Basic Calculus - 1
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Lecture 32 - Part 2
The Disk Method - Part 2

Let us take the next example. We have a region enclosed by y-axis, a curve $x = f(y)$ and the lines $y = a$ and $y = b$. We are taking the curve as $x = f(y)$ because we are looking at the region from the y-axis. Now, such a region is revolved around y-axis. It falls into place because x and y are interchanged; and that is the only trick here.

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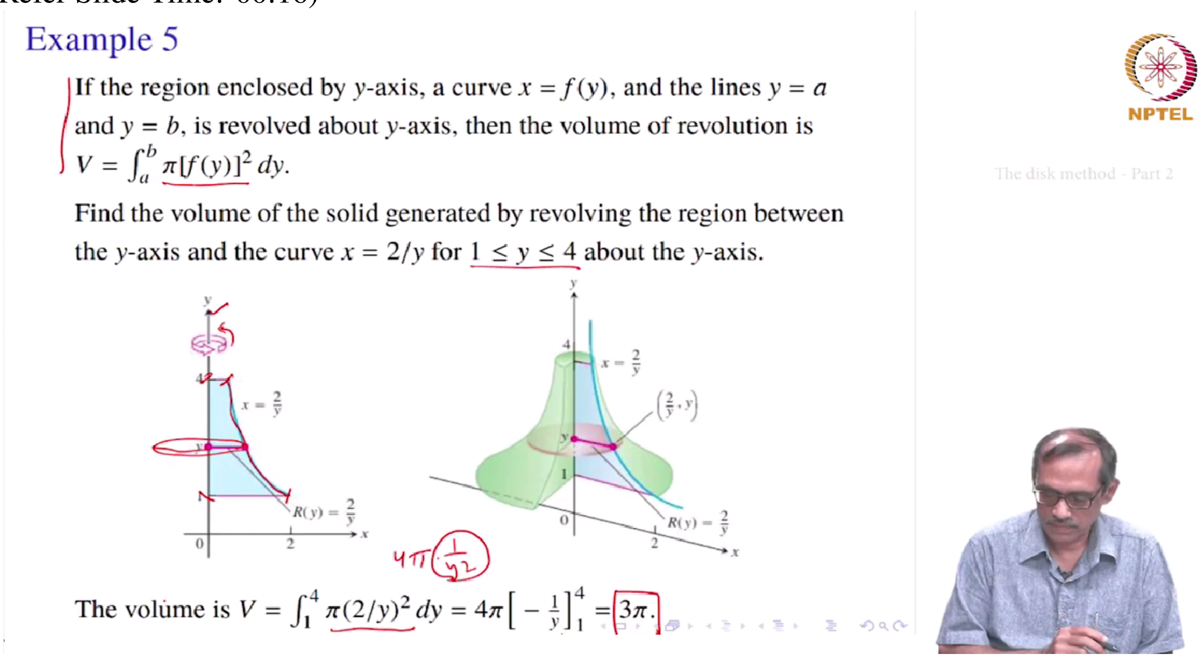

Then, the volume of revolution has to be the integral with limits in y of the area of the disk, which is $\pi[f(y)]^2$. That is clear from our earlier analysis. Instead of x, we are taking y here. By using this, we are going to solve the problem where the region is revolving around the y-axis instead of the x-axis.

The problem is to find the volume of the solid generated by revolving the region around the y-axis. This is the y-axis and the curve $x = f(y) = 2/y$. The relevant portion of the curve that is on one side of the revolving region is $x = 2/y$ for $1 \le y \le 4$. Then, you get the solid which is painted green here in the second picture. We want to find its volume. We go back to our formula. Since this is generating a disk here, that slice becomes a disk due to the revolution around y-axis. So, the volume is $\int_1^4 \pi(2/y)^2\,dy$. It is $4\pi\int_1^4 y^{-2}\,dy$. Since the integral of y^{-2} is $-1/y$, we have $-4\pi/y$ to be evaluated at 4 and 1, and then subtracted. If you simplify this, you would get the answer as 3π.

Now you see, we can also solve some problems where a region revolves around the y-axis.

Let us see one more problem. Here, we want to find the volume of the solid generated by revolving about y-axis, the region enclosed by the curve $y = (4/\pi)\tan^{-1}x$. The curve is given in terms as $y = f(x)$, but the region revolves about y-axis. And what is the region? The region is bounded by $y = (4/\pi)\tan^{-1} x$, the y-axis and the line $y = 1$. Everything is given in terms of y but the the curve is given $y = f(x)$.

That should not be a problem. We can write the curve as $x = \tan(\pi y/4)$. Now, everything is expressed in terms of y and the curve as x is a function of y. One of the limits for y is $y = 1$; and what is the other point? That will be the intersection of the curve and the y-axis, which turns out to be 0. So, we have found that this is the region. This region is revolved around y-axis. So you would get something like a cone; it is not exactly cone. The curve is $x = \tan(\pi 4/4)$. So, the volume should be equal to $\int_0^1 \pi \tan(\pi y/4)\, dy$.

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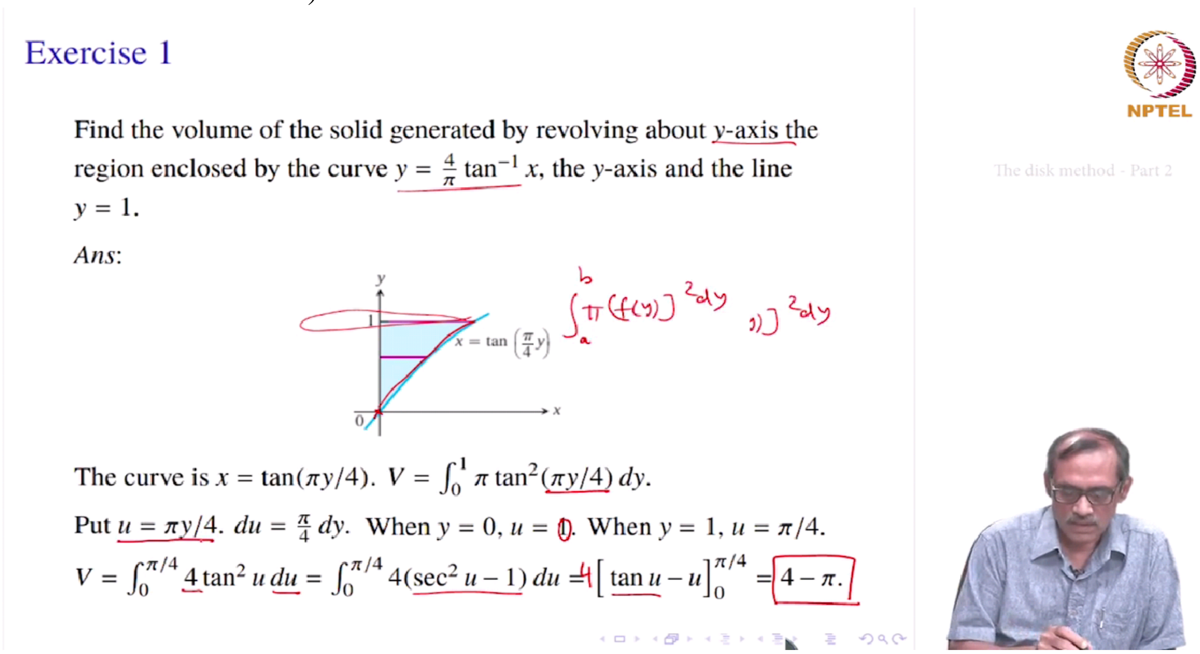

We have to evaluate this integral. We substitute$u = \pi y/4$. When $y = 0$, we get $u = 0$, and when y equal to 1, we get $u = \pi/4$. So the volume will be equal to the integral $\int_0^4 \pi \tan^2(\pi y/4)\, dy$. In terms of u, it is $\int_0^{\pi/4} 4\tan^2 u\, du$. As you see, dy becomes $4du$, and π was already there. It is all right. We just rewrite it, replacing $\tan^2$ with $\sec^2 - 1$, because we know the integral of $\sec^2$. The required volume is $\int_0^{\pi/4} 4(\sec^2 u - 1)\, du$. On integration, $\sec^u$ gives $\tan u$ and 1 gives u. So, it is $4(\tan u - u)$. This is to be evaluated at $\pi/4$ and 0, and then subtracted. It simplifies to $4 - \pi$.

Sometimes the curve might be given as $y = f(x)$, whereas you may need it in the form $x = f(y)$.

Let us take the next problem. Find the volume of the solid generated by revolving about the y-axis, the region bounded by the lines $x = 0$, $y = 0$, $y = 1$, and the curve $x = \sqrt{2y}/(y^2 + 1)$. Here, the limits of y are not given explicitly here, we may have to find out. But one of the lines is $x = 0$, which is the y-axis, and the region revolves around the y-axis. So that falls into place. We need the other limit of integration.

In the plot, it looks something like this. The curve is $x = \sqrt{2y}/(y^2 + 1)$ and that revolves about the y-axis. The region is bounded by the line $x = 0$, which is the y-axis, and the line $y = 1$, which is the top line here, and the curve.

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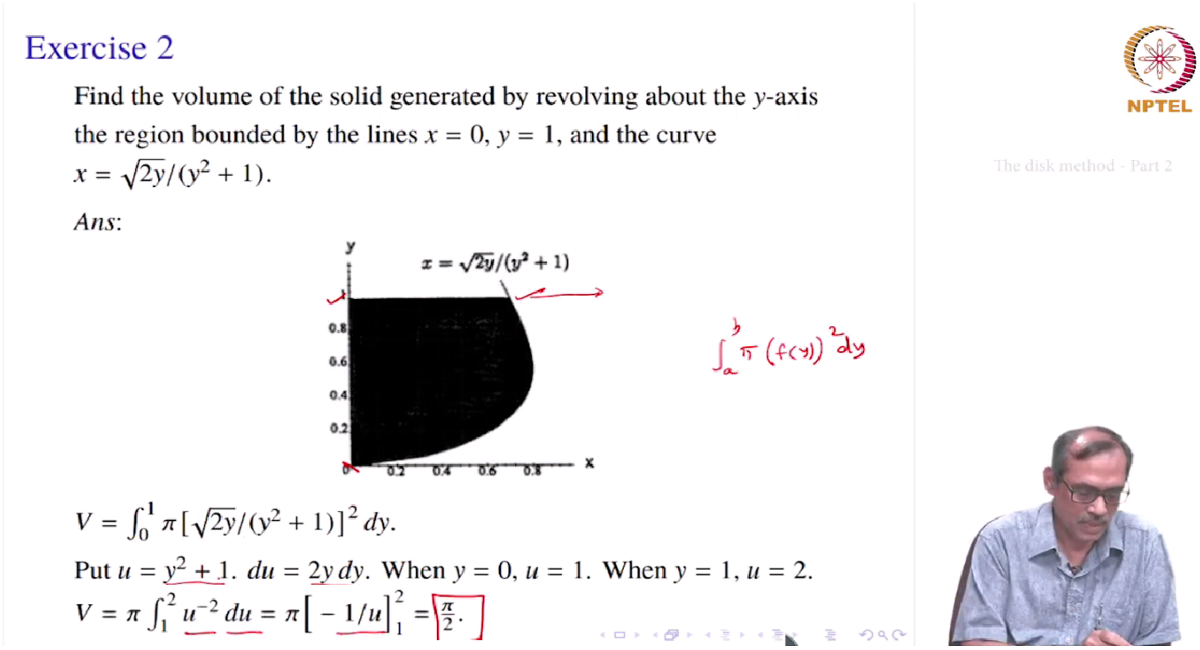

That gives the region nicely; and this region is revolved around the y-axis. Then, the volume will be the integral from a to b of $\pi[f(y)]^2$. What are the limits? On the top we have already 1 and the down we must see the intersection of this curve with the y-axis. The point of intersection is the origin. So, you would get $y = 0$ for the relevant limit for y. Therefore, the lower limit for y is 0 and the upper limit is 1. Then the volume is expressed as $\int_0^1 \pi[\sqrt{2y}/(y^2{+}1)]^2\, dy = \int_0^1 \pi[(2y)/(y^2{+}1)^2]\, dy$.

Now, it is a matter of computing the integral. How do we compute the integral? The term $y^2 + 1$ in the denominator which is creating the problem. We substitute $u = y^2 + 1$. Then, you get $du = 2y\, dy$. The limits for y are as follows: when $y = 0$, $u = 1$; and when $y = 1$, $u = 2$; that is right. We rewrite the integral now. It is $\int_1^2 \pi u^{-2}\, du$. Now, this can be integrated; u^{-2} gives $-u^{-1}$. We get the integral as $-\pi/u$, which is to be evaluated at 2 1nd 1, and then subtracted. That simplifies to $\pi/2$. So, this is how we will be proceeding.

Let us take one more problem. We want to find the volume of the solid generated by revolving a region about the line $y = \sqrt{2}$. It is not any axis; it is the line $y = \sqrt{2}$. And what about the region? The region is in the first quadrant bounded above by the line $y = \sqrt{2}$, the same line. So it is really touching that line; the axis of revolution is really one of the boundaries. And, the region is bounded below by the curve $y = \sec x \tan x$, and on the left by the y-axis.

So, how does it look? The region is in the first quadrant. It is bounded above by the line $y = \sqrt{2}$, below by the curve $y = \sec x \tan x$; and on the left by the y-axis. So, this is the region. This region is revolved about the line $y = \sqrt{2}$, not about the y-axis, the revolution is about this line. So, it may look something like this, and it is revolved around this, so the cross-sectional areas will be looking like this.

We want to find the volume of that solid. We have to take care of the fact that the axis of revolution is the line $y = \sqrt{2}$. Now, what is this cross sectional area? It is a disk with some radius. If this point is x, then what is the radius of this disk?

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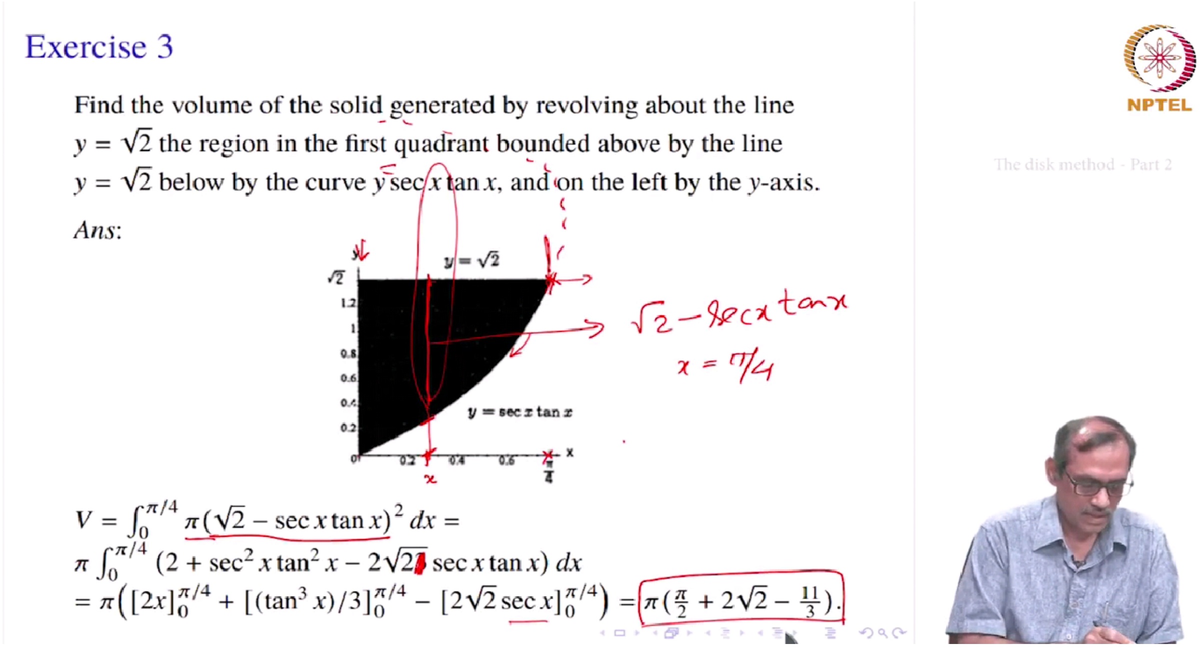

The radius of the disk is equal to $\sqrt{2}$ minus this side, which is $y = \sec x \tan x$. That is, the radius of the cross-sectional disk is $\sqrt{2} - \sec x \tan x$ at the point x. Then, the area of the cross-sectional disk is $\pi(\sqrt{2} - \sec x \tan x)^2$. Therefore, the volume of the solid is equal to $\int_0^{\pi/4} \pi(\sqrt{2} - \sec x \tan x)^2\, dx$. Why is the integral taken from 0 to $\pi/4$? Because the point of intersection of the line $y = \sqrt{2}$ and the curve $y = \sec x \tan x$ has the x-coordinate as $x = \pi/4$. You can verify easily as $\sec(\pi/4)\tan(\pi/4) = \sqrt{2}$. Once you get that point, you know the limits for x, which are 0 and $\pi/4$.

So, the volume of the solid is $\int_0^{\pi/4} \pi(\sqrt{2} - \sec x \tan x)^2\, dx$. We want to evaluate this integral. Here, π goes out, the limits 0 to $\pi/4$ remain, and we expand $(\sqrt{2} - \sec x \tan x)^2$. That gives $2 + \sec^2 x \tan^2 x - 2\sqrt{2} \sec x \tan x$. The integration of 2 is $2x$, of $\sec^2 x \tan^2 x$ is $(1/3)\tan^3 x$ and of $\sec x \tan x$ is $\sec x$. How does integration of $\sec^2 x \tan^2 x$ give $(1/3)\tan^3 x$? You take $\tan x = u$. Then, $du = \sec^2 x\, dx$ so that $\int \sec^2 x \tan^2 x\, x = \int u^2\, du = u^3/3$. Of course, if you differentiate $(1/3)\tan^3 x$, you get back $\sec^2 x \tan^2 x$ directly. And then, $\int \sec^2 x \tan^2 x\, dx = (1/3)\tan^3 x$.

Taking care of the constants that are multiplied there, we have the integral as $\pi[2x + (\tan^3 x)/3 - 2\sqrt{2}\sec x]$. This is to be evaluated at $\pi/4$ and 0, and the subtracted. That simplifies to $\pi(\pi/2 + 2\sqrt{2} - 11/3)$.

It is pretty straightforward. Though it is called the disk method, you may always think of the slice method. Remember that since the cross-sectional area becomes a disk, we call it the disk method.

Let us look at the next problem. Here, we want to find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$. This is the parabola $x = y^2 + 1$, and here is the line $x = 3$. The parabola and the line enclose the region pained blue. This region is revolved about the line $x = 3$.

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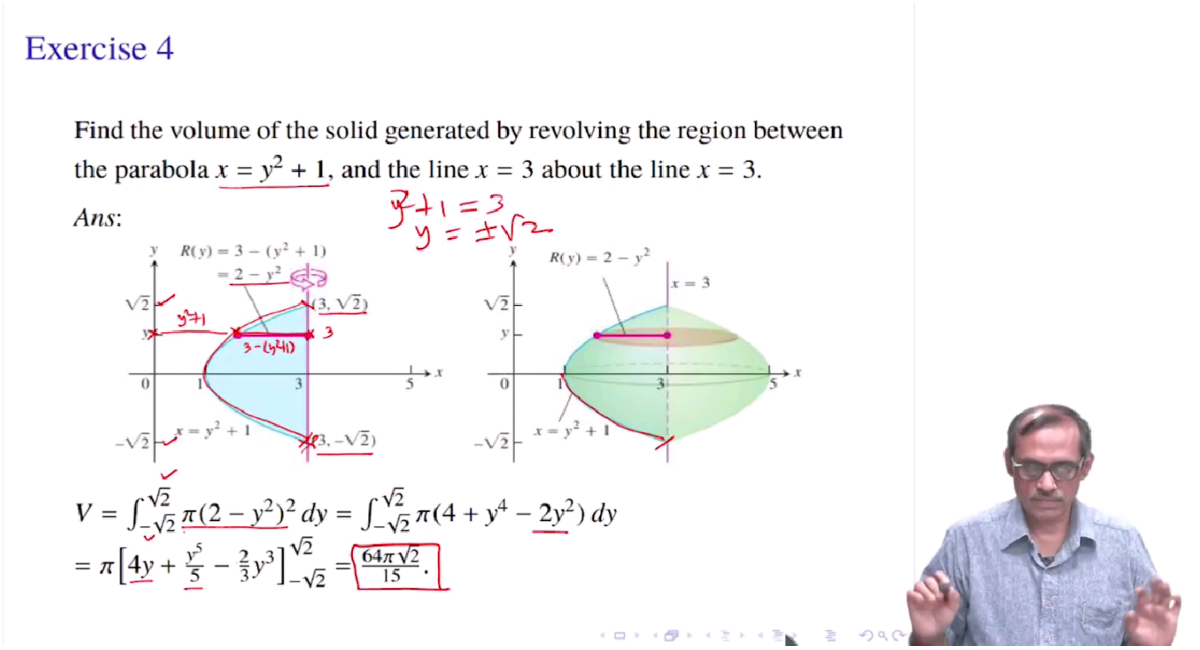

We should get the radius of the cross-sectional disk. This radius is painted pink corresponding to any point y. Since the curve is $x = y^2 + 1$, the radius is the height of this point (x, y) on the curve measured from the line $x = 3$. At any y, this length is equal to $y^2 + 1$; this length is 3; so the radius is $3 - (y^2 + 1) = 2 - y^2$. Thus, the cross-sectional disk has the area $\pi(2 - y^2)^2$.

Now, what are the limits of integration? The two points which give the limits of integration lie on the line $x = 3$. So, we find out where the line $x = 3$ intersects the curve $x = y^2 + 1$. You get $Y^2 + 1 = 3$, which implies $y = \pm\sqrt{2}$. These two values of y correspond to the intersection points. Of course, the corresponding x values will,be equal to 3. So, the intersection points are $(3, -\sqrt{2})$ and $(3, \sqrt{2})$. We are integrating with respect to y. So, we just take the limits as $-\sqrt{2}$ to $\sqrt{2}$.

The volume is equal to $\int_{-\sqrt{2}}^{\sqrt{2}} \pi(2 - y^2)^2\, dy$. We expand the integrand to get $\pi(2 - y^2)^2 = \pi(4 + y^4 - 2y^2)$. The integration of 4 is $4y$, of y^4 is $y^5/5$, and of $2y^2$ is $(2/3)y^3$. So, we get the integral as $\pi(4y + y^5/5 - 2y^3/3)$. This is to be evaluated at $\sqrt{2}$ and $-\sqrt{2}$, and then subtracted. If you simplify you would obtain $64\pi\sqrt{2}/15$ as the answer. Let us stop here.