Basic Calculus - 1
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Lecture 31 - Part 2
Volumes by Slicing - Part 2

Let us take the next example. Here, we want to find the volume of a solid, whose base is the disk $x^2 + y^2 \leq 1$ and the cross-sections by the planes perpendicular to the y-axis between $y = -1$ and $y = 1$ are isosceles right triangles with one leg in the disk. The disk is painted blue here. That is the disk; it is the circle along with the points inside it, that is why you call it a disk. The base of the solid is this disk. If you take any cross-sectional area, that is, a slice of the cross-section of the solid made by a plane which is perpendicular to y-axis (that is parallel to xz-plane), then you will get the limits for y as -1 and 1. That is how it happens on the y-axis. And, it forms an isosceles right triangle with one leg in the disk. So, this leg of the right triangle is equal to this leg. Also, one leg is in the disk.

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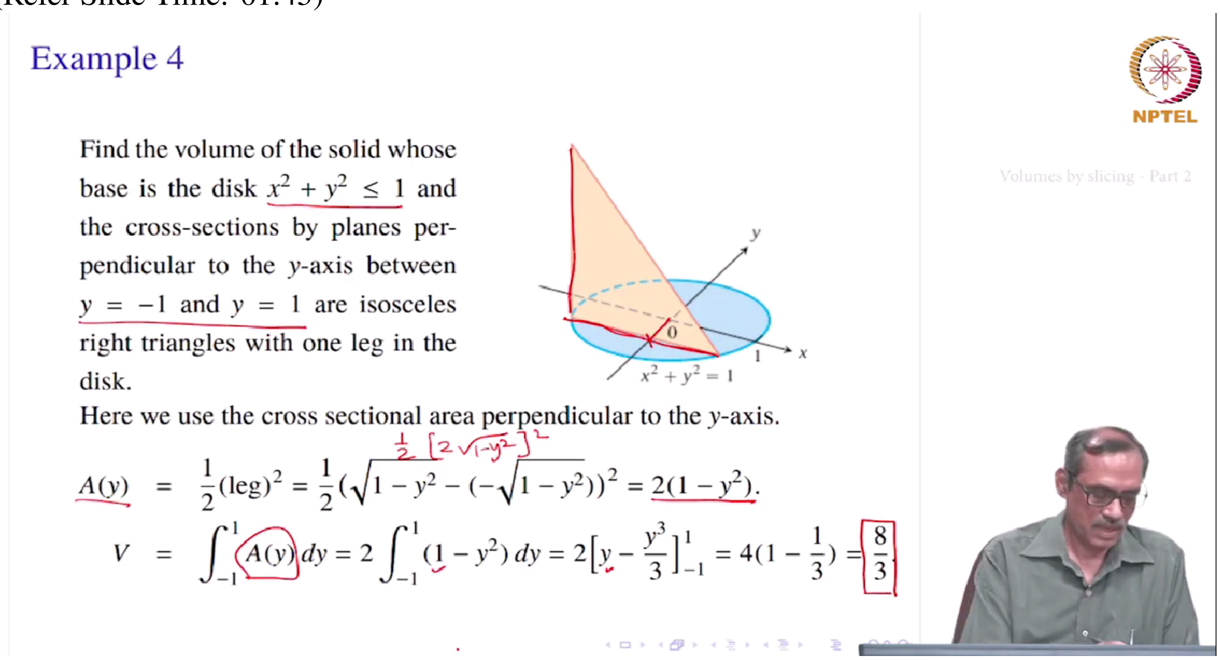

Here, we are using the cross-sectional area perpendicular to the y-axis instead of the x-axis. The integral representing the volume will be taken with respect to y, and this cross-sectional area has to be written as a function of y. Now, what is the cross-sectional area here? Since it is a right triangle, its area is half times half its base into the height. But the legs ahve the same length, so, it is half times leg square. But what is exactly the leg? The leg lies between two lines, $y = -1$ and $y = 1$, and it is inside the disk $x^2 + y^2 \leq 1$. If you take any y here, say, 0 to y this way, then this length is y, and this length will be $\sqrt{1 - y^2}$. Here we take the square root because it is in the circle $x^2 + y^2 \leq 1$. So, that is the x-coordinate. And then, on the other side we have similarly this length. So you may say it is two times $\sqrt{1-y^2}$. The cross-sectional area is half times $2\sqrt{1-y^2}$ into $2\sqrt{1-y^2}$. That simplifies to $2(1-y^2)$.

So, we know the cross-sectional area and we know the limits for y as -1 and 1. Then we can compute the volume. The volume will be equal to the integral from -1 to 1 of the cross-sectional area. Plugging those in, we have the volume as $\int_{-1}^{1} 2(1-y^2)\,dy$. For the integral, 1 gives y, y^2 gives $y^3/3$; so, this is to be evaluated at 1 and -1 and then subtracted. You compute and see that it is equal to $8/3$.

Sometimes you may have the need to take another axis instead of the x-axis. The problem itself dictates whether you take the x-axis or you take the y-axis or the z-axis. We will take one whichever is appropriate to the problem. In this problem, it is already given that the cross-section by planes perpendicular to the y-axis do such and such. That means, we have to choose the cross-sectional area in terms of y. We need to express that in terms of y for obtaining the volume.

Let us take some problems. This is the first problem. We want to find the volume of the solid that lies between planes perpendicular to the x-axis at $x = -1$ and at $x = 1$. You have the x-axis here, and these are the planes $x = -1$ and $x = 1$. The solid lies within these. That means its projection to x-axis lies between -1 to 1. Now, the cross-sections perpendicular to the x-axis are circular disks. Once you take any point x here, you will get a circular disk that is the cross-section whose diameter runs from the parabolas $y = x^2$ to $y = 2 - x^2$. The diameter is this one, which lies between these two parabolas. So it is something like this, thought not exactly. One end of that is in one of the parabolas $y = x^2$, and the other end is at the other parabola $y = 2 - x^2$. That means, To get the cross-sectional area, we have to compute first this diameter, then get the circular area.

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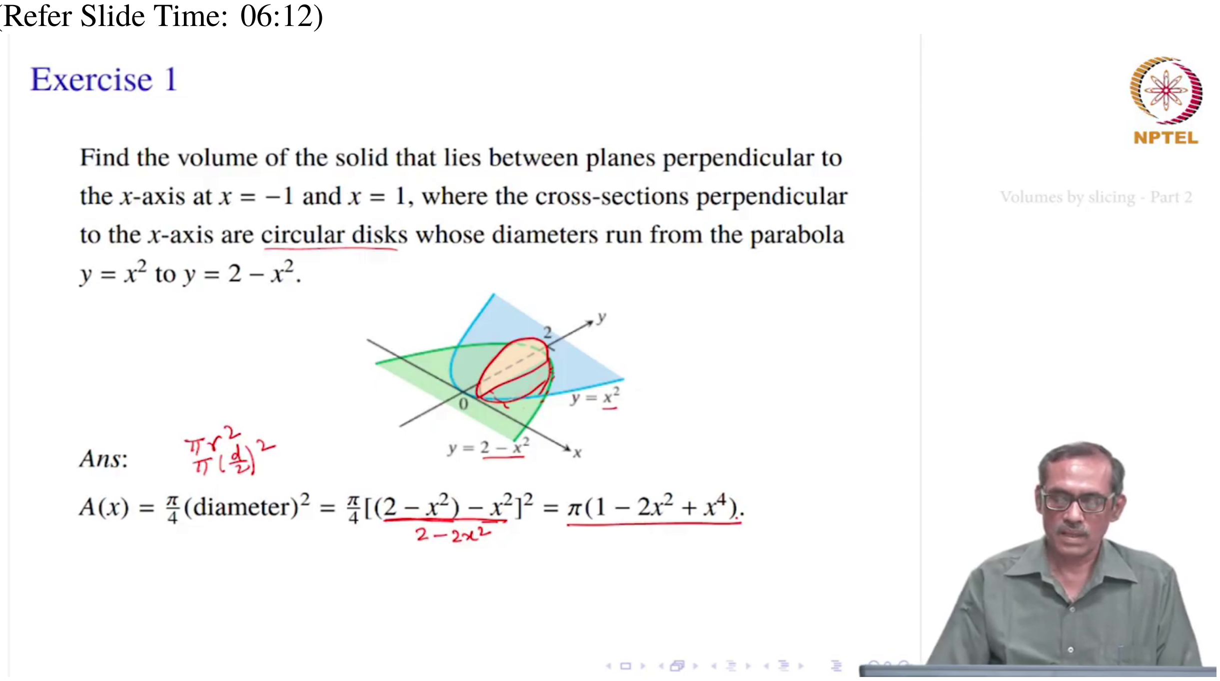

This is how it is: we have one parabola, which is $y = x^2$, another parabola which is $y = 2 - x^2$, and the diameter of that cross-sectional area lies between these two parabolas. The cross-sectional area here is a circular disk. Say, one parabola is here, another is there on the other side, and that is the diameter of the circle. Now, how do we get this diameter? The diameter will be the difference between the y-coordinates. That gives $2-x^2$ from the one parabola and the other one is x^2; so, it is the diameter is $2-2x^2$. So, the cross-sectional area, which is the area of that disk, is equal to π times diameter by 2 square.

The cross-section is perpendicular to the x-axis and it is a circular disk. Only half of that is shown here. That half will be equal to $(\pi/4)$ times radius square. However, we want the circle. Then, it is πr^2, which is equal to π times half of the diameter square; which is simply $(\pi/4)$ times diameter square. That is, the cross-sectional area is $(\pi/4)(2-x^2)^2$. That simplifies to $\pi(1-2x^2+x^4)$.

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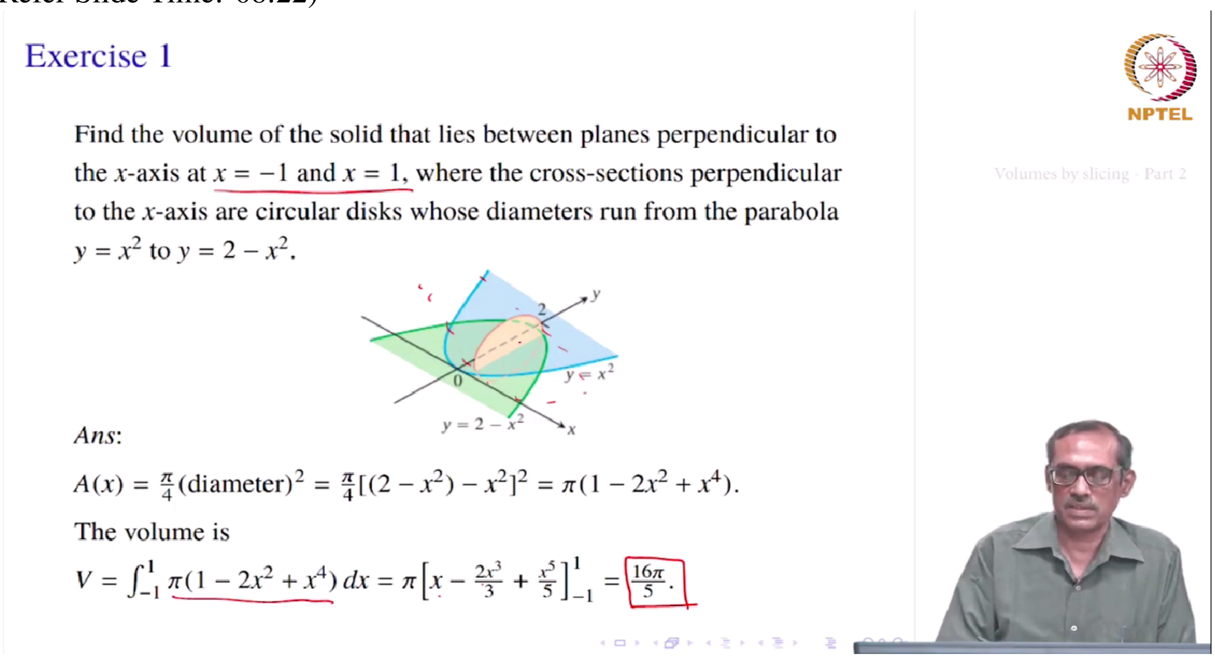

Once this is achieved, it will be easier to compute the volume; we just use a formula. The volume will be the integral from -1 to 1, which is already given in the problem, of the cross sectional area, which is $\pi(1-2x^2+x^4)$. The integral of 1 is x, of $-2x^2$ is $-(2/3)x^3$ and of x^4 is $x^5/4$. This is to be evaluated at 1 and -1 and subtracted. That simplifies to $16\pi/5$.

Here, you see that we have not plotted the whole solid. But we have already the information that it is from -1 to 1. The solid is something like a cylinder; but not exactly a cylinder; it is not uniform; at x if it is a circle, at another point x, it will be a smaller circle or maybe a bigger circle depending on where this x is. That is how it looks. To compute its volume, it is enough to find the cross-sectional area, because anyway it will be plugged in the integral. We need to plot that in the figure so that we may get the area of the cross-section.

Let us take the next problem. Find the volume of the solid that lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The solid now lies between two planes perpendicular to the x-axis at $x = -1$ and $x = 1$. We get the limits for x. We must have the cross-sectional area expressed in terms of x.

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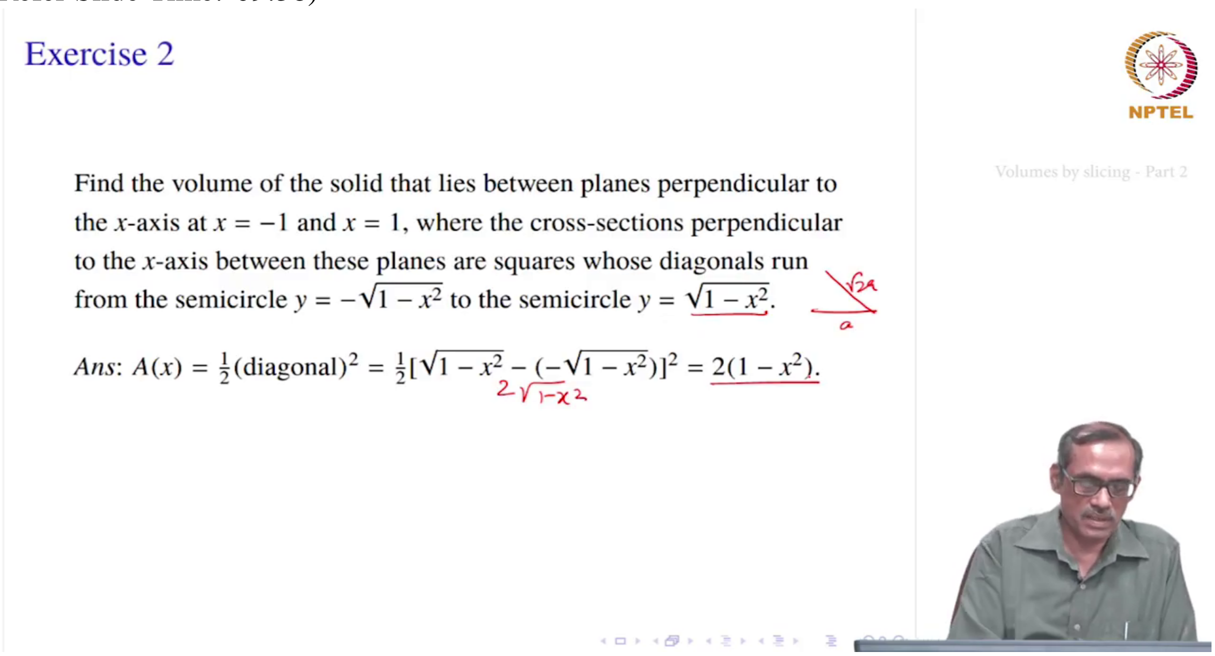

The cross-sections perpendicular to the x-axis between these plans are squares. So, it is a simpler one. The diagonals of such squares run from the semicircle $y = -\sqrt{1-x^2}$ to the semicircle $y = \sqrt{1-x^2}$. So, it is a square whose diagonals are of this type, not the sides, fine. What do we do we want? We want to find the area of that cross-section. Since it is a square, its area will be equal to half into diagonal square. Because, if the side is a and its diagonal is $\sqrt{2}a$, then $a^2 = (1/2)(\sqrt{2}a)^2$.

A diagonal will be between $y = -\sqrt{1-x^2}$ to $y = \sqrt{1-x^2}$. Then, the length of the diagonal will be $\sqrt{1-x^2} - (-\sqrt{1-x^2})$, which gives $2\sqrt{1-x^2}$. We have to take its square and multiply with half. That gives $(1/2)(2\sqrt{1-x^2})^2 = 2(1-x^2)$. That is how we get the cross-sectional area.

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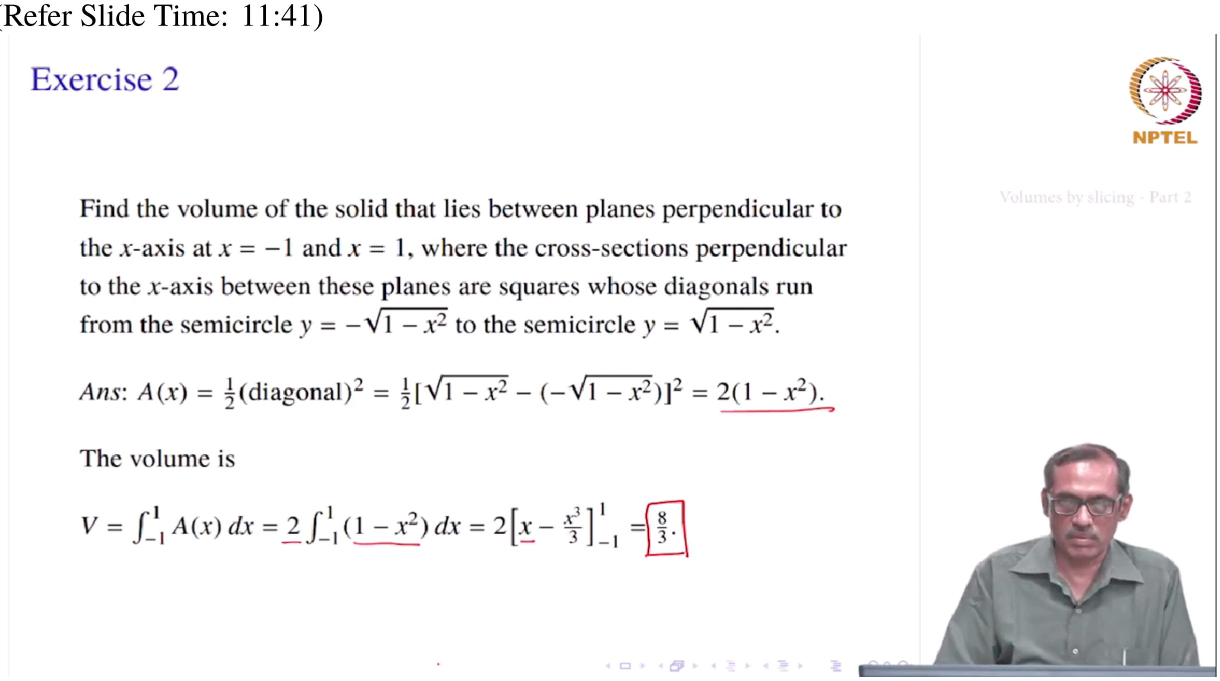

Once the cross-sectional area is obtained and we know already the limits, the volume will be equal to the integral of the area between these limits. That gives the volume as $\int_{-1}^{1} 2(1 - x^2)\, dx$. The integral of 1 is x, of x^2 is $x^3/3$. So, it is $2(x - x^3/3)$ evaluated at 1 and -1, and then subtracted. That simplifies to $8/3$.

Here, of course, the problem simplifies itself a lot. We have not even found out the picture. We just look out for the diagonal of that square, which is the cross-section. Since everything is given in the problem, we did not have the need to plot it and that became easier.

Let us take another problem. We want to find the volume of the solid that lies between planes perpendicular to the y-axis. That means, we may have to express the cross-sectional area in terms of y. The planes are $y = 0$ and $y = 2$, between which the solid lies. So, we have the limits for y as 0 and 2.

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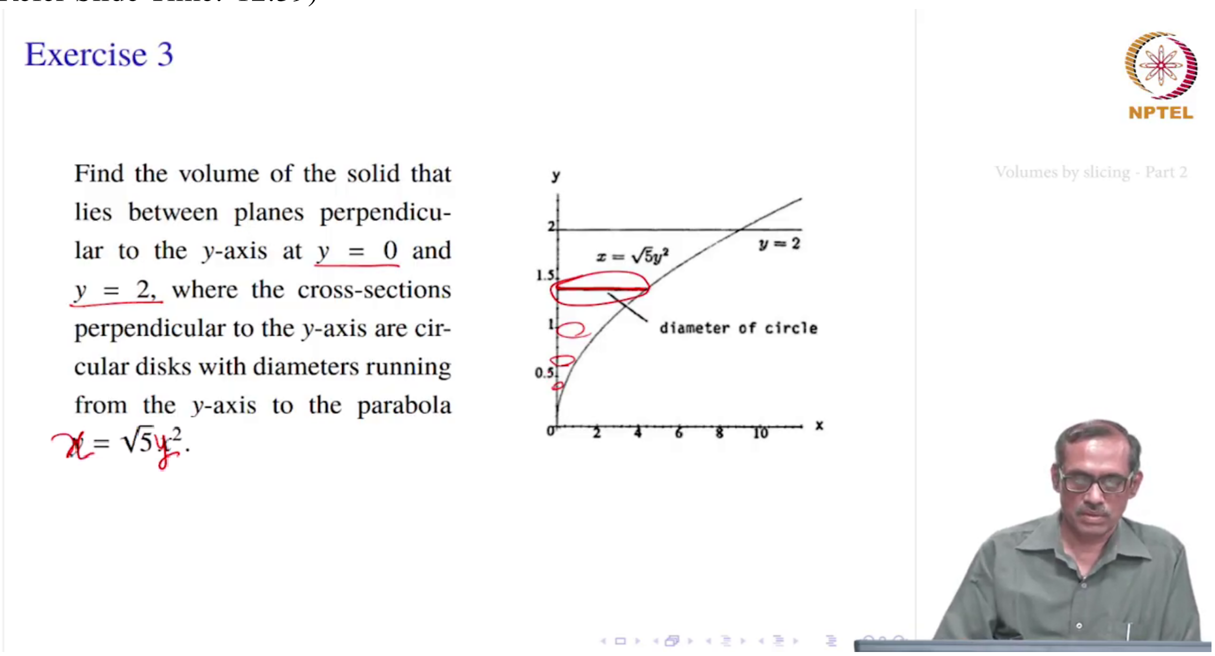

There is a correction here. The parabola is not $y = \sqrt{5}x^2$; it is $x = \sqrt{5}y^2$ as shown in the picture.

Now, what about the cross-sections? The cross-sections perpendicular to the y-axis are circular disks. That means, at any point y, the cross-section is a circular disk; that is how the cross-sectional area looks. And what is the constraint on that circular disk? Their half diameters run from the y-axis to the parabola. Tt is not this way. The diameter of the circular disks runs from y-axis to the parabola $x = \sqrt{5}y^2$. That means, this exactly will be the diameter, and this is our parabola $x = \sqrt{5}y^2$. Volume is positive. So, we take the diameter this way, and plot the picture on the first quadrant. Once this is the diameter of the circular disk, that is how the solid is. In fact, everywhere it is circular. The solid now looks like this. It is a solid which looks something like a cone but it is not cone, it only looks like that. We want to find the volume of this solid.

All that we have to do is compute this cross-sectional area. It is the disk whose diameter is given by this length. This length is running from y to this point on the curve. So, that is $x = 0$ to $x = \sqrt{5}y^2$. For any point y, its length will be root $\sqrt{5}y^2 - 0$. With this, we find the diameter to be this point minus that 0, that is, $\sqrt{5}y^2 - 0$.

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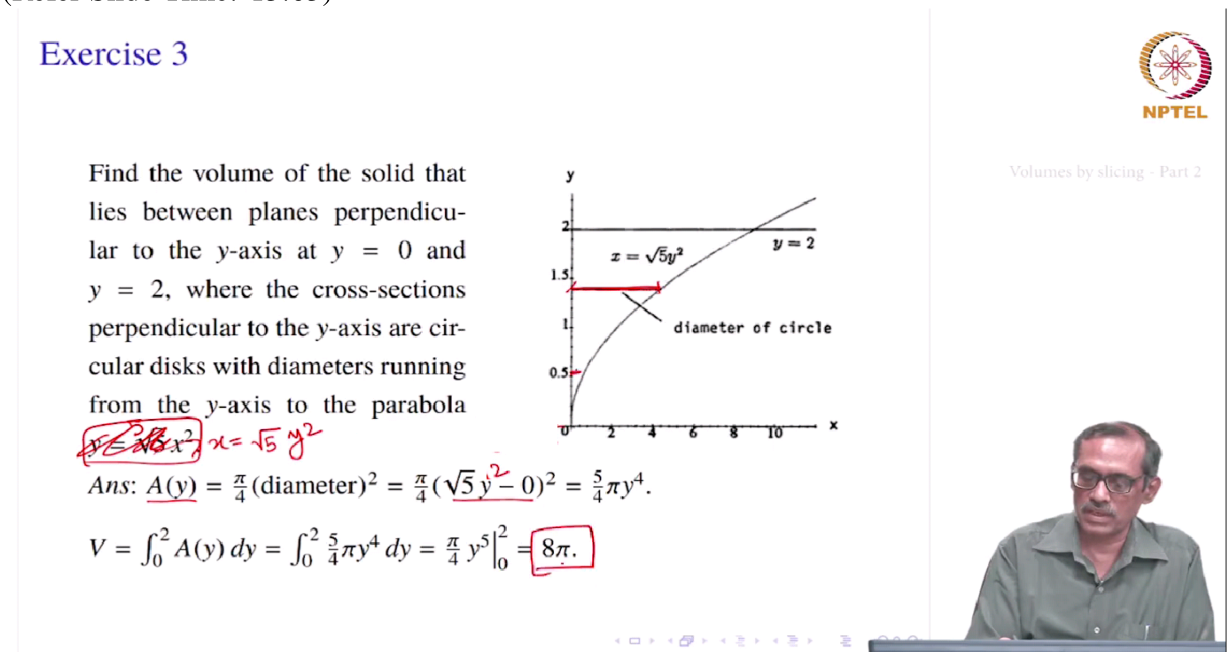

The cross-sectional area $A(y)$ is a circular disk with diameter as $\sqrt{5}y^2 - 0$. So, $A(y)$ is equal to $(\pi/4)$ times the diameter square, which is $(\pi/4)(\sqrt{5}y^2)^2$, or $(5\pi/4)y^4$. Then, the volume of the solid will be equal to the integral $\int_0^2 A(y)\, dy$. This is $\int_0^2 (5\pi/4)y^4\, dy$. As the integral of y^4 is $y^5/5$, the 5 cancels, and we have $(\pi/4)y^5$ evaluated at 2 and 0, and then subtracted. That gives 8π. Let us stop here today.