To plot the region in the first quadrant, we draw the circle $x^2 + y^2 = 3$. It is not up to scale here; y looks a bit more shortened, it should have been here. It is this portion of the circle $x^2 + y^2 = 3$ that lies in the first quadrant. You get the points on the x-axis as $\sqrt{3}$, where y is 0. And on the y-axis, where x is 0, you get y as $\sqrt{3}$ also. These are the points of intersection of the circle with the axes.

This region is bounded on the left by this circle and on the right by the line $x = \sqrt{3}$. It is this line. The region is bounded above by the line $y = \sqrt{3}$, which is this line. So, this is the region; and it is to be revolved about the y-axis. It would go this way, since it is revolved about the y-axis. If you take any point on the y-axis, we need to find the corresponding outer radius and the inner radius. It is the outer and inner radius of the washer. For the washer now, you have the outer radius as the whole thing, which is equal to y. It would go up to this, and that is $\sqrt{3}$. And the inner radius will be up to this point. Since that is on the circle $x^2 + y^2 = 3$, you get x equal to that length on the x-axis, which is $x = \sqrt{3 - y^2}$. And what are the limits? The limits for y are clearly 0 and then $\sqrt{3}$. That describes the region.

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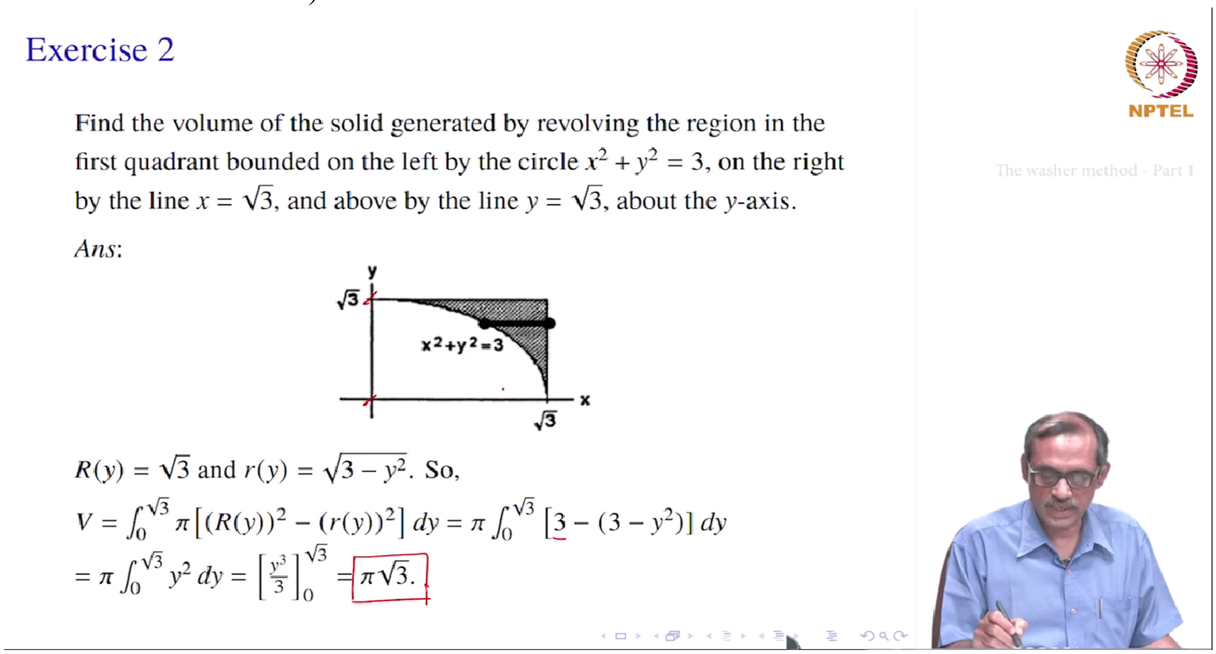

So, we have $R(y) = \sqrt{3}$ and $r(y) = \sqrt{3 - y^2}$. Then, the volume is equal to $\int_0^3 \pi[R^2(y) - r^2(y)]\, dy = \int_0^3 \pi[3 - (3 - y^2)]\, dy$. That gives $\pi \int_0^3 y^2\, dy$. This gives π times $y^3/3$ evaluated at $\sqrt{3}$ and 0, and subtracted. The answer is $\pi\sqrt{3}$.

Initially when you see the story or the statement of the problem, it was not obvious how to go about it. But once you plot it, the things become easier.