Basic Calculus - 1
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Lecture 32 - Part 1
The Disk Method - Part 1

This is lecture 32 of Basic Calculus - 1. In the last lecture, we have discussed about computing volumes of certain kinds of solids by using the slice method. Today, we will be talking a particular case of that slice method, which is called the disk method. This is applicable to a subclass of those solids, which we have discussed last time. Specifically, we will be using the solids which are generated by revolving certain area around a straight line. They are called solids of revolution. The slice method for such solids is called the disk method.

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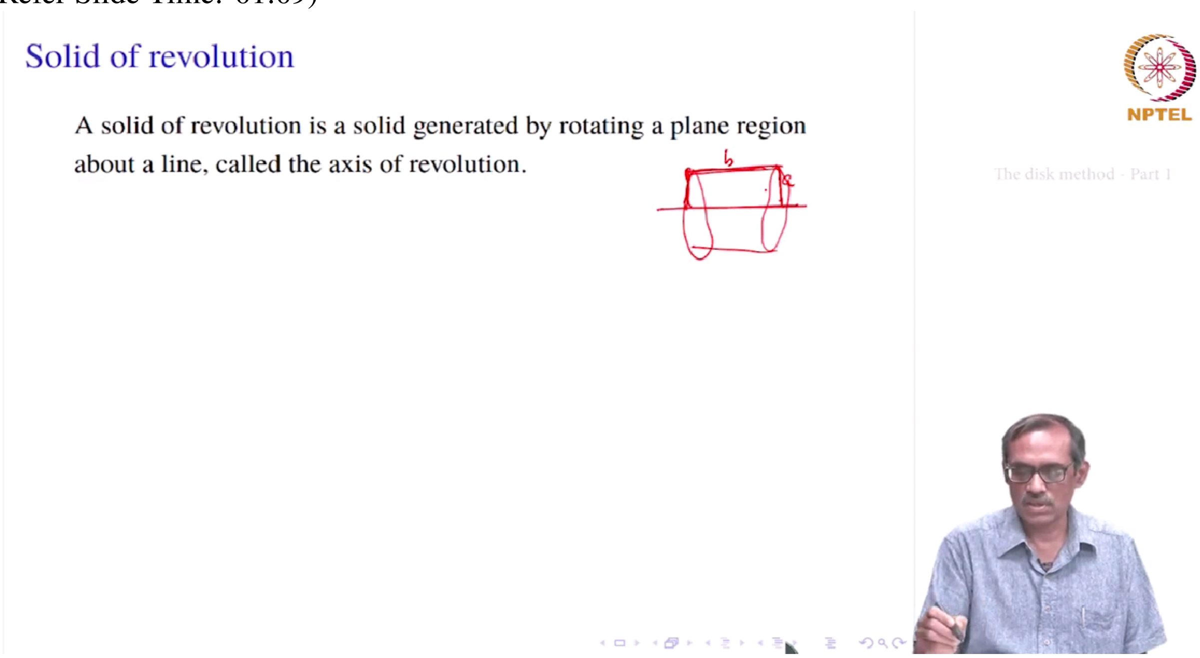

First, what a solid of revolution? A solid of revolution is a solid, which is generated by rotating a planar region about a line. That line is called the axis of revolution. For example, you can take a line and raise one rectangle on it. Suppose you take this as the axis of revolution, and this rectangle having sides as a and b. When you revolve it around this straight line, you would get one cylinder. That is how you will be having a cylinder of height b; and the circular base is having radius as a. This is an example of a solid of revolution.

In general, it need not be this rectangle; it can be some planar region and the straight line (axis) may not be intersecting or even touching that plane region; it can be somewhere else. We will see how do you get a solid of revolution if you revolve the area around this line. And then, we want to compute the volume of such a solid.

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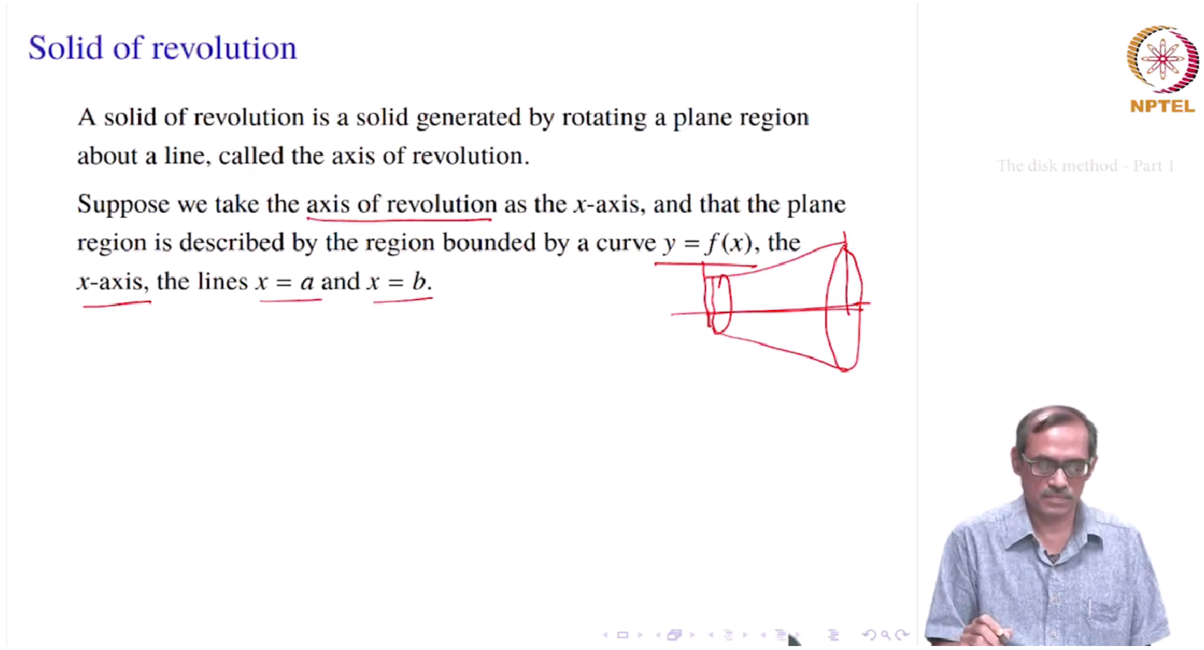

You have a solid, which is obtained by revolving a certain plane region around a certain line. Then, how do we go about computing its volume? That straight line is called the axis of revolution. To make it convenient, let us take that axis of revolution (that straight line) as our x-axis in the plane. And suppose the plane region can be described now with this axis of revolution and another line perpendicular to this; we call that as y-axis in the plane. Suppose this is a region bounded by a curve $y = f(x)$, the x-axis, the lines $x = a$ and $x = b$.

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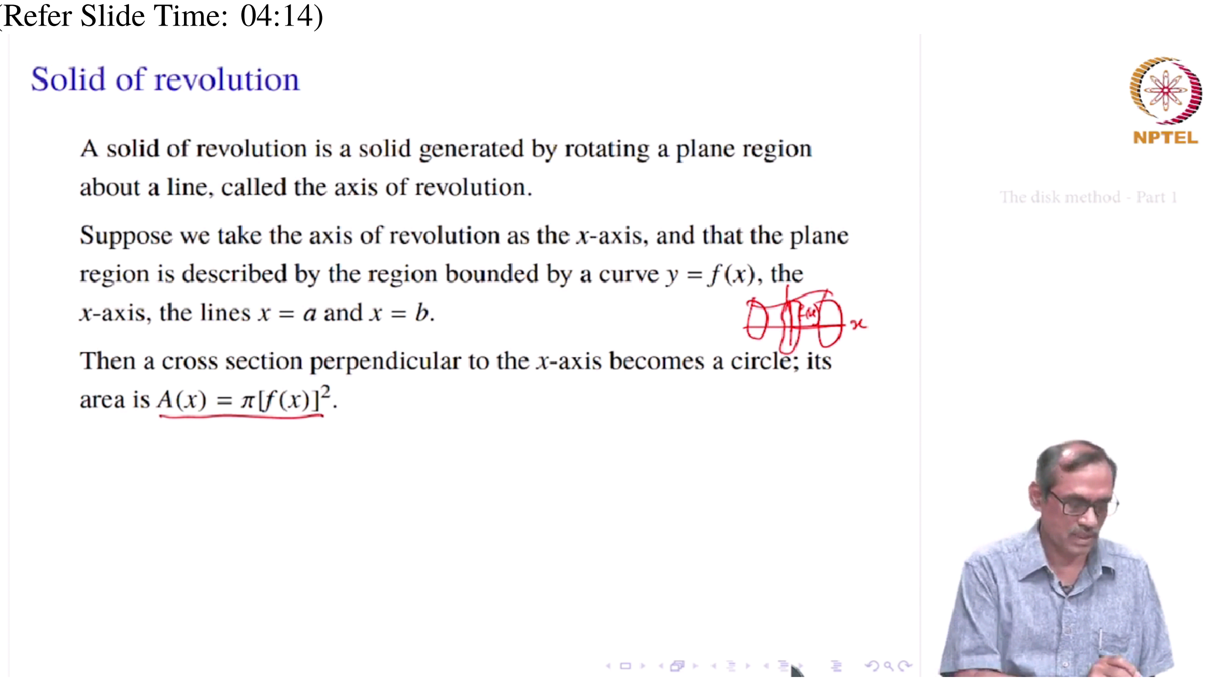

We are taking a very particular case. We will come to a slightly general case later. It is something like the axis of revolution and you have a curve $y = f(x)$, you have a line $x = a$ and you have a line $x = b$. You are revolving it around this line. So, you would get some solid of this kind.

This is how the solid of revolution are formed. And we are taking a particular case where the line itself, the axis of revolution consists of this line, and the area is bounded by these two lines $x = a$ and $x = b$.

For this kind of solids, what do we do? First, we take a cross-section perpendicular to the x-axis; for we want to use the slice method. Once you have some solid of revolution here, we take its slice here. That means, we take any cross-section perpendicular to the x-axis. That is our x-axis now. At any point c you take the area of this cross-section. Since it is x, this one is equal to y or $f(x)$; it is that height or the radius of that circle which is generated here. It will have a radius as $f(x)$. Therefore, its area will be π times $[f(x)]^2$. So the cross-sectional area $A(x) = \pi[f(x)]^2$. Therefore, the volume of the solid by using the slice method is equal to $\int_a^b \pi[f(x)]^2\, dx$.

We call it the disk method, because the slice here is a disk. And, this will be our general formula in the disk method. Under all these conditions only we write this formula. The main condition is that $\pi[f(x)]^2$ is an integrable function. Usually, we assume that $f(x)$ is continuous.

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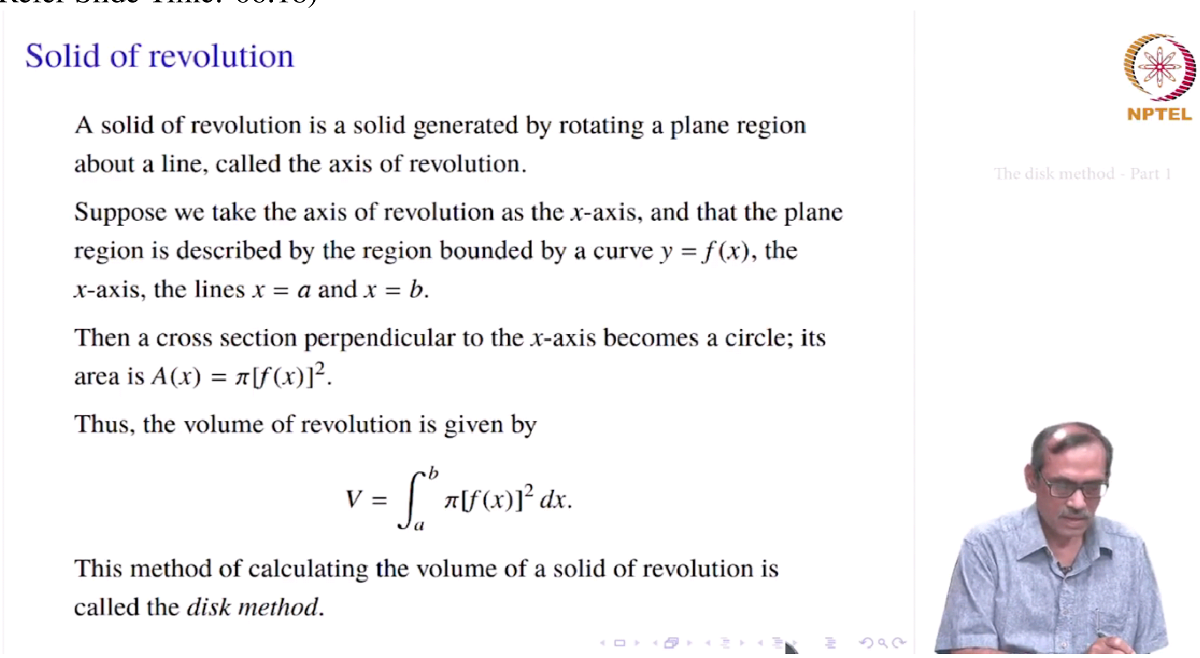

What kind of solid we are generating for which it is the volume? We must remember that, it is the region bounded by $y = f(x)$, the x-axis, which is the axis of revolution, the line $x = a$, and the line $x = b$. This region is revolved around the x-axis to get the solid. The limits of integration are obtained from these two lines $x = a$ and $x = b$. The area of the cross-section is $\pi[f(x)]^2$ so that the volume will be the integral as given above. Since the slice or the cross-sectional area becomes a disk, the slice method for these solids is called the disk method.

Let us see an example. The region has x-axis on the bottom, the curve $y = \sqrt{x}$ on the top, the line $x = 0$ on the left, and the line $x = 4$ on the right. This region is revolved around x-axis. It is same as revolving the curve around x-axis, and then consider all points inside by taking $0 \le x \le 4$. We want to find the volume of this solid of revolution. This is how it looks. The curve $y = \sqrt{x}$ is painted blue here, and then x lies between 0 and 4. You get some solid here. That solid is obtained by revolving this curve; in fact, revolving this region about x-axis.

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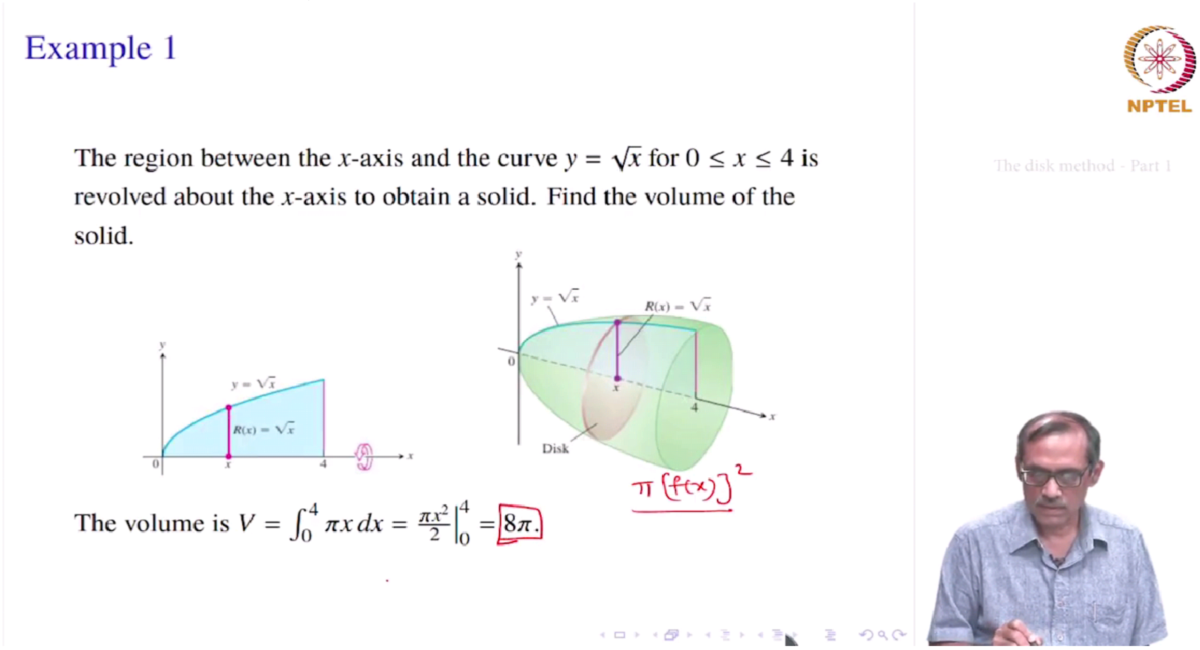

According to our formula, the volume should be equal to the integral from 0 to 4 of $\pi[f(x)]^2$. Now, $f(x) = \sqrt{x}$; its square is x, then you integrate it from 0 to 4. It is $\int_0^4 \pi x \, dx = \pi x^2/2\big|_0^4$. That gives π. The volume of this solid is equal to 8π.

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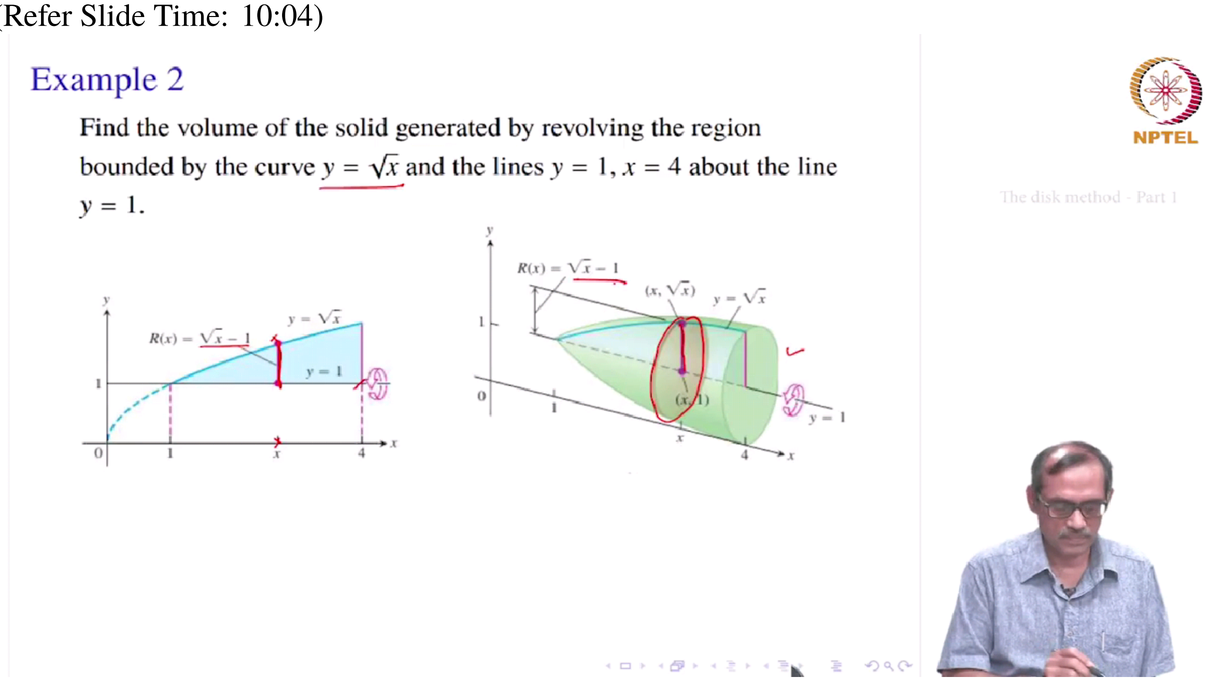

We take one more example. Sometimes, through the examples we will introduce other kinds of regions, which when revolved give rise to different kinds of solids. But all of them will be called solids of revolution.

Here, we want to find the volume of the solid generated by revolving the region bounded by the curve $y = \sqrt{x}$ like the earlier one. But now, the line is not starting at the x-axis. The region is now bounded by this curve, the lines $y = 1$ and $x = 4$. This region is revolved around the line $y = 1$ instead of around $x = 0$. Now, we are revolving the region around the line $y = 1$, and the region is bounded by this curve, the lines $y = 1$ and $x = 4$.

So how the figure would look like? We have $y = \sqrt{x}$, which is the blue one here. This is the line $y = 1$ around which the region is revolved. What is the region? It is bounded by these three; so it is this region painted blue. Then, you would get this solid, which is painted green here, in the second picture. We want to find the volume of this solid so generated.

The first thing is: we have to find the area of this disk; and that should be coming from our formula. It is $\pi[(f(x)]^2$. But it is not just $f(x)$ here because the line is now $y = 1$ not the x-axis. You have to get the radius correctly. What is the radius of the generated disk? It is this line here which is painted pink. Since this is $y = 1$ and this point is $y = \sqrt{x}$, the height is now $\sqrt{x} - 1$. That is really the radius of that disk. Since the radius of this disk is $\sqrt{x} - 1$, our $A(x)$ will be $\pi[\sqrt{x} - 1]^2$.
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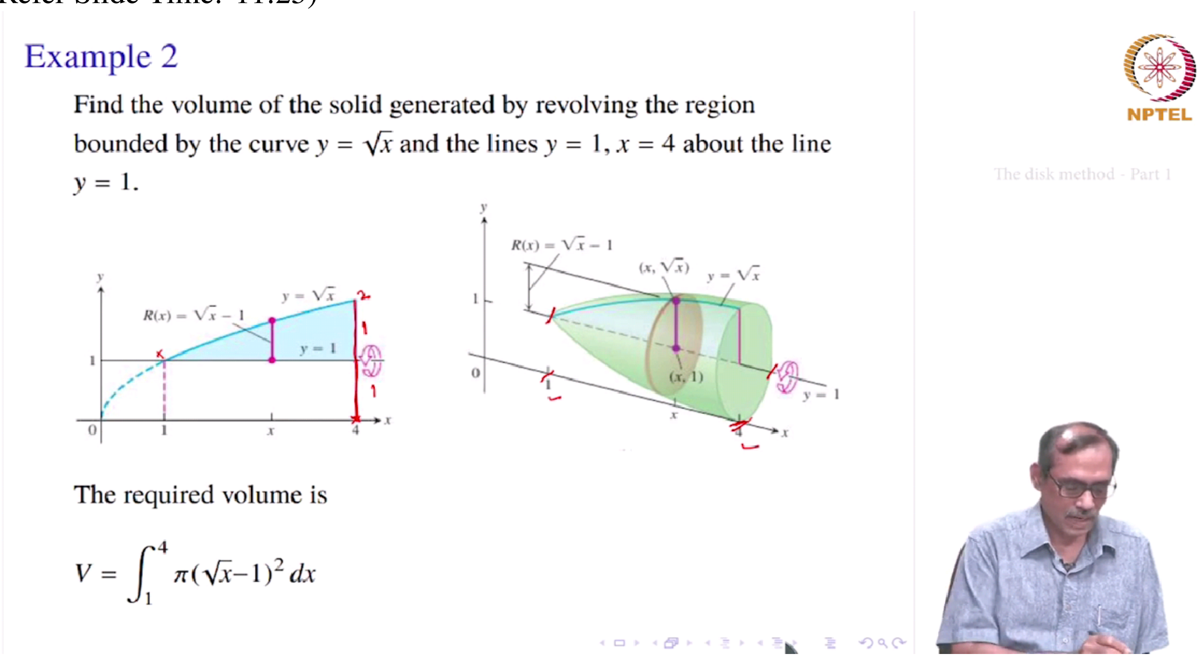

And the area of the disk is to be integrated from where to where? It is the line $y = 1$ as one limit, and what is the other limit? The other line that encloses the region is $x = 4$. So, what are the limits of the integration? Once you revolve it around this, this value is 4 so that this is 2; the height is 2. Therefore, once the region revolves around $y = 1$, this is the other height which is also 1. So, the total height is 2. Notice that once it is revolved around it will touch the x-axis as shown in the green picture here.

Since it is revolving around $y = 1$, we must find out what are the limits for this integration. The integration is with respect to x now, since the disks swap parallel to the x-axis. At this point, it is $x = 4$ and on the left side, you have $x = 1$, because this is the point where it crosses the y-axis and that height is $y = 1$. You can also get from $y = \sqrt{x}$ and $y = 1$ that $x = 1$. So, x varies from 1 to 4. Therefore, volume is equal to $\int_1^4 \pi[\sqrt{x} - 1]^2\, dx$.
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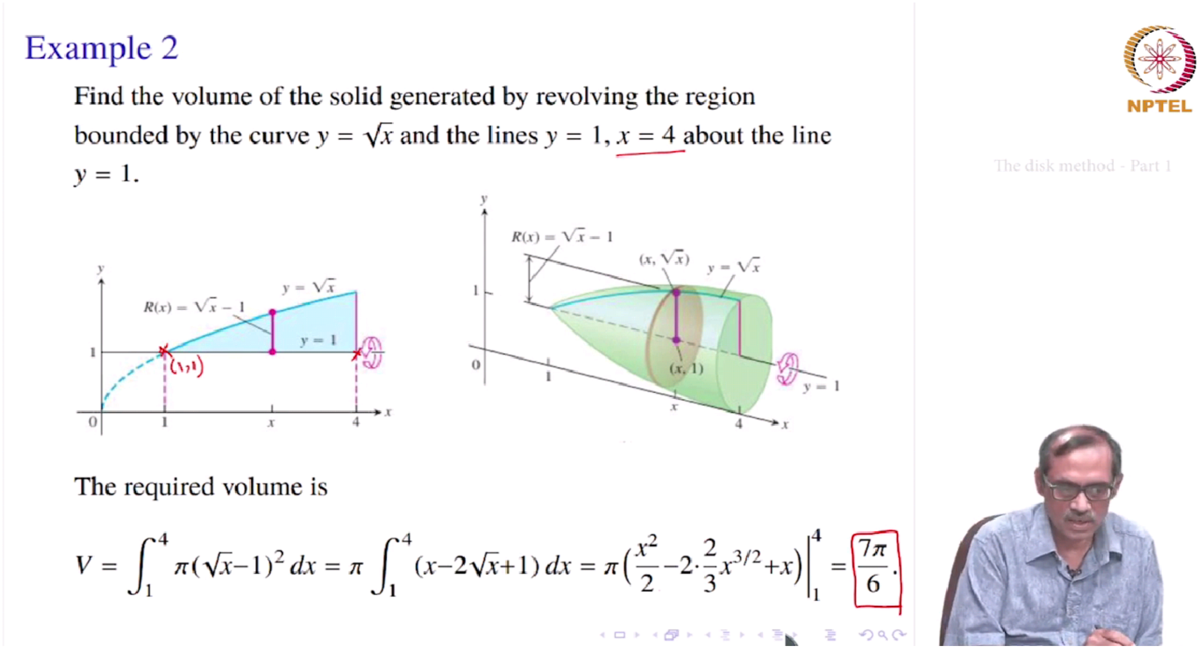

Now it is a matter of integration. We expand the integrand; it is $\pi(x - 2\sqrt{x} + 1)$. Integration of x gives $x^2/2$, of $2\sqrt{x}$ gives $2x^{3/2}/(3/2)$ and of 1 is x. So, this is the correct expression. It has to be evaluated at 1 and 4 and then subtracted. After simplification, you would obtain the volume as $7\pi/6$.

As you see, to find the limits of integration we had to look at the intrsection of the curve and the line $y = 1$. In some problems these limits may be given directly: "from this line to this line"; but here, it was not given. We had to find it out. And that turned out to be where the line $y = 1$ crosses the curve $y = \sqrt{x}$. It was found to be $(1, 1)$. So, we got the limit for x as 1 and 4 on the other side, as it was given in the problem. Next, we had to get the radius of that cross-sectional area which was a disk, and then plug it in the formula.

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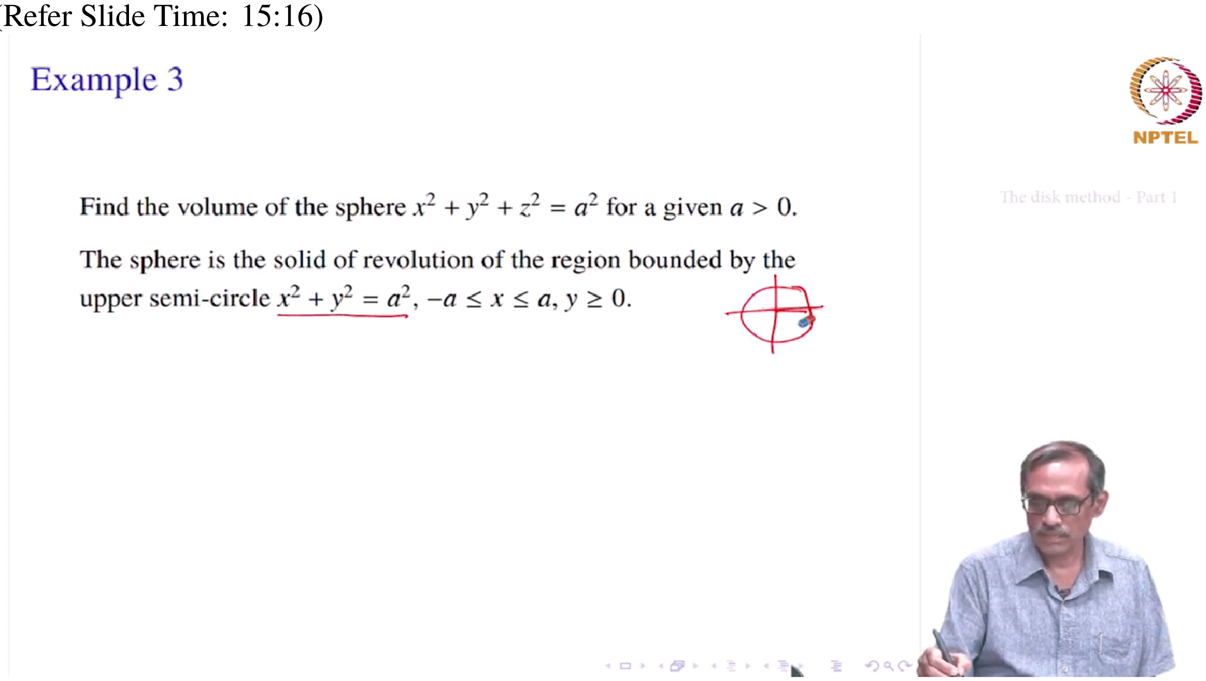

Let us consider another example. Here, we want to find the volume of the sphere. Of course, we know this from other places. But we want to see how our method is applied to compute the volume of the sphere. The sphere is given as $x^2 + y^2 + z^2 = a^2$ for some positive number a. We want to find the volume of this. That means we have to think of this sphere as a solid of revolution.

This sphere has radius a and how do we take it as a solid of revolution? We look at the semi-circle $x^2 + y^2 = a^2$. It has radius a. Since we take only the semi-circle, we have $y \geq 0$. Which means, we just considered this semi-circle where this is 0 and this is the x-axis. And then, we revolve this semi-circle around x-axis to obtain the sphere. If you take the whole circle and revolve, you also get this sphere, but you get twice of this sphere, because they are repeated. Now, there is no repetition in obtaining the solid of revolution. So, you take the semicircle. Moreover, since you take the semicircle, this really is a function, which you can write as $y = \sqrt{a^2 - x^2}$. That is the curve.

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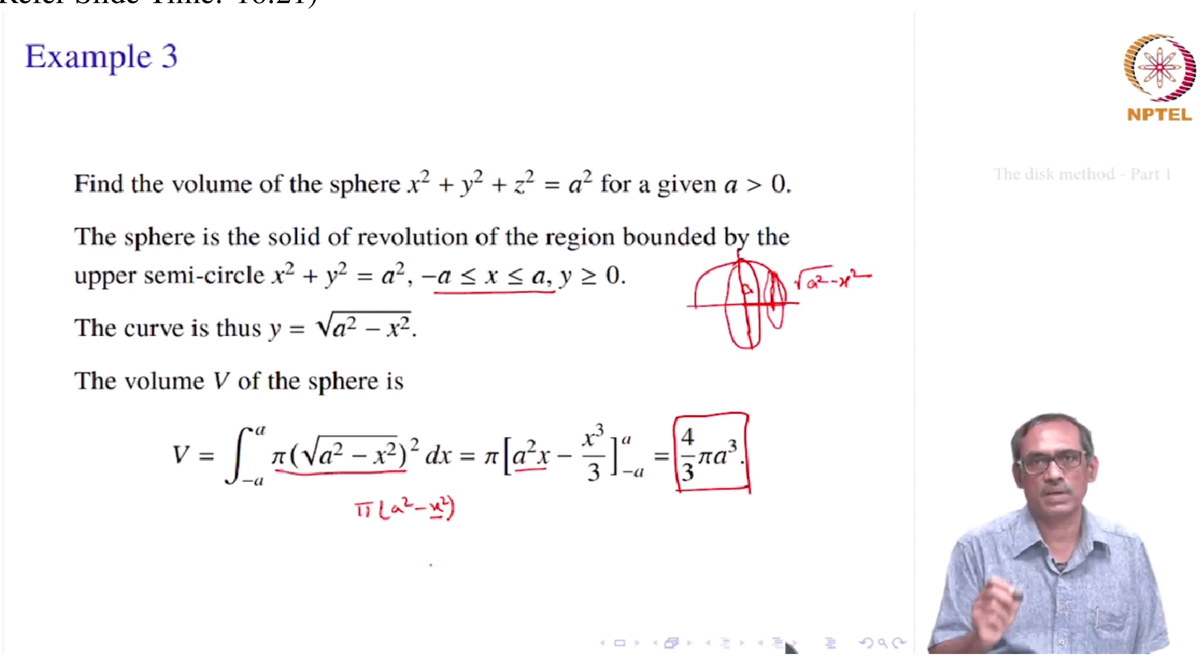

This curve $y = \sqrt{a^2 - x^2}$ is revolved around the x-axis with obvious limits as $-a$ to a. Therefore, volume of the solid will be equal to the integral from $-a$ to a of the cross-sectional area, or the disk. What is the disk now? The disk is generated here; and this is the disk. It has the same radius a. But we must find the radius of the disk at any point x not only in the middle. At any point x, it would look something like this. If this is x, then its radius will be this y component, which is $\sqrt{a^2 - x^2}$. Then, the cross-sectional area or the disk so generated is $\pi\left(\sqrt{a^2 - x^2}\right)^2$. The volume of the sphere is than equal to $\int_{-a}^{a} \pi\left(\sqrt{a^2 - x^2}\right)^2 dx$.

That simplifies to $\int_{-a}^{a} \pi(a^2 - x^2)\, dx$. The integration of a^2 is a^2x and of x^2 is $x^3/3$. So, it is $\pi(a^2x - x^3/3)$ evaluated at $-a$ and a, and then subtracted. You simplify it to get $(4/3)\pi a^3$. This conforms to our formula for the volume of the sphere of radius a.

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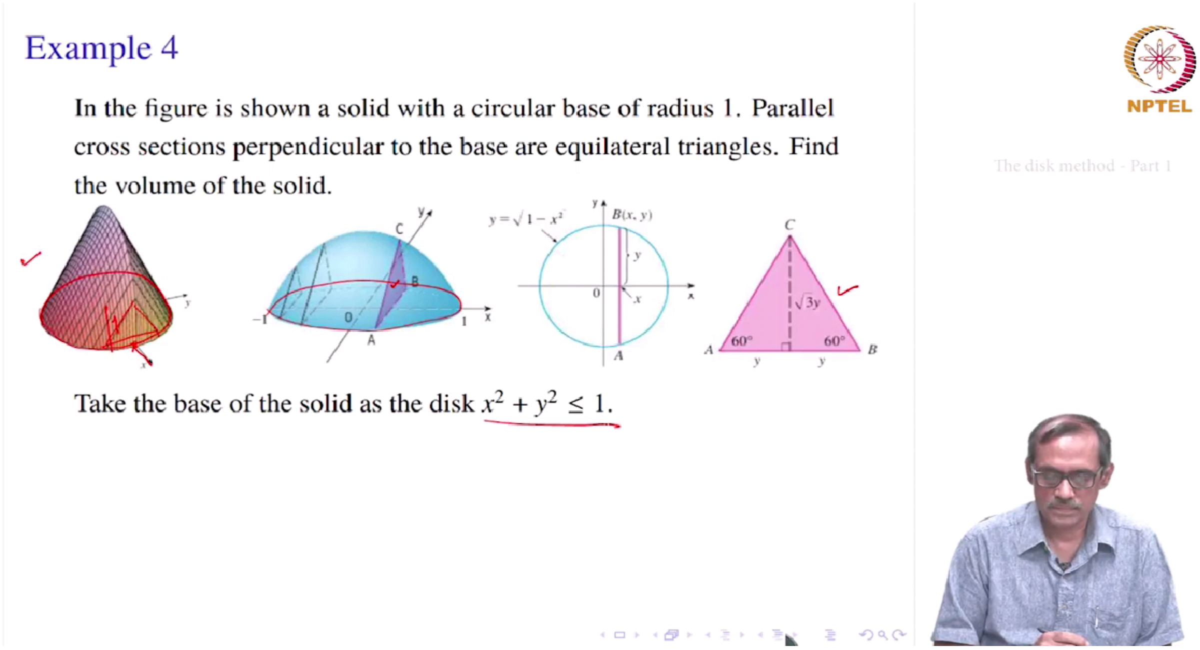

Let us take another example. It looks complicated; let us read it slowly. In the figure is shown a solid with a circular base of radius 1. Look at the first picture which is in multicolor: brown, red and slight blue. It has a circular base of radius 1. Here, is the base, it is a circle of radius 1. This length is 1. These are the cross-sections here. It is given that these cross-sections perpendicular to the base are equilateral triangles. That means, if you take the cross-section it will not touch the circumference of the circle, it will go like this; that becomes an equilateral triangle. The triangles will be looking like this, where it is pink; these are equilateral triangles. At every point you take the cross-section parallel to that, you will get an equilateral triangle.

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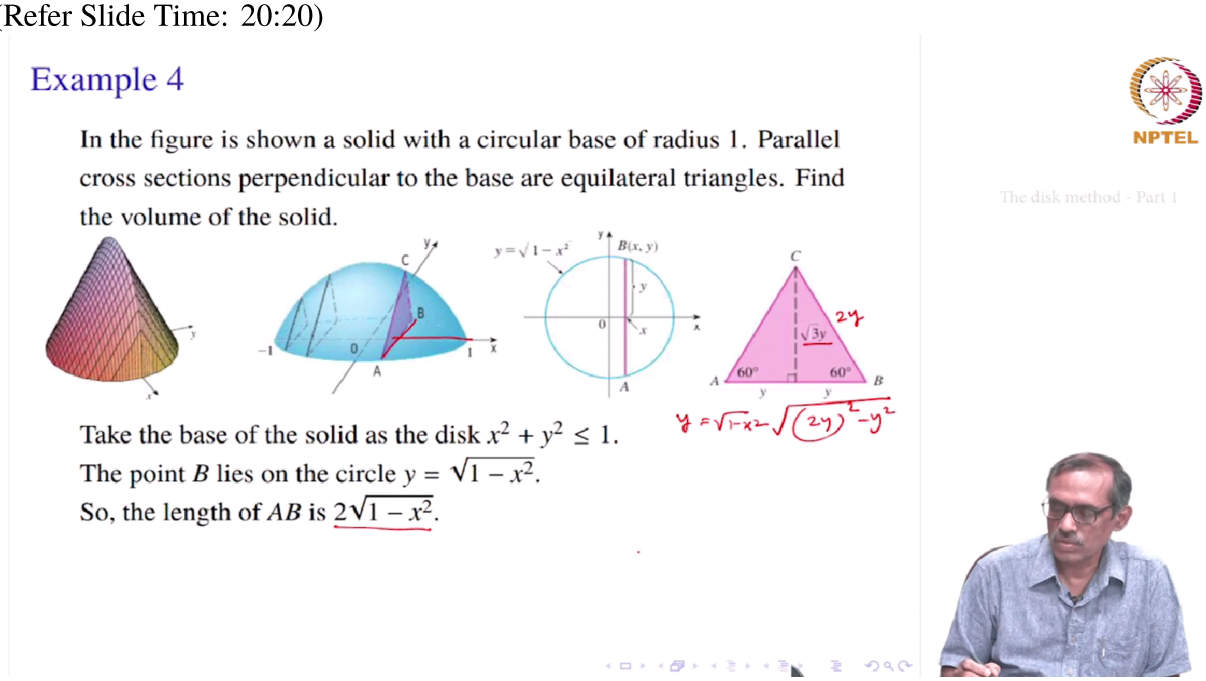

What do we do? We look at it in a different way. We take the axis as x-axis which goes through the middle in the base. Let us take that as the x-axis. Then you look at this triangle which is formed; it is colored pink in the second picture. That triangle is an equilateral triangle and everywhere, similarly, it becomes an equilateral triangle. That is how it looks. I think the pictures are depicted correctly. Then, we have to find out the volume of this solid.

First thing is, we have to see it as a solid of revolution. And then, of course, we can apply our formula. The base of the solid is the disk $x^2 + y^2 \leq 1$. Since this is the x-axis, the base is the disk $x^2 + y^2 \leq 1$. We take any cross-sectional area painted violet in the second picture. Let the points common to the cross-sectional area and the base be A and B. These points lie on the circular base $x^2 + y^2 = 1$. Therefore, in the xy-plane, these points satisfy $y = \sqrt{1 - x^2}$. We want to find the length of AB, the base of that equilateral triangle. Since the x-axis goes in the middle of AB, the length of AB is equal to twice of $\sqrt{1 - x^2}$. Since the cross-sectional area is an equilateral triangle ABC, each of its side has length $2\sqrt{1 - x^2}$.

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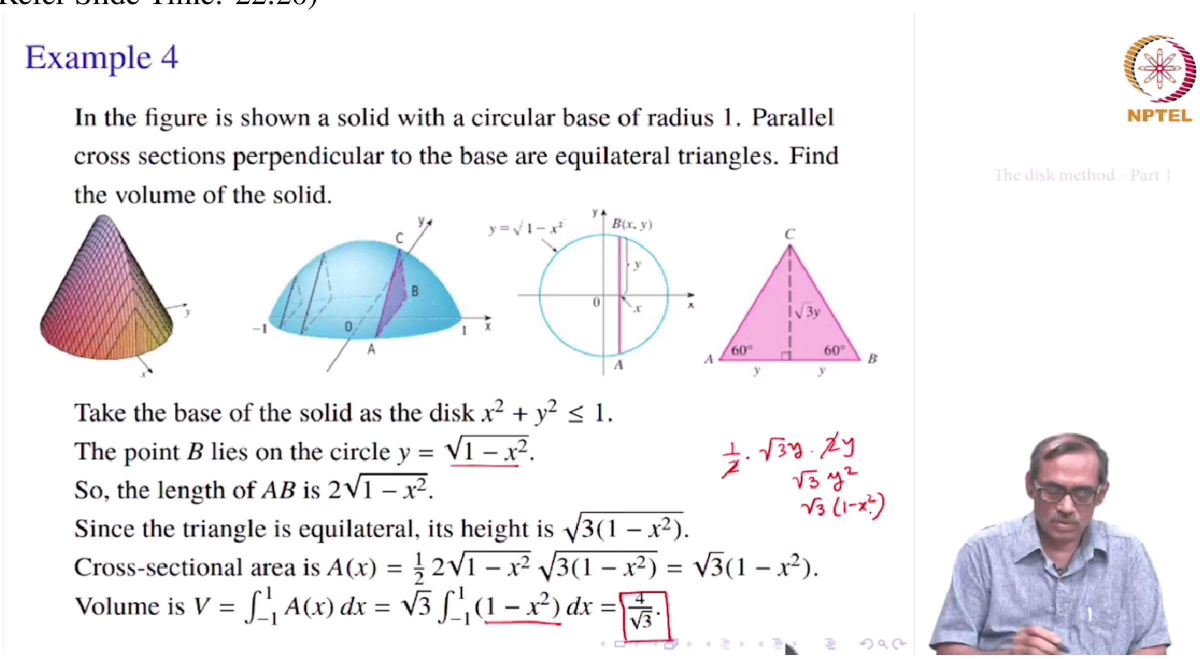

We need to find the area; this is the cross-sectional area. We can use the slice method directly because disk method is anyway coming from the slice method. Let us see it that way. Once each side of this equilateral triangle is $2\sqrt{1 - x^2}$, what will be its height? We come back to this pink picture. If this side is y, where $y = \sqrt{1 - x^2}$, then the length of this perpendicular is $\sqrt{(2y)^2 - y^2} = \sqrt{3}y$. Hence, the area of triangle ABC is $(1/2)\sqrt{3}y \times 2y$. After canceling 2, we get $\sqrt{3}y^2$. And what is y? It is equal to $\sqrt{1 - x^2}$. So, the cross-sectional area is $\sqrt{3}(1 - x^2)$.

Once you have got the area of that cross-section, we get the volume by integrating it with correct limits. Here, the limits are -1 and 1. Hence, the volume of the solid is $\int_{-1}^{1} \sqrt{3}(1 - x^2)\,dx$. This turns out to be $4/\sqrt{3}$.