Basic Calculus - 1

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Lecture 31 - Part 1

Volumes by Slicing - Part 1

Well, this is lecture 31 of Basic Calculus - 1. In the last one or two lectures we had discussed about how to compute area of certain type of regions, especially, area between two curves with a line $x = a$ on the left and a line $x = b$ on the right. Such an area can be converted to an integral, a definite integral from a to b of the function $f(x) - g(x)$, where $f(x)$ is the top curve and $g(x)$ is the down one. Basing on that we have solved some problems.

Today, we will be talking about another such application of integrals. It will be volumes of solids. Specifically, we will be computing volumes by slicing. There are also other methods, and we will cover them slowly. Let us look at this particular method: Volumes by Slicing.

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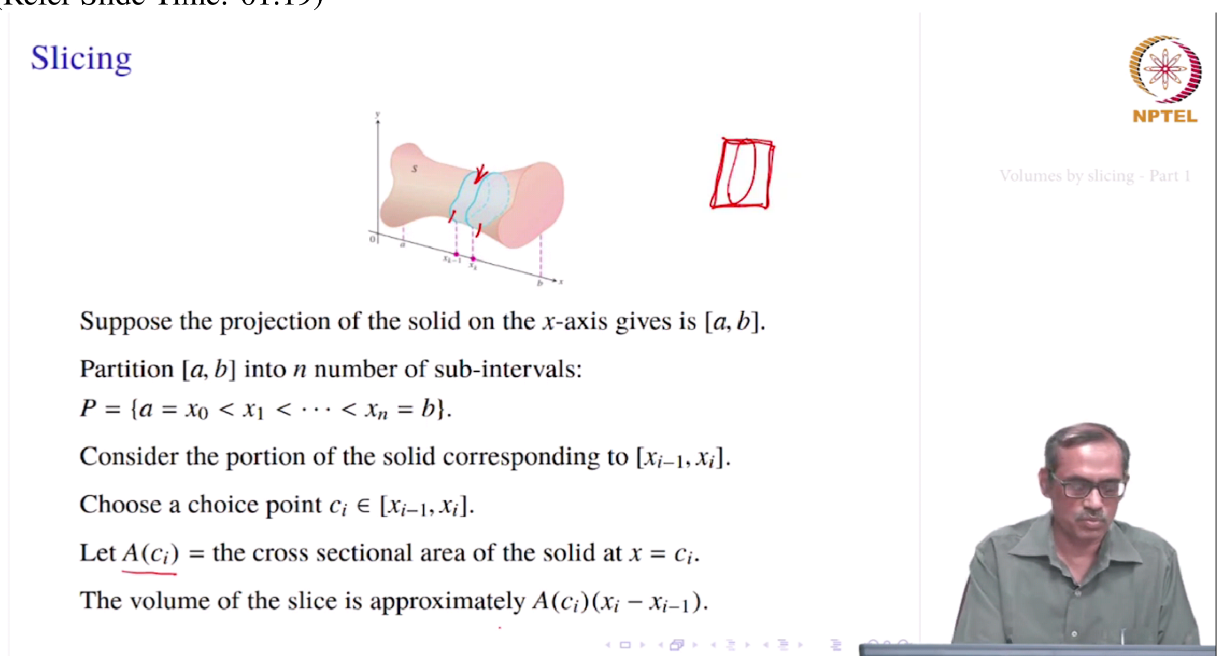

Consider a solid, whose projections on the x-axis lies within the planes $x = a$ and $x = b$. Notice that $x = a$ is a plane in 3D; it is perpendicular to the x-axis, that is parallel to the yz-plane. And, $x = b$ is similarly another plane parallel to the yz- plane. The solid is lying between these two parallel planes perpendicular to the x-axis. By projecting this we get the points, say, the left hand point is a and the right hand point is b.

Then what we do? As earlier, we will be formulating a Riemann sum for the volume of this solid. The volume of the solid will be something like taking a small part of this solid and then looking at it as if it is sliding from a to b. It will give rise to some sub-interval, say, x_{i-1} to x_i forming a partition of the interval $[a, b]$.

Let us start with a partition of $[a, b]$ into n number of sub-intervals. The partition points are a itself which is x_0, and b which is equal to x_n, and in between we choose some other points so that, $x_0 < x_1 < \cdots < x_n$ becomes a partition of the interval $a, b]$. In the sub-interval $[x_{i-1}, x_i]$, we choose a point c_i. The portion of the solid, which is between x_{i-1} to x_i will be approximated by the area at that point c_i times the length $x_i - x_{i-1}$. It will be a cylindrical volume. Instead of taking this exact volume, we will be taking $c_i(x_i - x_{i-1})$, area at c_i times this length.

If that area is something like this and it moved, it generates a cylindrical solid. So, this particular portion will be approximated by such a solid, which we find easily. Of course, it requires that we know how to get the area or the cross-sectional area at the point $x = c_i$, which is a point chosen inside the sub interval $[x_{i-1}, x_i]$.

Recall that these points c_i comprise a set which we call as a choice set while you introduce the integral for the area below the curve $f(x)$ and above the x-axis. We are taking a similar approach here. Instead of $f(c_i)$ we are now taking the area at c_i. This area at c_i is the cross-sectional area of the solid at c_i. It is somewhere here; it is that cross-sectional area. Now the volume of the slice will be approximated by the area at c_i times the thickness $x_i - x_{i-1}$ as we told.

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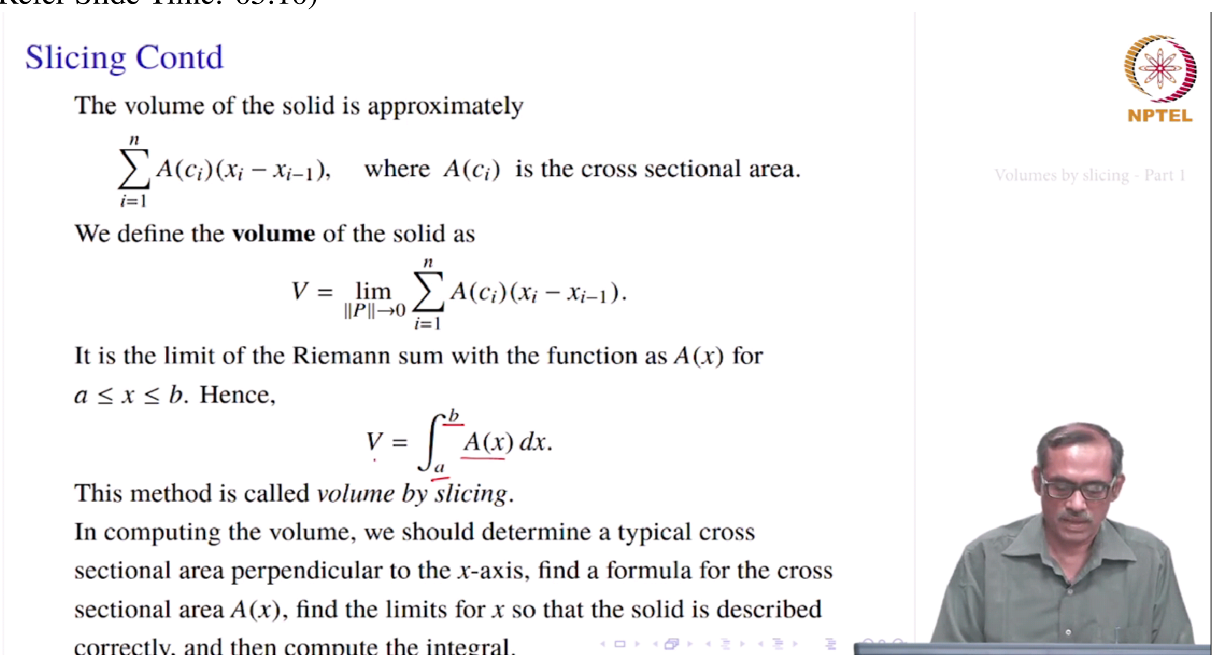

Then, we form the Riemann sum. The volume of the solid is approximately equal to the summation of all these slices. Each slice has the volume $A(c_i)(x_i - x_{i-1})$. So, the Riemann sum $\sum_{i=1}^{n} A(c_i)(x_i - x_{i-1})$ is an approximation of the volume of the solid. Now, we will declare that this is equal to the volume of the solid when the lengths of these sub-intervals approach 0. With the norm of the partition as the maximum of all these lengths $x_i - x_{i-1}$, we see that when the norm goes to 0, we should get the volume V. That is, the limit of the Riemann sum $\sum_{i=1}^{n} A(c_i)(x_i - x_{i-1})$ as $\|P\| \to 0$ is the volume.

When $\|P\| \to 0$, whichever point c_i we choose between x_{i-1} and x_i it will not matter; that is the idea in the limit concept. If this limit exists for all possible choices c_i. then that limit will be called as the volume of the solid.

You see that it is a Riemann sum where the function is $A(c_i)$; it is not f, it is $A(c_i)$. Then, it gives rise to the definite integral $\int_a^b$ of the function of x lying in $[a, b]$, which is the cross-sectional area at x. This approximation and the limit process says that the volume of the solid can be written as the integral from a to b of the cross-sectional area at any point x on the x-axis, where the interval $[a, b]$ has been obtained from the projection of the solid to the x-axis. Since we are taking slices we will tell this method of computing the volume as Volume by Slicing.

We will discuss some problems on how we go about it. But first we must find out the requirements of computation. In computing this volume, first thing is we should find a typical cross-sectional area perpendicular to the x-axis. This should give us the points a and b on the x-axis. It would guarentee that the solid lies within these two planes parallel to the yz-plane. After finding those two points $x = a$ and $x = b$, we find the cross-sectional area. At any point $x \in [a, b]$ we should be able to express that slice or the cross-sectional area as a function $A(x)$. And then, the volume will be equal to the integral of $A(x)$ from a to b. That definite integral will be called as the volume of the solid.

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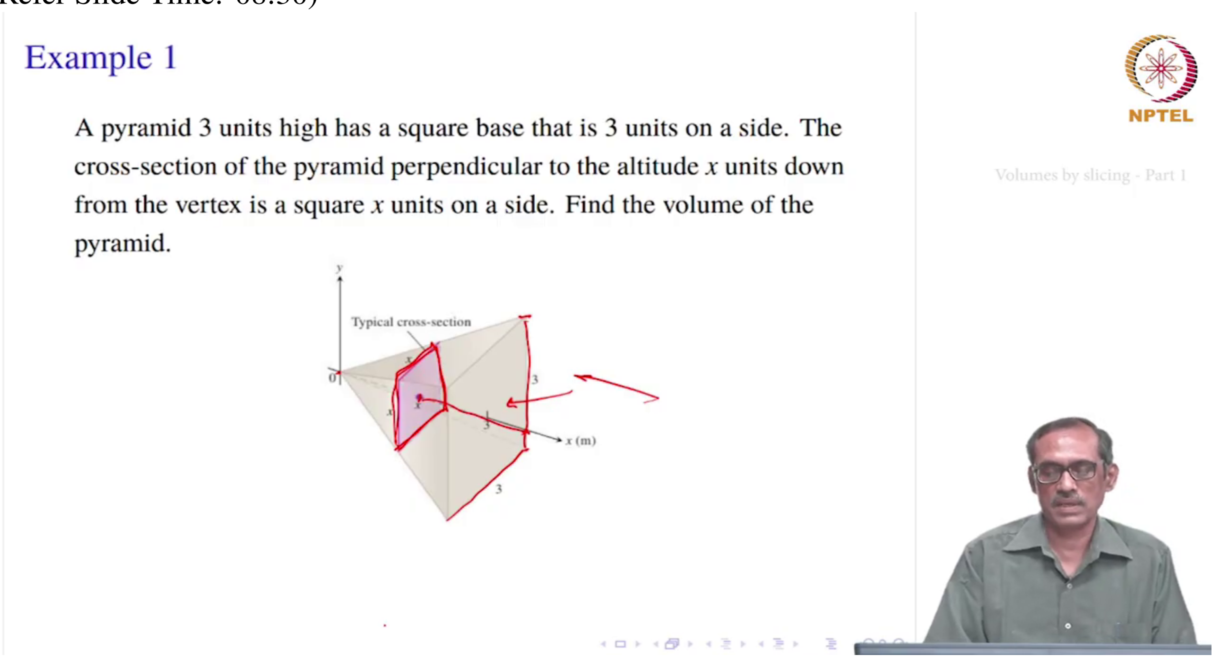

Let us take an example. Suppose we have a pyramid of three units high. So, its height is three units. This is three units. And, it has a square base. So, this is the base. We are showing the base on the right side because we are taking the x-axis this way. Its square base is three units on each side. That is also three units; this is also three units. And, it is starting from the origin, and this is the tip of that pyramid. We take a cross-section of the pyramid perpendicular to the altitude x. We are looking it this way, so the altitude is on the x-axis. This is the base.

On this base we take a perpendicular to the altitude. So, that becomes a plane perpendicular to the x-axis. This slice will be a square with each side as x. If you correctly draw it, then you can see that the cross-sectional area is a square with each side measuring to x units. We want to find the volume of the pyramid.

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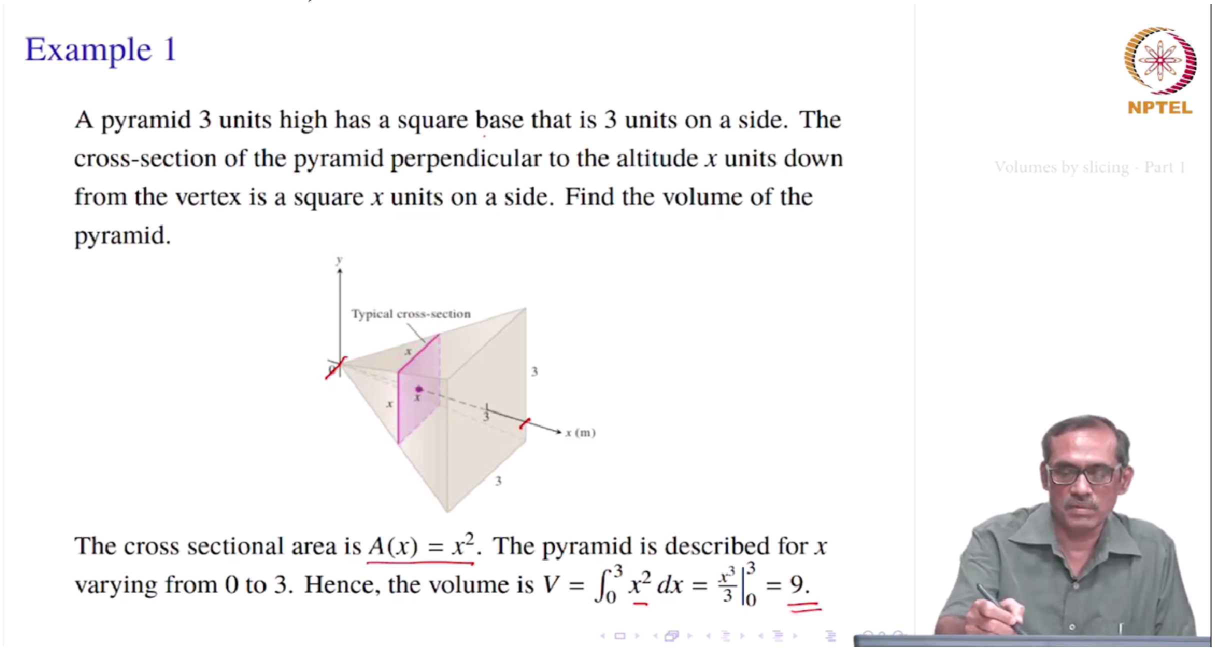

We have the cross-sectional area as $A(x) = x^2$. Since it is a square of side x, the cross-sectional area is x^2. The pyramid is described for x varying from 0 to 3. Hence, the volume will be the integral $\int_0^3 x^2\,dx$. That turns to be $x^3/3$ evaluated at 0 and 3, and subtracted. That gives the answer as 9.

The main problem is to have the plot or a figure which corresponds to the story given in the problem. Once that stage is over, it is just a formula we have to apply.

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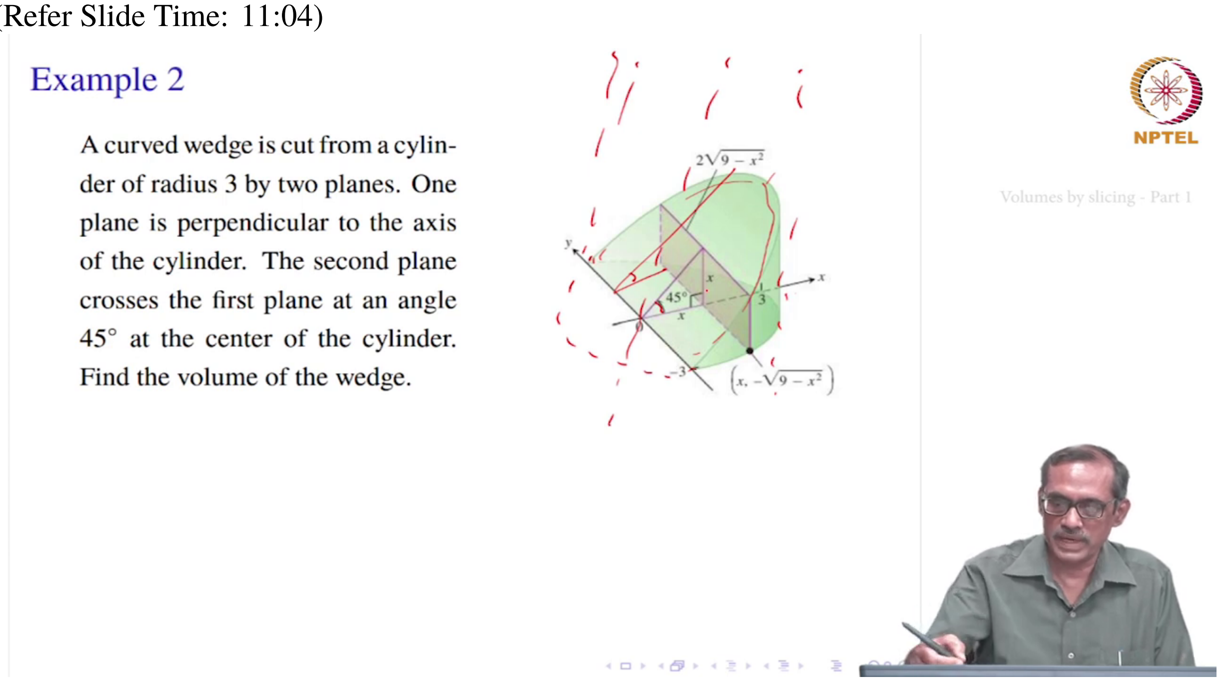

Let us go to Example 2. Here, a curved wedge is cut from a cylinder of radius 3 by two planes. So we have a cylinder; imagine it is going this way, standing like this. On the xy-plane you have its base and it is standing on the xy-plane. That is how a cylinder will look like. And it is a circular cylinder, so its base a circle, a circle of radius 3. Such a cylinder is cut by two planes, where one plane is perpendicular to the axis of the cylinder. The axis of the cylinder is this one. Since the plane is perpendicular to that, we will take that as the xy-plane. We are taking z-axis as the axis of the cylinder. In that plane, which is the xy-plane now, let us choose our x-axis and y-axis.

And, the second plane crosses the first plane at an angle of 45 degrees at the center of the cylinder. That means the other plane is this one, this plane. That is cutting the cylinder at an angle of 45 degrees. If you take any straight line on this and any straight line down, then that will be 45 degrees at the center. You want to find the volume of the wedge. So, we get only one side; we do not see the other side. Whatever is really painted, that corresponds to this problem. Now, the question is how do we compute this volume?

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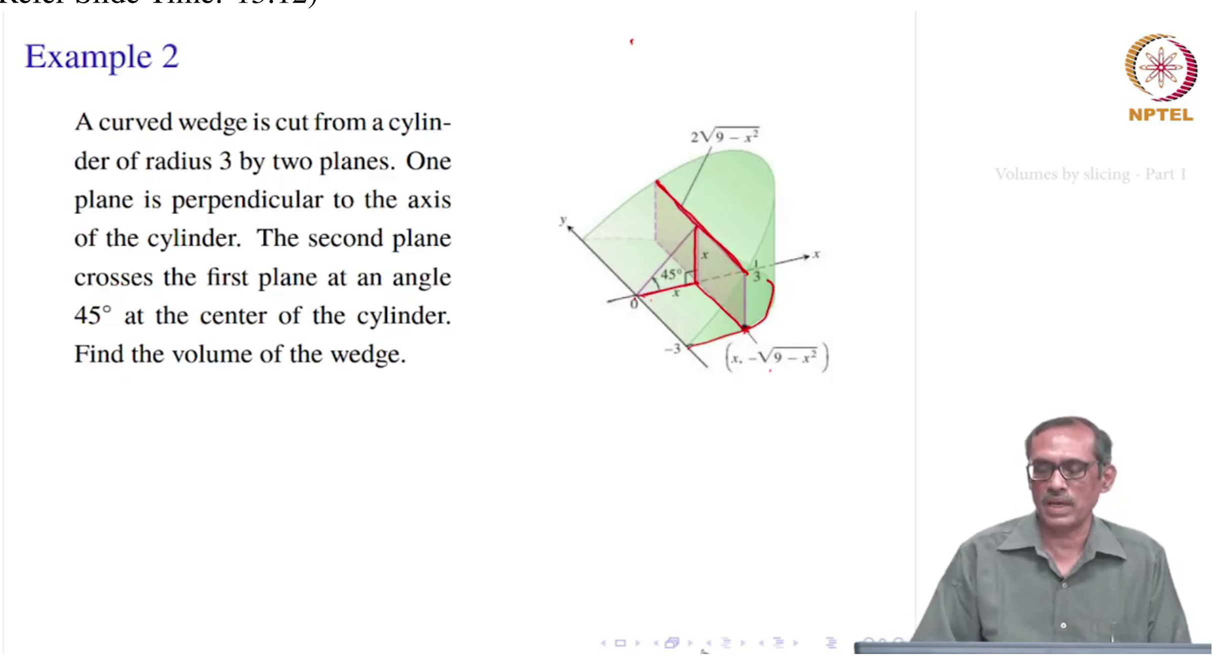

Let us erase these things; this will be clearer. Now, the picture of the wedge looks like this with our choice of the planes. We have the base and the x-axis is there. This radius is 3, and the other plane is making 45 degrees with this plane. We want to compute the volume by slicing.

So, let us take a plane perpendicular to the x-axis, which is parallel to yz-plane to make a slice from this. Since it is 45 degrees, and from the base it is the point x, the height of the slice is also x there. And what is its length? The length will be twice of this length. And, what is this length? Since it is a circular base, we have the half circle here. Then, for any point x here, the y-coordinate will be $\sqrt{9-x^2}$ because $x^2+y^2=9$ as the radius is 3. So, this is $\sqrt{9-x^2}$. Then, the slice is now a rectangle with one side as $2\sqrt{9-x^2}$ and the other side as x.

Therefore, the cross-sectional area is $A(x) = 2x\sqrt{9-x^2}$. And x varies between 0 and 3. So you should have the volume as the integral $\int_0^3 A(x)\,dx = \int_0^3 2x\sqrt{9-x^2}\,dx$. That is the integral. Now, we just have to evaluate this integral. To evaluate this, we can substitute $u = 9-x^2$. Then, its differential du will be equal to $-2x\,dx$. When $x=0$, $u=9$ and when $x=3$, $u=0$. The integral will have a minus sign here because there is $-2x$ here. That means, it is $-\int_9^0 \sqrt{u}\, du$. That would give $u^{3/2} \times (2/3)$ and then evaluated from 9 to 0. Then, this minus sign goes away, and this will be from 0 to 9.

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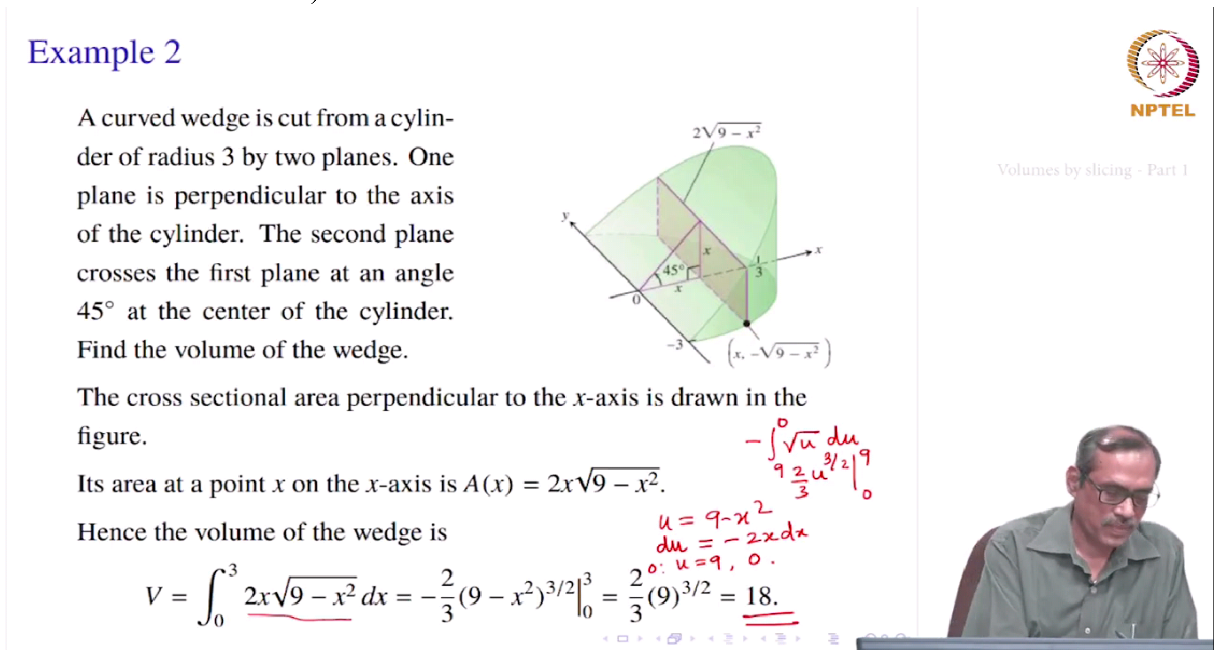

You can also write the same thing in terms of x. It is $\int_2^3 2x\sqrt{9-x^2}\, dx$. As $dx = -d(9-x^2)$, the integral is $-(2/3)(9-x^2)^{3/2}$ and it is to be evaluated at 3 and 2, then subtracted. This simplifies to 18.

So, the main problem here is reading the story given in the problem and translating it to the figure. Once this figure is obtained, we can get immediately the integral, and then compute the integral.

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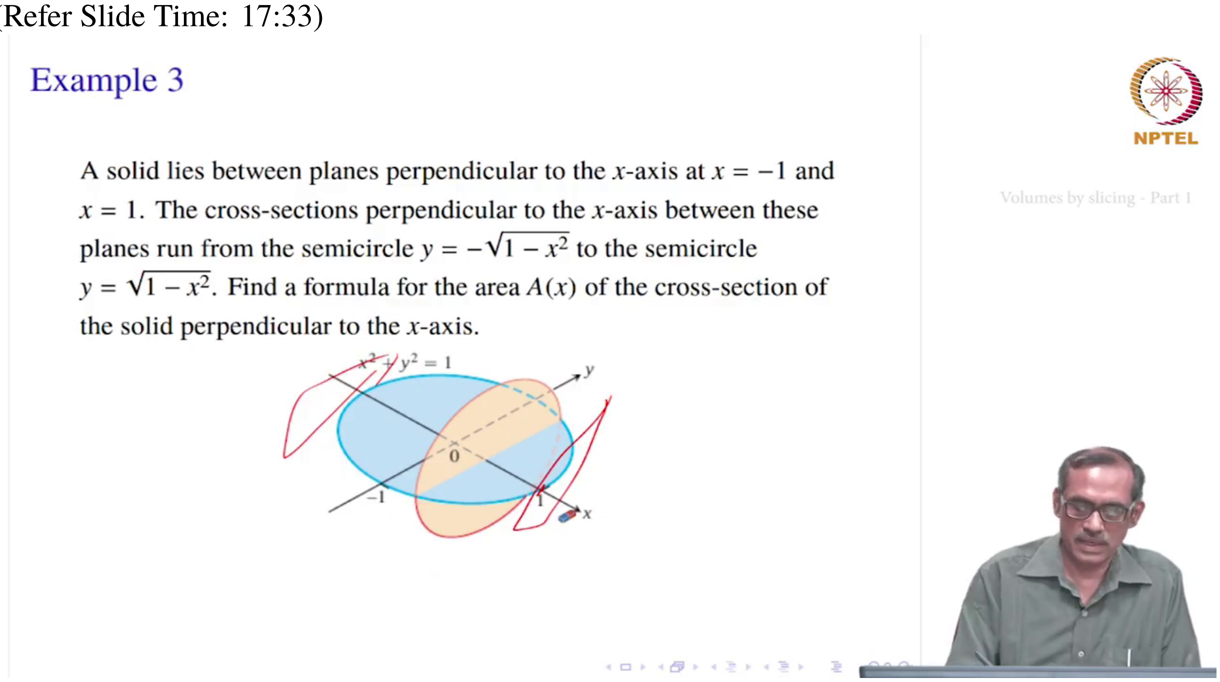

Let us take another example. Here we have a solid that lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. Let us take the x-axis here. We have planes $x = -1$ and $x = 1$. The solid lies between these two planes. That means its projection on the x-axis yields two points $a = -1$ and $b = 1$. These are already given. The cross-sections perpendicular to the x-axis between these two planes, that is between a and b, run from the semi-circle $y = -\sqrt{1-x^2}$ to the semi-circle $y = \sqrt{1-x^2}$. That means, we have a circle $x^2 + y^2 = 1$. The cross-sections perpendicular to this are drawn between these two; so, we have a circle $x^2 + y^2 = 1$. The cross-section is perpendicular to the x-axis. If we take the cross-sections, these planes, they run from this circle to this. That encloses the cross-sectional area.

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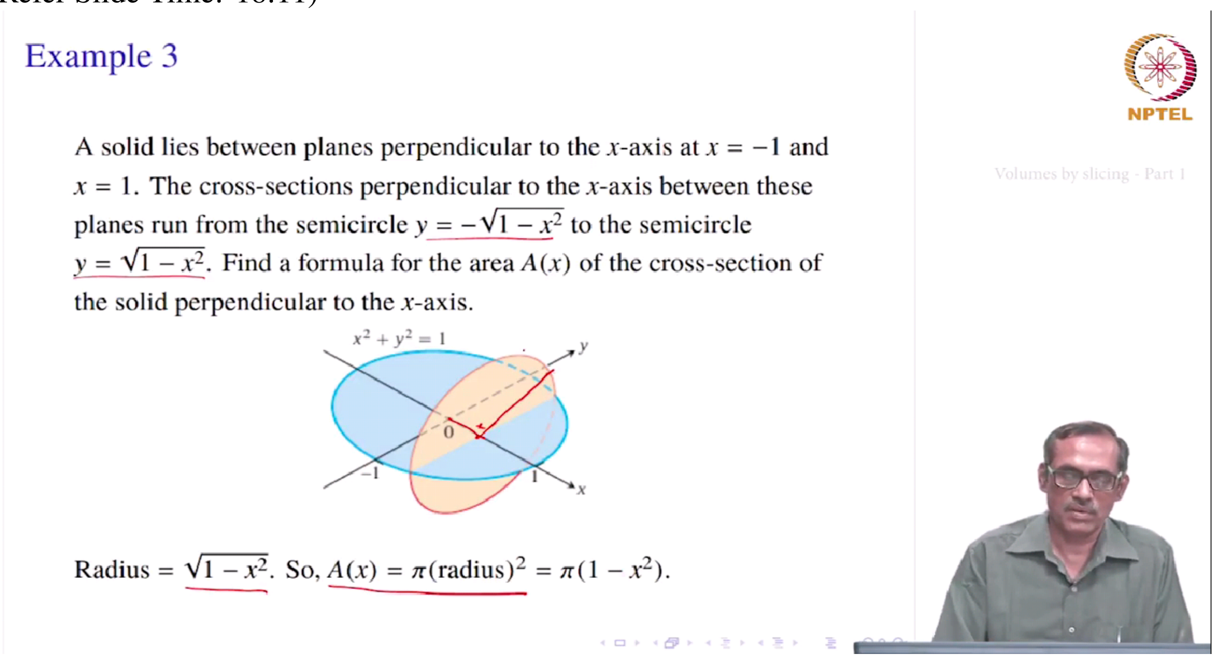

You want to find a formula for the area of the cross-section of the solid perpendicular to the x-axis. The cross-sectional area is painted here in brown. The solid is such that always it will lie between this circle, that is you have to take the cross-section within this circle. So, we nee to find its radius, and that would give the answer. The radius will be $\sqrt{1-x^2}$ because these are the two limits at any point x. So, it is the circle $x^2 + y^2 = 1$. Once you take the cross-sectional area, then that will have the same radius. We go up to x here and this one will be y, which is now equal to $\sqrt{1-x^2}$, the radius of the circular cross-sectional area. Once you get its radius, the cross-sectional area is $\pi r^2 = \pi(1-x^2)$.

Of course, you want to find the volume of the solid. So, you integrate this one from -1 to 1. That would give the volume of the solid.