Basic Calculus - 1

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Lecture 6

Functions - Part 2

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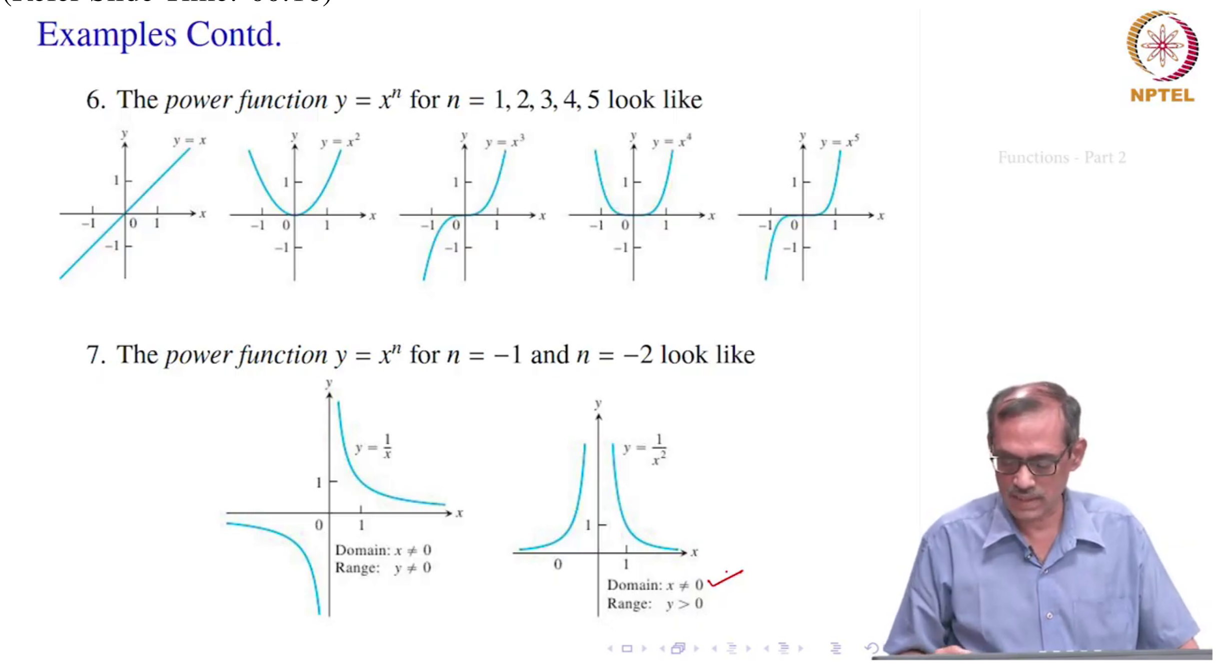

So, we continue with some more examples. Let us see the power function $y = x^n$. When $n = 1$, it is really the function $y = x$. Earlier, we did it. It is the the identity function on the whole of real line. It is the straight line $y = x$. That is how it will look.

When you take $n = 2$, it is $y = x^2$. That is the second one, which looks like a parabola now: $y = x^2$, at 0 it is 0, at 1 it is 1, at 2 it is 4, at minus 1 it is 1, at minus 2 it is 4, and so on. It is a parabola.

If we take $y = x^3$ and plot it, maybe with a graphing calculator or in the computer, you would see the graph would look like this. There is some portion here which is matching with the x axis. It is not really matching. It is really joining with 0. If $x < 0$, then its value is also less than 0. Similarly, when it is greater than 0, its value is also greater than 0. But it is a bit flawed, like the x-axis at 0. So, the graph is plotted like this. At 0 only it touches here, crosses the y-axis, and again it goes.

Now come to $y = x^4$. It will also look like this. But it is not like your $y = x^2$ parabola. It is bit flatter here at 0. However, if x is smaller than 0, then it is not 0. Again if x is slightly bigger than 0, it is not also 0, at 0 only it is 0. And, if we take $y = x^5$, it is similar to $y = x^3$, but it is a bit flatter. We will see how it is coming to be flatter, or what is the meaning of that. Right now it is nonsense. So, $y = x^5$ looks something like $y = x^3$ This is the plot.

Now, let us take some more examples, say, the power function with $n = -1$. You would get the function as $y = 1/x$, which we have seen earlier. It is not defined at 0. Except at 0, everywhere else it is defined. It looks something like this. Its domain does not include 0 and its range also does not include 0. But all positive and all negative numbers are there. So its range is $\mathbb{R} - \{0\}$ and domain is also $\mathbb{R} - \{0\}$, as we have discussed earlier. Its graph will look like this hyperbola.

If you take $n = -2$, then you would get $y = 1/x^2$. Again, at 0, it is not defined. Its domain excludes this 0; and range now cannot be negative because this is $1/x^2$, which is greater than 0 always, because x is never 0 here, so $1/x$ can also never be 0. The range is really all positive real numbers. And the graph would look something like this. At $-x$ and at x, it has the same value. It would look something like this.

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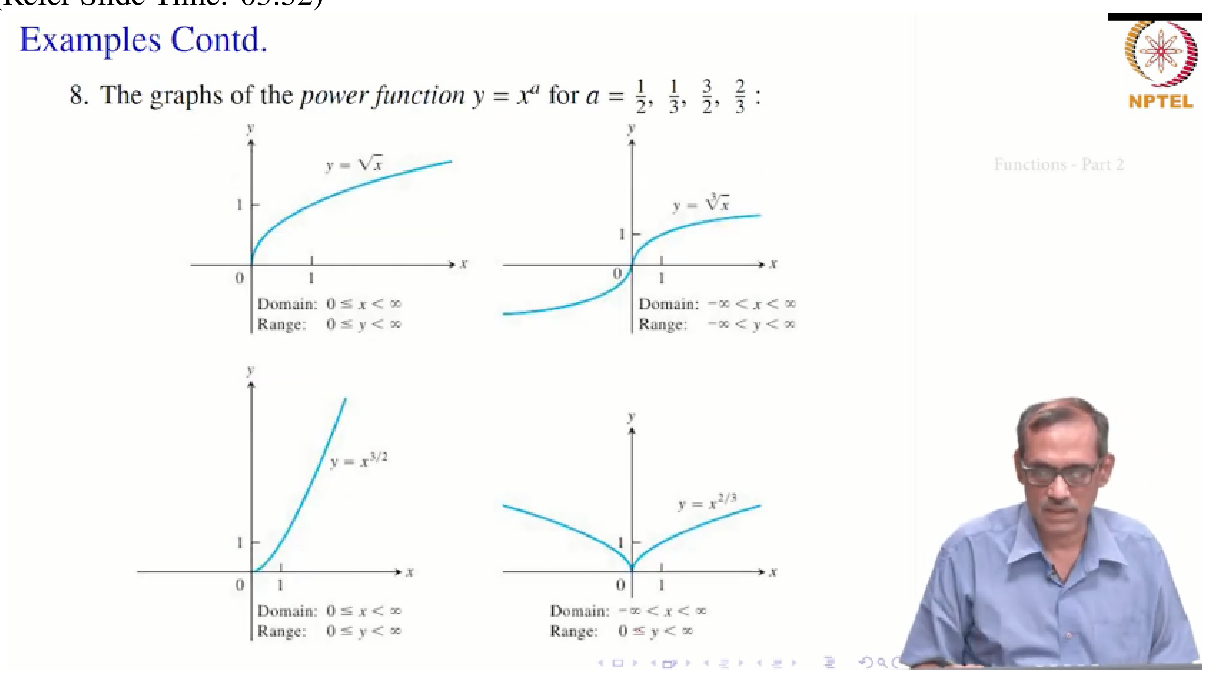

Let us take some rational powers, the fractional powers. Suppose we take the power function $y = x^a$, where a is some fraction like: $1/2$, $1/3$, $3/2$, $2/3$ etc. We will give four examples here, just to have the feeling for how do they look. For $a = 1/2$, let us take $y = \sqrt{x}$. It has domain, as we know, as non-negative real numbers, range is also non-negative real numbers; and it looks like this. At 0, again it is 0.

Take $y = x^{1/3}$, where $a = 1/3$. Then you would get y to be equal to the cube root of x. Now, if you go to $x = y^3$ and look at it in a different way, that is, take the horizontal as y, just rotate it 90 degrees, then the graph of $y = x^3$ and that of $y = x^{1/3}$ look similar now. If you rotate it, it will be giving $y = x^3$ curve. This is your y-axis, think of going along this, and this is the vertical one. Turning from south to north x, it looks something like this. In this way if you take $x^{1/3}$, it looks a bit flatter. This curve looks like bit flatter.

Let us take $a = 3/2$; that is, $y = x^{3/2}$. It is really the square root of x cubed. If you plot it, then it would be looking like this. Its domain is again non-negative real numbers; range is also non-negative real numbers; and it looks something like a portion of a parabola.

Now come to $x^{2/3}$; it is the cube root of x squared. So, it would come this way, meet at 0 and again, goes this way. Its domain is the whole of real numbers because cube root is defined for everything and range of course includes 0. The range is the set of all non-negative real numbers.

Now, let us look at very simple and useful ones, say, polynomial functions. We write a polynomial as $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, where these coefficients are given. For a particular polynomial, like $y = 1 + x + x^2$, all of a_0, a_1 and a_2 are 1.

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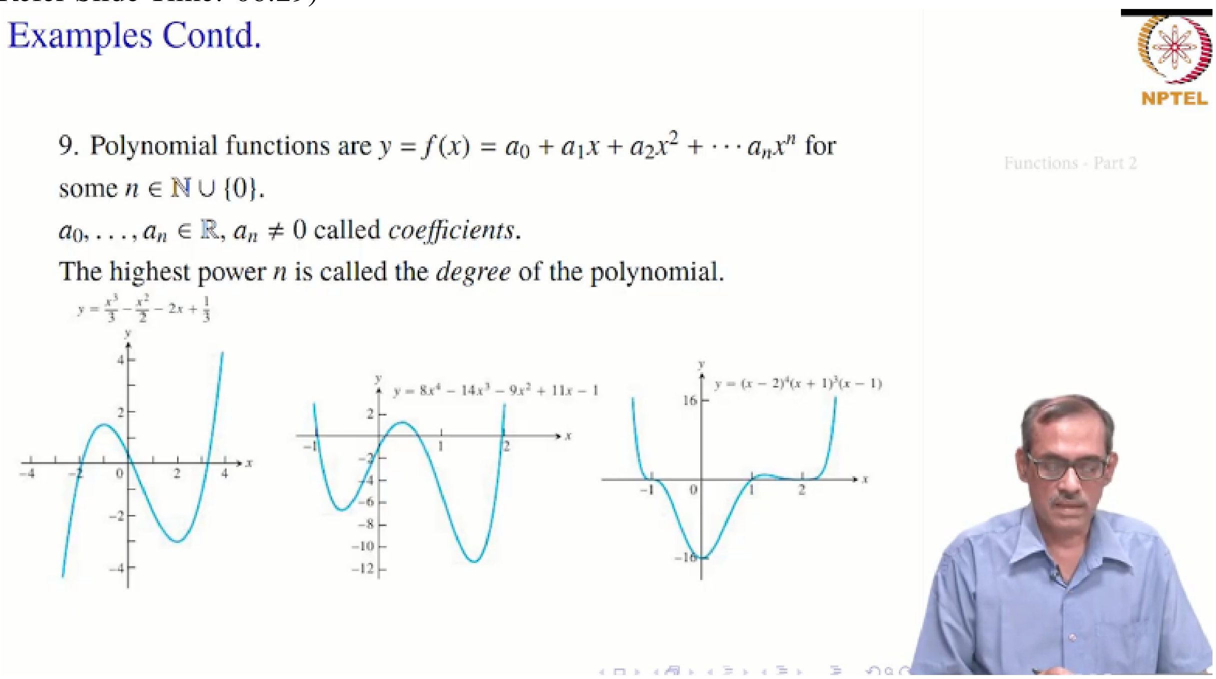

So, this is a generic one, where these $a_0, a_1, a_2, \ldots$ are real numbers. They are called the coefficients of the powers of x. Now this n can be 0. When n is 0, you get the constant function $f(x) = a_0$. If n is 1, you get a straight line $y = a_0 + a_1 x$, and so on. Let us see some graphs.

The highest power of n is called the degree of the polynomial. So, degree is 0 means constant, degree is 1 means linear, degree is 2 means quadratic, and so on. The first one gives a cubic, a cubic generically looks like this. When the coefficient of x^3 is positive, it would look something like this. The graph of $y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + {}_{3}1$ looks like this. It becomes 0 somewhere between 0 and 1, becomes 0 somewhere between minus 2 to 1, close to minus 2, and again between 3 to 4 close to 3. You can factor it and check.

Let us see a quadratic polynomial, or of degree 4. It looks something like this. It has again a zero, it vanishes here at -1 and between 0 to 1, and again near 2. So, this is how a degree 4 polynomial looks like. We took another, which is really of degree 8: $(x-2)^4(x+1)^3(x-1)$. That again looks this way. These are all graphs drawn in the computer. Just have the feeling of how do they look like.

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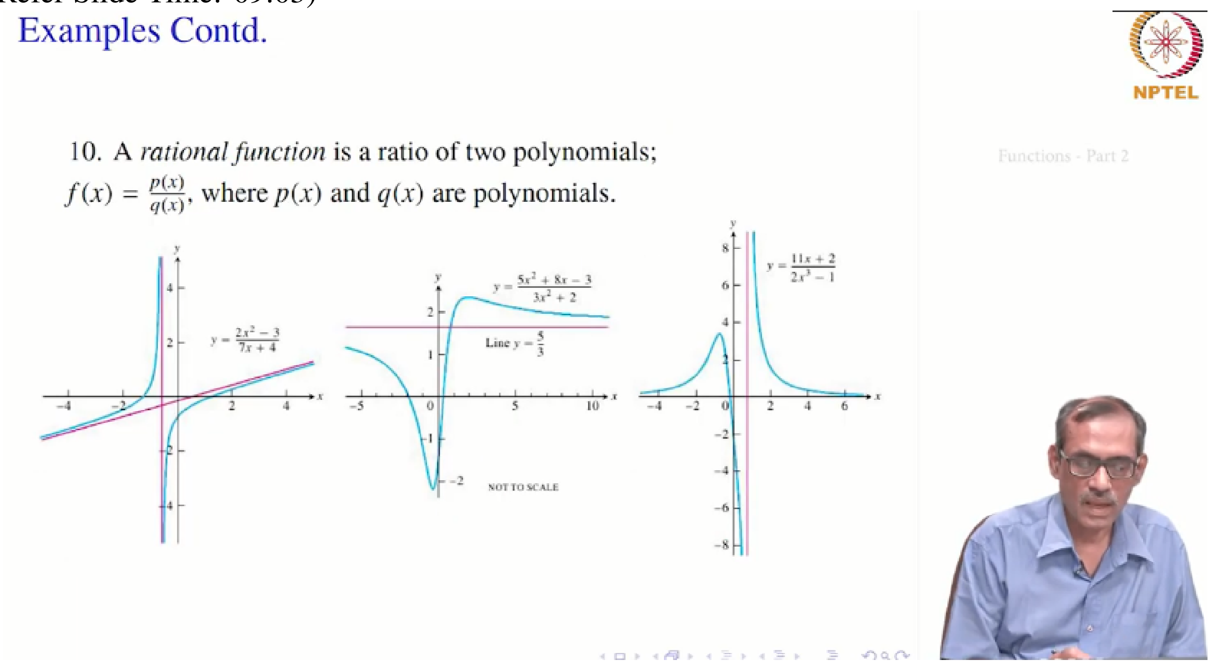

Let us see a rational function. A rational function is the ratio of two polynomials. Here $p(x)$ is a polynomial and $q(x)$ is a polynomial. We have the function $f(x) = p(x)/q(x)$. Naturally this rational function will not be defined for all those x, where $q(x) = 0$. Because we cannot divide by 0. So, except those points, its domain will be everything else, that is, the whole of real numbers minus the zeros of $q(x)$.

For example, $y = (2x^2 - 3)/(7x + 4)$. It is not defined for $x = -4/7$. Except that point it is defined everywhere else. Its graph looks like a hyperbola. They are a bit tricky; they can change shapes. They are not so uniform like the polynomials. So, it looks something like this. This line which is seen as the pink vertical line, meets the curve at infinity. It meets at two points here on the down and also on the top. They are called asymptotes. But we will not be talking about them; we are talking very much in advance now. We will be coming to that slowly after some time. Finding the asymptotes, or where some straight lines meet the curve will give some idea about the shape of the curve.

Let us look at the second one: $y = (5x^2 + 8x - 3)/(3x^2 + 2)$. That is a curve which looks something like this. Look at the blue lines. It is no way connected to the second one. We had seen the first one, because the $q(x)$ there is $3x^2 + 2$. It is not defined when $x^2 = -2/3$, which is not possible in reals. So, the function is defined for real numbers. Its domain is the whole of real numbers. You cannot see any gap here, in the other one you had some gap: $-4/7$. It was not defined at $-4/7$. But here there is no such thing. For every real number it is defined.

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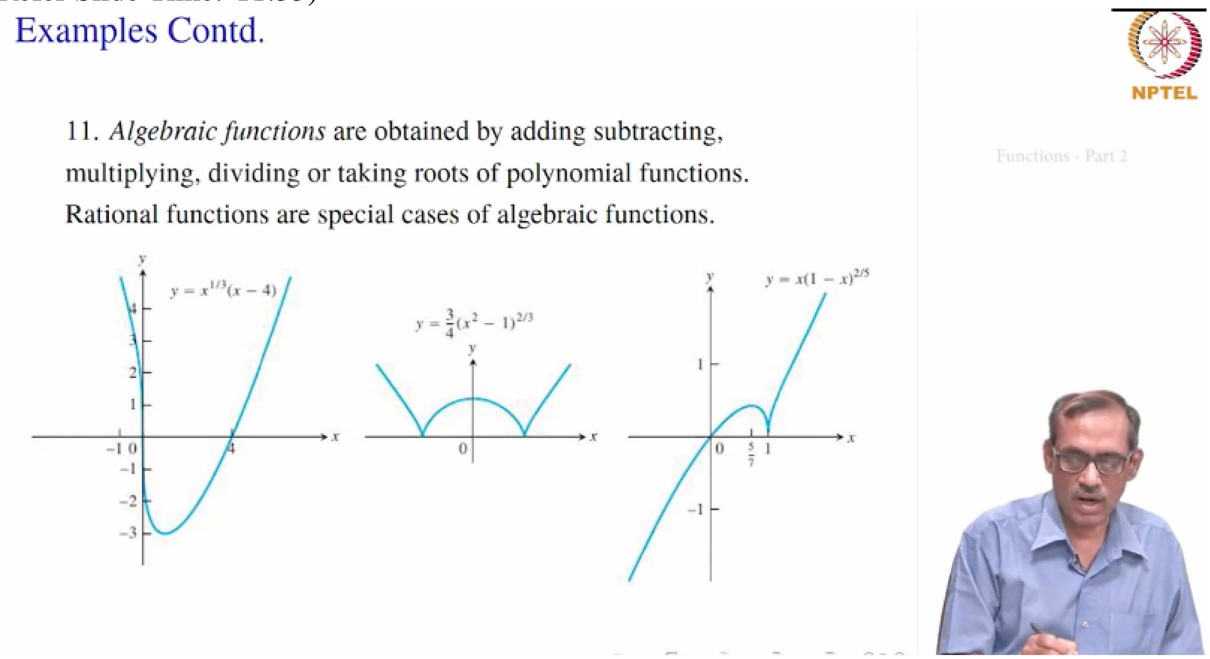

Similarly, take the third one: $y = (11x+2)/(2x^3-1)$. Here, $2x^3 - 1$ can be 0. Where? When $x^3 = 1/2$, or $x = \sqrt[3]{1/2}$. So except this point, it is defined everywhere else. That is how it looks.

These are some facts about the rational functions. In fact, all of these are called algebraic functions. Algebraic functions are those which are obtained by adding, subtracting, multiplying, dividing or taking roots of polynomials. Rational functions are special cases of that. Here are some other plots: $y = x^{1/3}(x-4)$, $y = \frac{3}{4}(x^2-1)^{2/3}$, $y = x(1-x)^{2/5}$. We should try some problems now.

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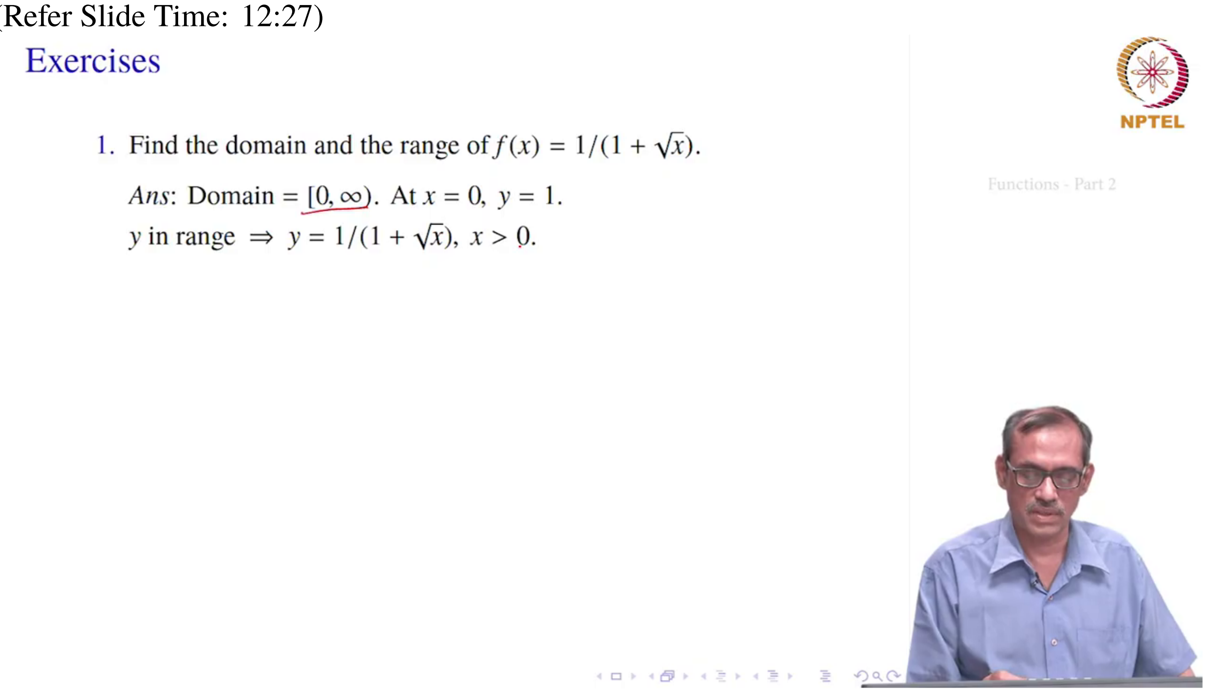

As I told, sometimes we will just give the formula for $f(x)$ and try to see what could be the domain, where this function can be defined, and then what could be the range. For example, here is one. We are asked to find the domain and the range of this function, which is given by the formula $f(x) = 1/(1+\sqrt{x})$. First thing, we should find out where it is not defined, and then see what happens. It is not defined when $\sqrt{x} = -1$. But $\sqrt{x}$ cannot be -1. Can it be? Because by definition, the square root is non-negative. So, that is ruled out. But then, once $\sqrt{x}$ is there, this x has to be non-negative. That is exactly the domain. So, the domain is equal to non-negative real numbers here. At $x = 0$, we get $y = 1$; and y is in the range means that y must be equal to 1 divided by 1 plus root x. We know that $x > 0$ or if $x = 0$, then $y = 1$. We have taken that case first. So, let us take the second case: when $x > 0$.

In this case, we observe something. As x increases, y decreases, since it is 1 by, but it remains positive. So, it cannot go beyond this 1. Therefore, the range must be 0 to 1. But whether it is equal to or not, for that you have to see something more. For each $y \in (0, 1]$, closed 1, open 0, a semi open interval, you should find a corresponding x. And that is quite easy to get. So, this is the range.

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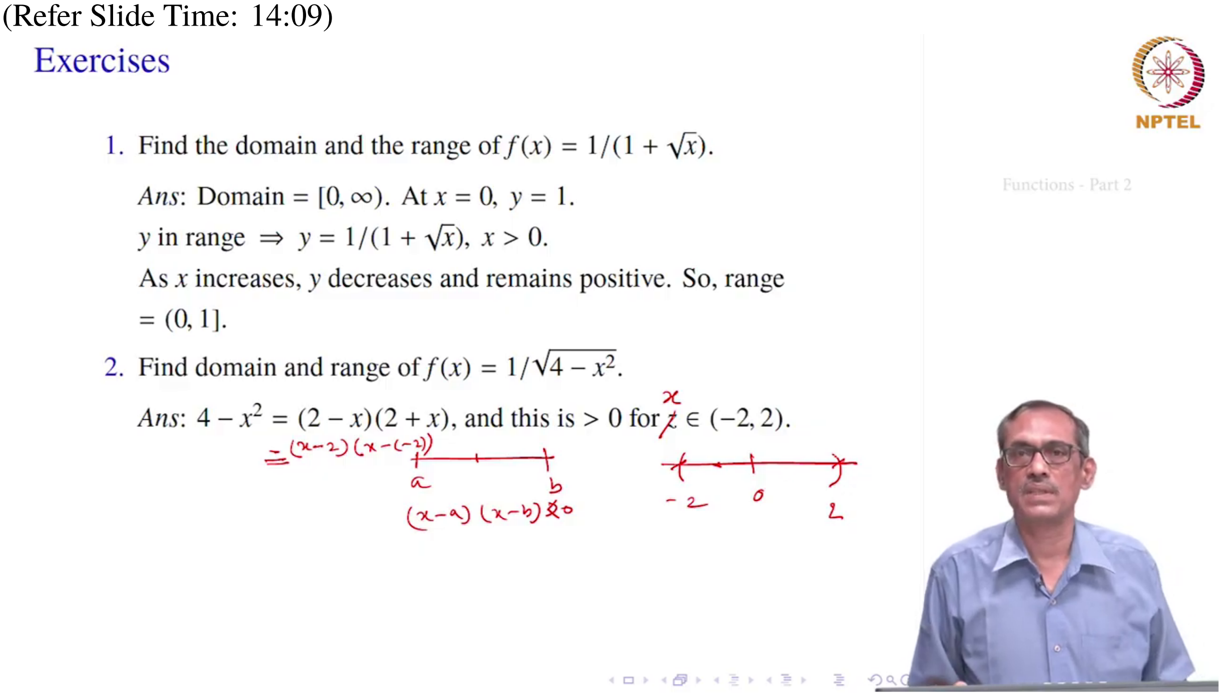

Similarly, here in the second problem, we have to find the domain and range of $f(x) = 1/\sqrt{4-x^2}$. Here again, $4 - x^2$ has to be greater than or equal to 0. But also it is 1 divided by, so it cannot be equal to 0. So, this must be greater than 0. Now if you factor it: $4 - x^2 = (2-x)(2+x)$. So, this must be bigger than 0. For which ones, which x, that is bigger than 0? We would find that for every x, which is between this -2 and 2, this is bigger than 0. You can check it. See that $(-2, 2)$ is an open interval. If x is anywhere here, then it is 2 minus x. This is positive and $2 + x$ is also positive. So, this is bigger than 0. It is really helpful to write in this form. If you take two numbers a and b, you see that $(x-a)(x-b)$ will be positive once x is here. Because, $x - a$ is positive, but $x - b$ is negative, then you would get it to be less than 0, if it is inside this.

Now using the same form, you can write this $x - 2$ with a negative sign into $x - (-2)$. So, $(x-2)(x-(-2))$ will be remaining negative. If it is between minus 2 to 2, and there is a negative sign here, so, it is positive when it is within that. That will be easier to apply. So, the domain should be equal to the open interval $(-2, 2)$.

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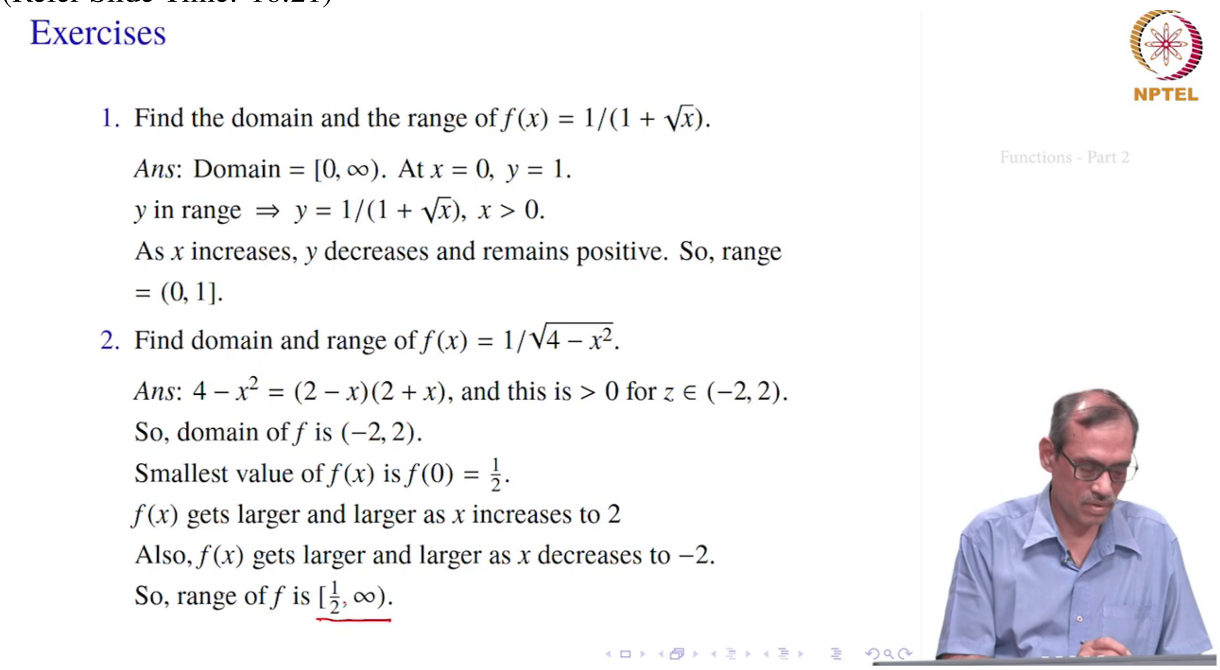

Then what should be the range? For that, we may have to observe something more. The smallest value can be half, you can see. Check from that by inspection. And then $f(x)$ gets larger and larger as x increases. But it can increase only up to 2, not more than that, and it gets larger and larger as x decreases to -2 also. Because it is that factor $(2 - x)(2 + x)$.

So, both of them are decreasing; range cannot go below half. So, it has to be half closed. That is any number bigger than half and also can be equal to half, because at 0 it is equal to half. Therefore, the range is $[1/2, \infty)$, half to infinity where half is closed. That is how it would look.

Let us take one more example. Say the point P in the first quadrant lies in the graph of the function $f(x) = \sqrt{x}$. If you know the graph of $f(x) = \sqrt{x}$, then express the coordinates of P as functions of the slope of the line joining P to the origin. We are asked to find the coordinates of P.

So, P will be a point say, (a, b). There are two numbers here; these should be expressed as functions of the slope of the line joining P to the origin. If you take a point on that curve, you have the origin, then the slope of this line joining those is $\tan\theta$. We have to express the coordinates as some functions of this slope, that is what is being asked.

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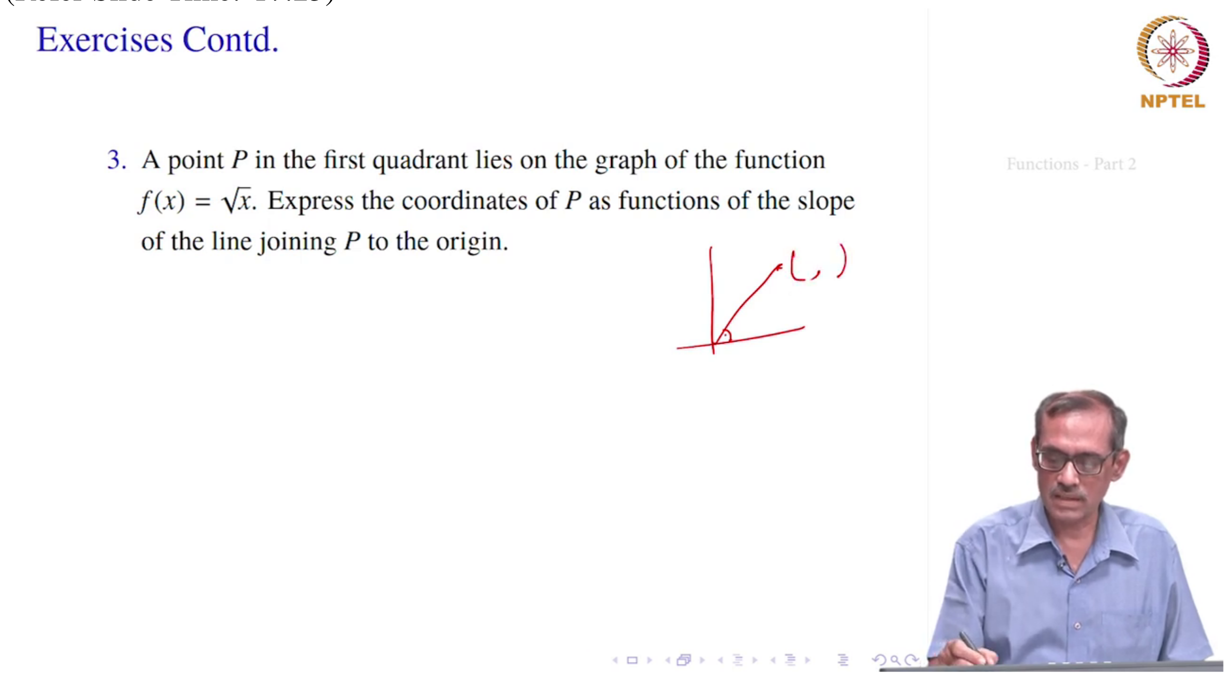

Let us say P is a point on the curve $y = \sqrt{x}$. What is the slope of the line joining P to the origin? It is $(y_2 - y_1)/(x_2 - x_1)$. That gives you $(\sqrt{x} - 0)/(x - 0)$, that is, $1/\sqrt{x}$. So, x has to be greater than 0. Because $\sqrt{x}$ is there anyway. Because the slope $m = 1/\sqrt{x}$ and $P = (x, \sqrt{x})$ we can now write it as $P = (1/m^2, 1/m)$. So, that is the answer.

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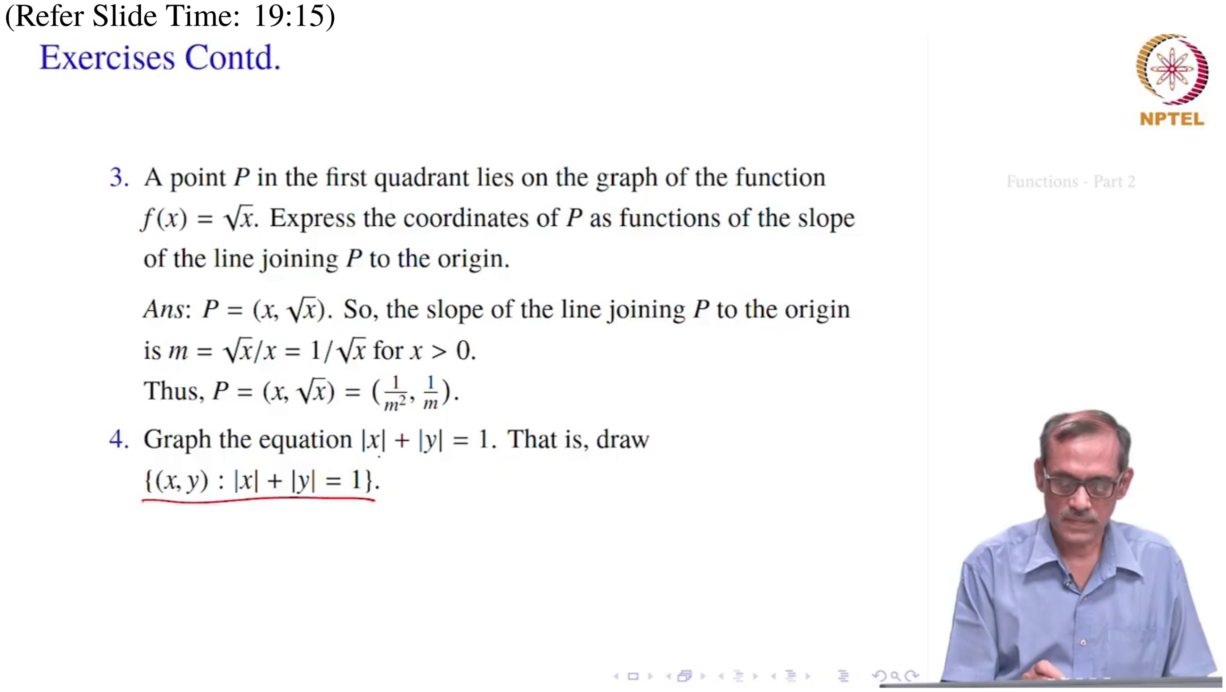

Let us look at the next problem. It asks to plot the graph of the function, which is defined by $|x| + |y| = 1$. It is really an equation. That means you want to find out all those points satisfying this equation: $|x| + |y| = 1$. We have to really plot this set, the set of all (x, y) such that $|x| + |y| = 1$. At some places it will be a function y equal to a function of x. But in total, it may not be a function, like your circle. So, how do you go about it?

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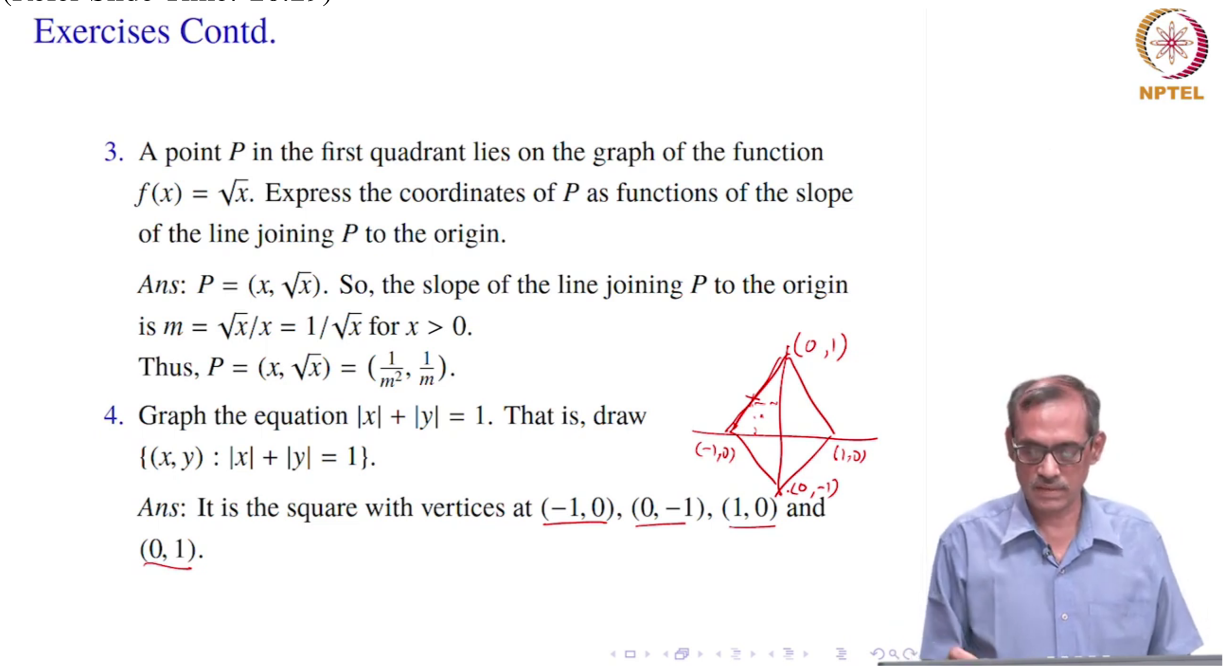

Let us look at some points. When x is 0, y can be 1 or -1, and when y is 0, x can be 1 or -1. So, our points on this are $(-1, 0)$; $(0, -1)$; $(1, 0)$; and $(0, 1)$. If you take the square with these vertices, it looks something like this. This is your $(1, 0)$, this is $(-1, 0)$, this is $(0, -1)$ and ythis is $(0, 1)$. If you take any point here, then that also satisfies $|x| + |y| < 1$. If you take any point on the straight line that again satisfies $|x| + |y| = 1$. So, it is really the square joining these as the vertices. These are the four points at the vertices of this square. So, let us stop here.