Basic Calculus - 1
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Lecture 30 - Part 1
Area between Curves - Part 1

Well, this is lecture 30 of Basic Calculus 1, recall that we had introduced the integral through a particular type of area. Suppose $y = f(x)$ is a curve. And we wanted to find out the integral $\int_a^b f(x)\,dx$. So, we take the area between x-axis, the curve and the lines $x = a$ and $x = b$. This area was defined as the integral $\int_a^b f(x)\,dx$, a definite integral. Now, we turn it around. We take a different kind of area; (another type of area, not of this type) but it is bounded between two curves and the same lines.

Then how to compute this area? Now that we have the fundamental theorem of calculus, we know how to integrate sum of the functions. Then, we use those techniques and express these new types of area as some definite integrals and evaluate the definite integrals for computing the area. That is what we will be doing for areas between curves.

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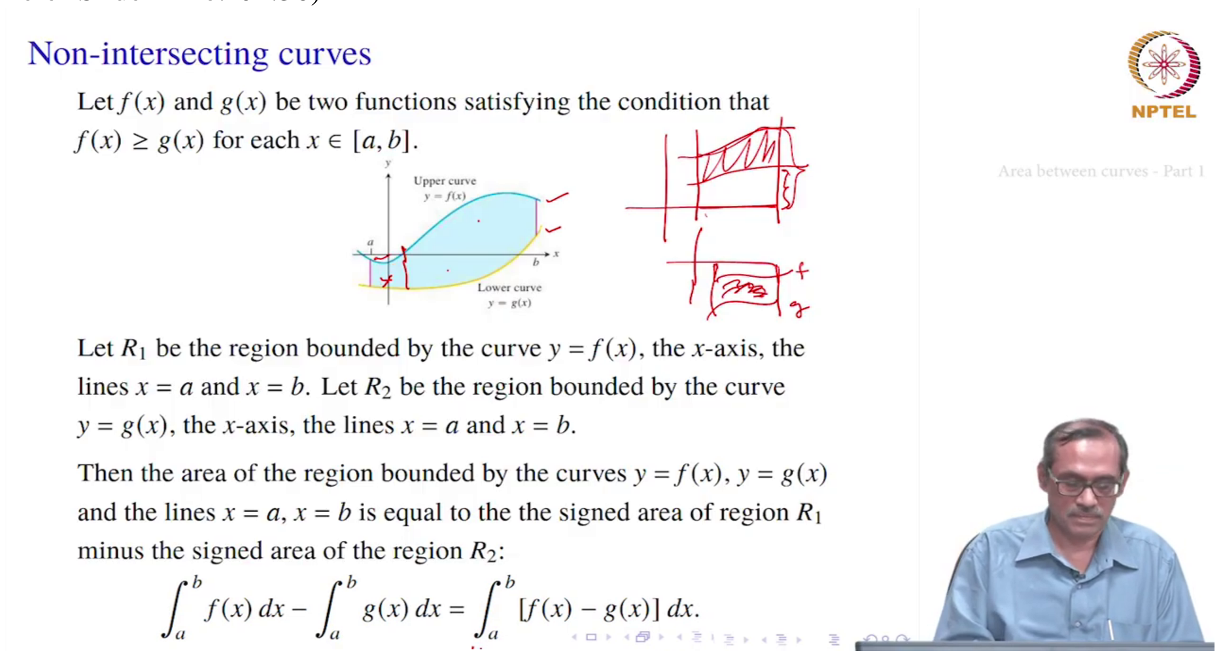

Let us first take the case of non-intersecting curves. Suppose, we have two functions say, $y = f(x)$, which is in blue here, and $y = g(x)$, which is in yellow here. We want to compute the area between these two curves, on the left bounded by the line $x = a$, and on the right bounded by the line $x = b$. This is the area which is painted blue here. How to compute the area of that region? That is our problem now. All the while we have assumed that for any x inside this interval $[a, b]$, $f(x) \geq g(x)$, it is to the top.

We start with this: $f(x)$ and $g(x)$ are two functions satisfying the condition that $f(x) \geq g(x)$. Then, we take R_1 to be the region bounded by the curve $y = f(x)$, the x-axis, lines $x = a$ and $x = b$.

That means, it is this area, where R_1 is the region bounded by $y = f(x)$ and other lines as given. So, it is the union of the required region and this extra region. Let R_2 be the region bounded by the curve $y = g(x)$, the x-axis, and lines $x = a$, $x = b$. It will be the union of these two areas. Now, how to get the area of the region bounded by these two curves with $x = a$ on the left and $x = b$ on the right? You can see that it is the signed areas. So, the required area is that area taken for $f(x)$ minus the area taken for $g(x)$. That is how you would get that area.

It might not be always in this form. You can think of some other types such as $f(x)$ and $g(x)$ both lie above the x-axis, and you have $x = a$ and $x = b$ as they are. Then you can see that the area of this region will be equal to area of which is bounded by $f(x)$ and the x-axis. Again, the required area will be this minus the one which is bounded by $g(x)$ and the x-axis.

that is easier to see in the picture. Sometimes it may happen that both of them are downside, that is, below the x-axis. Here, one can be f another is g. like this. Then again, to get the area you take this area minus the whole area. It is the signed area we are taking about. That will again give us the area of this region. Look at the picture here. It is given that one of them is lying below the other, and here, it is also below the x-axis.

Look at the other one. It has some part above the x-axis and some part below the x-axis. Still you can get the area by taking $f(x) - g(x)$. Look at this area. It is for $f(x)$ and this is for $g(x)$. When you subtract the signed areas, it will be plus of this. And on the other side, you have $f(x)$ bounded by this one and minus of the signed area from where again you get back this area. So, they are plus; which is same thing as $f(x) - g(x)$.

That is how we get $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$. These are these two areas, and that would give us $\int_a^b [f(x) - g(x)]\,dx$. That will be the area painted blue here. We will be using this in many examples and see how one curve is lying above the other, on the top of that, so you get the condition $f(x) \geq g(x)$ for each $x \in [a, b]$. This condition is required.

And of course integrability conditions should be satisfied, we usually assume that $f(x)$ and $g(x)$ are continuous functions. You may say they are continuous here so that these integrals are well defined. Let us apply this. All that we remember is if $f(x) \geq g(x)$ for every $x \in [a, b]$ and you can integrate them, then the area bounded by those two and the lines $x = a$ and $x = b$ is the integral $\int_a^b [f(x) - g(x)]\,dx$.

Let us see the other case, when the two curves intersect. Here, our assumption is that they do not intersect. But if they intersect, then what can we do? Suppose something like this. Here is $f(x)$ and it is $g(x)$ something like this. Then what we do? We break that region into two parts, wherever the intersection point is. Here, the first one has $f(x) \geq g(x)$ and the second one has $g(x) \geq f(x)$. You may have the integrals from a to c and from c to b. The area between those two curves can again be obtained. This will be the integral $\int_a^c [f(x) - g(x)]\,dx$ plus the integral $\int_c^b [g(x) - f(x)]\,dx$. So, once they intersect, this condition will be satisfied that in one of the intervals $f(x) \geq g(x)$ and on the other $g(x) \geq f(x)$. Then we can find out the areas and add them. That is what we will be doing.

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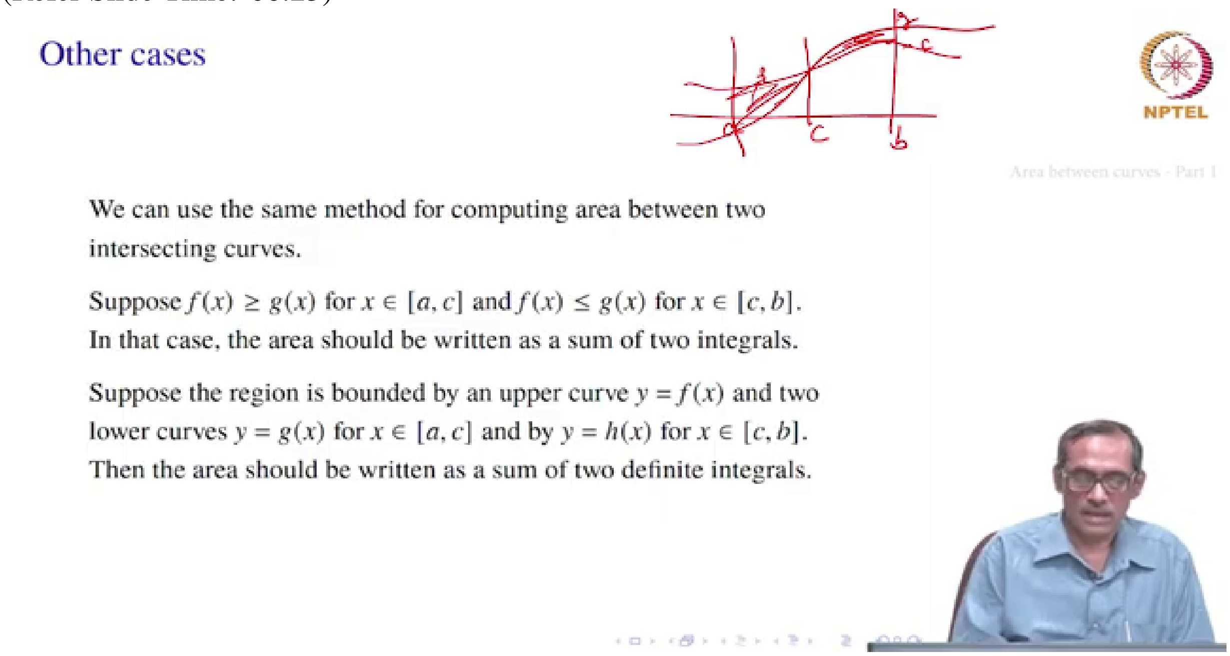

Let us try one example. Find the area of the region enclosed by the line $y = -x$ and the parabola $y = 2 - x^2$. Here, $y = -x$ is this line, in yellow. And the parabola $y = 2 - x^2$ is the blue one. We are not given anything else; no line on the left like $x = a$ or on the right like $x = b$. That means, it is assumed that these two curves intersect at (at least) two points. Between those two points you can find out the region. And it so happens that one is a line another is a parabola.

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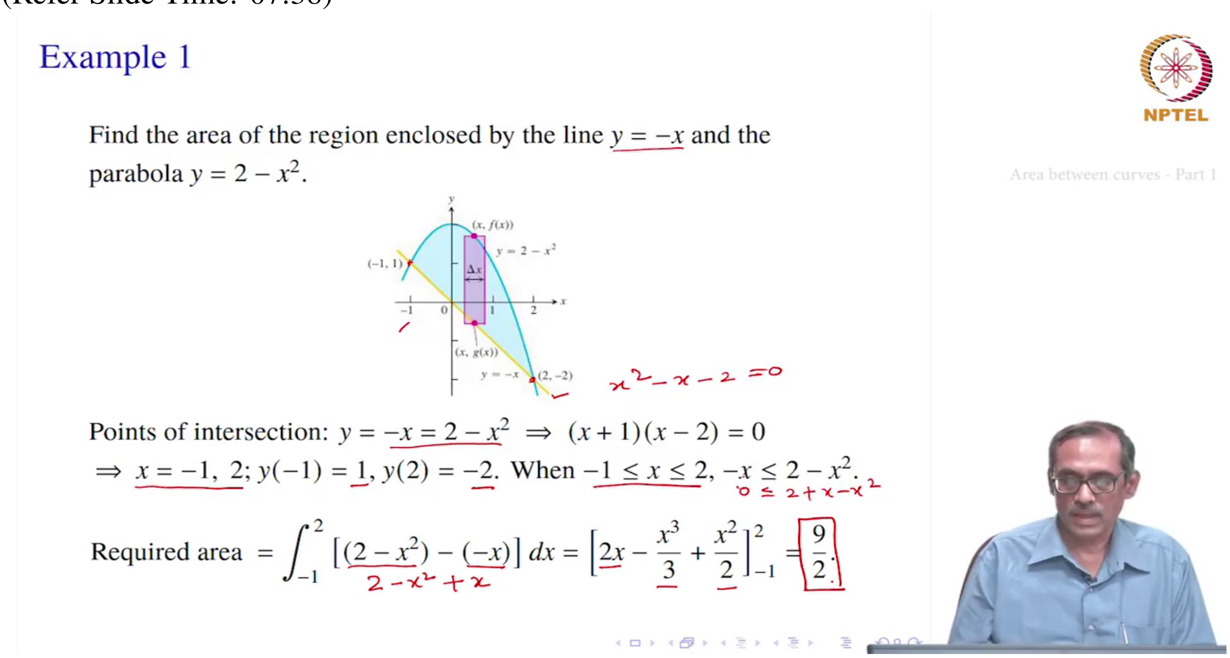

We find that they intersect exactly at two points. You get one point here $x = -1$ and the other point is $x = 2$. The picture may be misleading sometimes, we do not know whether it is plotted correctly or not. Let us find them out by solving the equations. To find the intersection points of $y = -x$ and $y = 2 - x^2$, we eliminate y. We get $-x = 2 - x^2$. That gives $x^2 - x - 2 = 0$, which is

$(x + 1)(x - 2) = 0$ To verify it, just multiply. This gives $x^2 + x - 2x - 2 = 0$. That is all right. We get the intersection points as $x = -1$ and $x = 2$. These are the two intersection points as the plot shows. And we need the values at those points, which we find out from any of them. So, $y = -x$ gives $y(-1) = 1$ and $y(2) = -2$. You have the points of intersection as $(-1, 1)$ here, and $(2, -2)$ here.

Now, you want to find the area of the region, which is painted blue here. We just go for the formula. But before that we should see whether one of the curves is lying always on the top of the other or not. It is from -1 to 2; the plot also says that. But we should verify. When x varies from -1 to 2, we see that $-x \leq 2-x^2$; how? We have $-x-(2-x^2) = x^2-x-2 = (x+1)(x-2)$. When x lies between -1 to 2, this remains less than or equal to 0. So, for these values of x, $-x-(2-x^2) \leq 0$ which implies $-x \leq 2 - x^2$. That means the curve $y = 2 - x^2$ the parabola lies on the top of the curve $y = -x$, y equal to 2 minus x square is greater than or equal to this curve $y = -x$, which is the straight line. So, this condition is satisfied.

Therefore, the required area will be $\int_{-1}^{2} [(2 - x^2) - (-x)]\, dx$. as $2 - x^2 \geq -x$ for $x \in [-1, 2]$. When you get the integral, it is $\int_{-1}^{2} (1 + x - x^2)\, dx$. When integrated, 2 gives $2x$, x gives $x^2/2$ and $-x^2$ gives $-x^3/3$. This is to be evaluated at 2, evaluated at minus 1, and then subtracted. When simplified that becomes $9/2$.

This is how we will be computing areas. But it might so happen sometimes that, you have to look it from the y-axis side instead of x-axis. It may become easier sometimes. We will see that in one of the problems.

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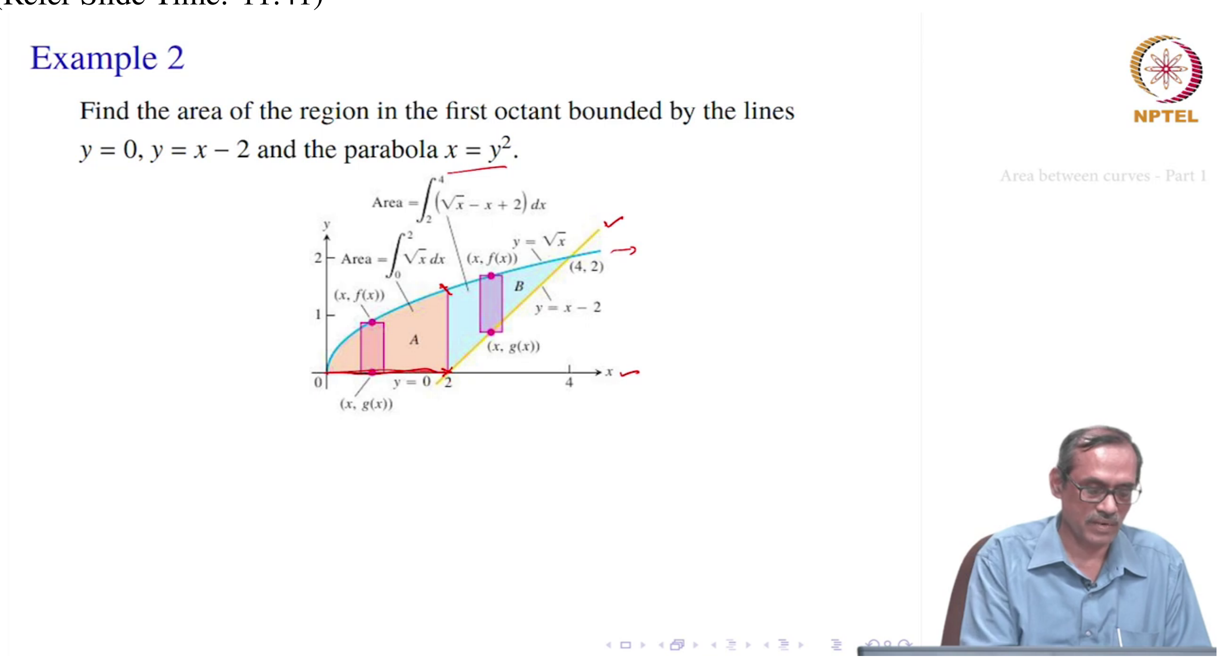

Here is another problem. We want to find the area of the region in the first octant. These curves might intersect somewhere else, but we are not worried, we only want the region in the first octant, which is bounded by the lines $y = 0$, that is, the x-axis, $y = x - 2$, that is this yellow line, and the parabola $x = y^2$, which is the blue curve. This is the region of which we need to compute the area.

You want to find the area of this region. This area is not just bounded by two lines; it is bounded by the vertical lines, and the two curves, as in our earlier problem. But there is something else here. On the below, you have another line, the x-axis. That means we break at the point 2, and we pose this area as the sum of two areas. One is this A, which is painted brown, and another is B, which is painted blue.

For the region A, we have the curve $y = \sqrt{x}$, which is your $x = y^2$ in the first octant. Then, $y = \sqrt{x}$ on the top and $y = 0$ on the bottom. And for B, we have $y = \sqrt{x}$ on the top and $y = x - 2$ below. Those two integrals have to be obtained separately and then added together. That is how we will be getting the area of the region required.

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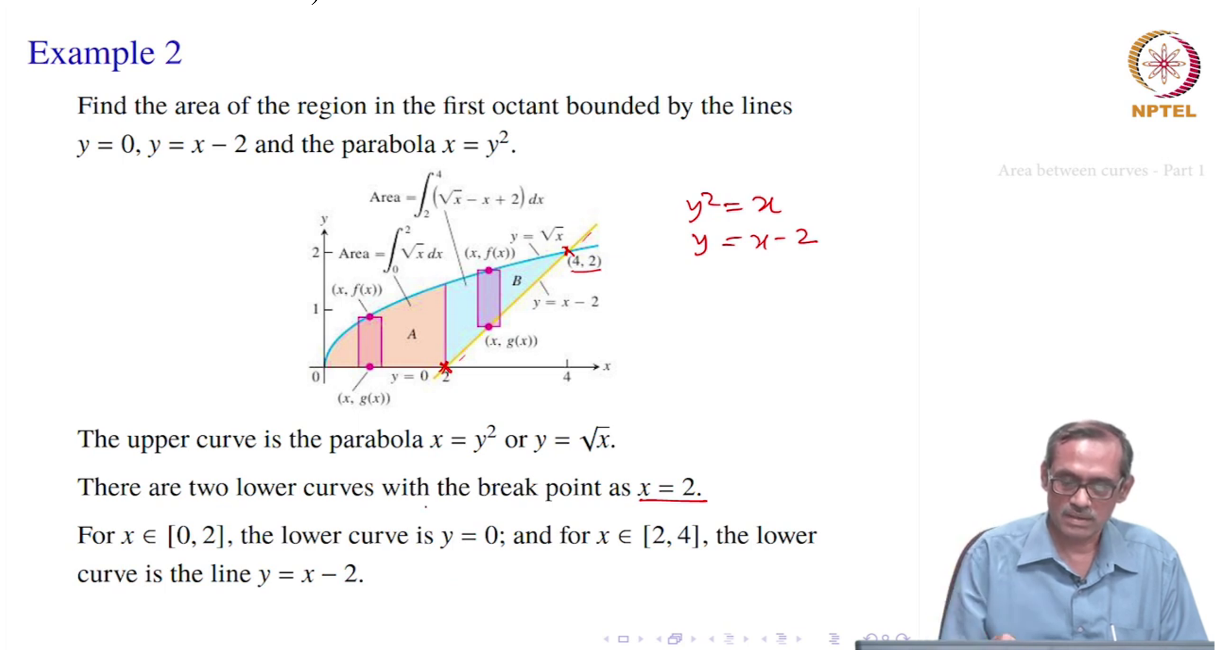

We have the upper curve as $y = \sqrt{x}$. And it has two lower curves with breakpoint at $x = 2$, that is what we mentioned. The point of intersection where this blue and the yellow curves meet is the point $x = 4$. It has to be found out of course algebraically; and that is easy. You have $y^2 = x$ and $y = x - 2$. If you solve them, you get the point as $(4, 2)$. That is how we have obtained the break point at $x = 4$. And this region is again divided into two with the we have another break point at 2; where two different regions are coming. Here, the upper curve and the lower curve are different.

So, when x varies from 0 to 2, the lower curve is $y = 0$, and the upper curve is $y = \sqrt{x}$. That is what we discussed. And on the other interval, when x varies from 2 to 4, the lower curve is the line $y = x - 2$, the yellow one and the upper one is again $y = \sqrt{x}$.

We thus divide the region and then the required area will be equal to the sum of two integrals. It is the integral from 0 to 2, where $\sqrt{x}$ is lying to the upper part of the x-axis, which is $y = 0$. That will correspond to the integral $\int_0^2 [\sqrt{x} - 0]\, dx$. And from 2 to 4, it will correspond to the integral $\int_2^4 [\sqrt{x} - (x - 2)]\, dx$.

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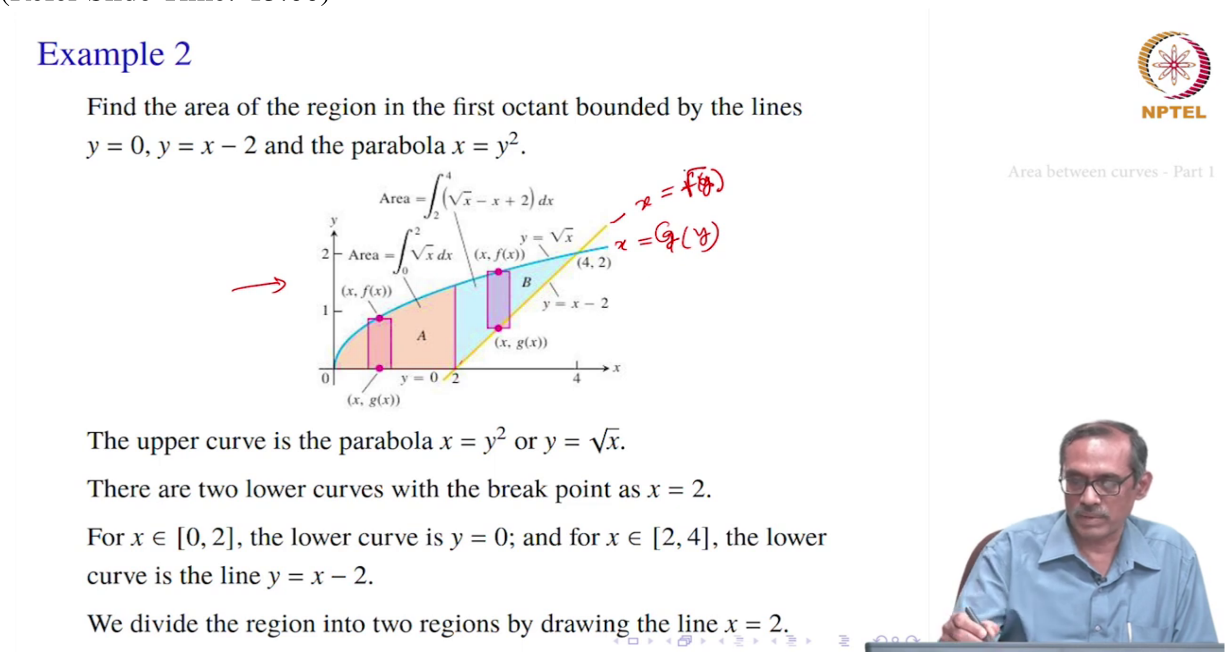

This step is crucial. After that it is usual computation with the integrals. When you take $x^{1/2}$, its integration gives $x^{3/2}/(3/2)$, which is $(2/3)x^{3/2}$. This is to be evaluated at 0 and 2. Similarly, in the other one, $\sqrt{x}$ gives $(2/3)x^{3/2}$ and $x - 2$ gives $(x-2)^2/2$. The whole expression is to be evaluated at 4 and 2, and then subtracted. We then get the answer as $10/3$.

But there is another way of looking at it. Just look at the picture. If you look at the region from the y-axis; that is, view it from the side of y-axis. Then you can see that you have the curve x equal to $f(y)$, a function of y, which is the yellow one. This is x as a function of y; and then you have the parabola x as a function of y again; let us write it as $g(y)$. You see that this is lying from this side. If you look from the side of y-axis, the yellow one is on the top of the blue one.

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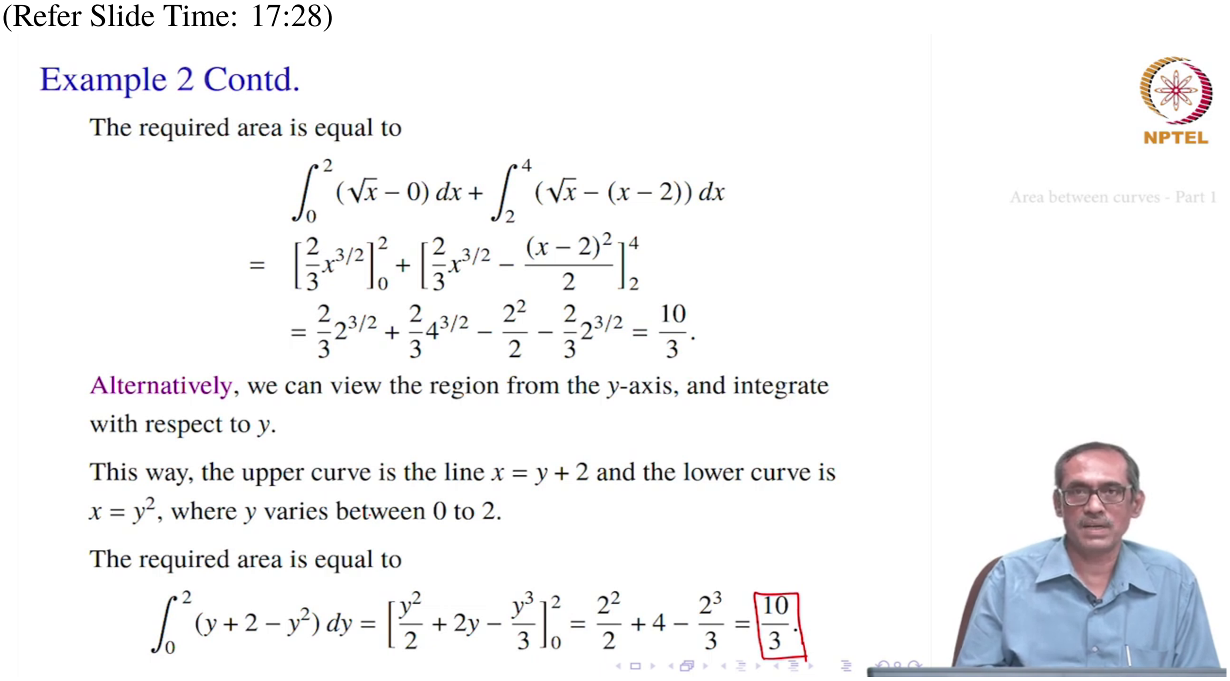

The whole region can be thought of as integrating with respect to y. And that will be the integral of $g(y) - f(y)$. That is what we do in the alternative way.

The upper curve is the $y = x - 2$, which we write as $x = f(y)$, that is, as $x = y + 2$. The lower curve is $x = g(y) = y^2$, where y varies from 0 to 2. That is what you see from the picture. Since the intersection point is $y = 2$, we have y varies from 0 to 2.

So, we write the integral correspondingly. It is $\int_0^2 [(y + 2) - y^2]\, dx$, since $x = y + 2$ is on the top, and $x = y^2$ is in the bottom. We integrate: y gives $y^2/2$, 2 gives $2y$, and $-y^2$ gives $-y^3/3$; and then this is to be evaluated at 0 and 2 and subtracted. That simplifies $10/3$ as earlier.

Seen from the side of x-axis, the area was expressed as a sum of two areas. Whereas, looking at it from y-axis, we could write it as one integral. Sometimes one maybe easier and the other may be difficult; we have to see the picture of the region and then decide which way to move.

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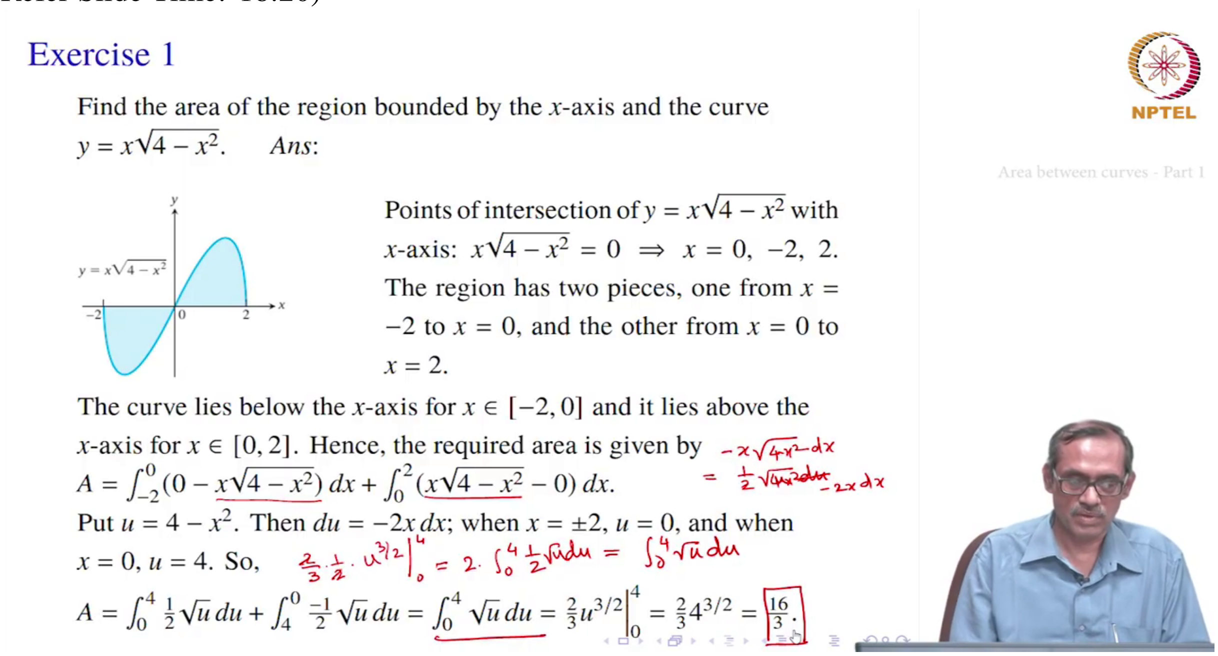

Let us take another problem. Here, we want to find the area of the region bounded by the x-axis and the curve $y = x\sqrt{4 - x^2}$. It is asking to find the region bounded by these two curves, where one is a line, the x-axis and the other one is $y = x\sqrt{4 - x^2}$. Implicitly it is assumed that the curve $y = x\sqrt{4 - x^2}$ intersects the x-axis, at least at two points. Let us find out when does it intersect the x-axis.

To get the intersection, we have $y = x\sqrt{4 - x^2}$ and $y = 0$. Eliminating y we get $x\sqrt{4 - x^2} = 0$. That gives us three points, not only two. Hence, the region bounded by this curve and the x-axis now has two pieces, one from -2 to 0 and another from 0 to 2. Area of each of these pieces have to be computed and then added together.

Now, the area of the first region has x varies from -2 to 0. For this region, the top curve is $y = 0$ and the bottom curve is $y = x\sqrt{4 - x^2}$. So, its area is $\int_2^0 [0 - x\sqrt{4 - x^2}]\, dx$. Of course, by definition, it will be simply the modulus of that, the modulus of $x\sqrt{4 - x^2}$. Since x remains negative here, the modulus will be minus of that. This also gives the the same expression.

And then we add the other area, where x varies from 0 to 2. Here, the upper curve is $x\sqrt{4-x^2}$ and the lower curve is $y = 0$. The area is $\int_0^2 [x\sqrt{4-x^2} - 0]\, dx$.

Now, we evaluate the integrals. We have to integrate $x\sqrt{4-x^2}$. But how? We find that $x\, dx$ is half of $d(x^2)$; or even, we can take $4 - x^2$ directly, since the constant 4 has derivative 0. And, one negative sign will be added. So, we substitute $u = 4 - x^2$. With this u, we find its differential $du = -2x\, dx$. We have to talk about the limits. When $x = -2$, $u = 4 - x^2 = 4 - (-2)^2 = 0$ and when $x = 0$, $u = 4 - 0^2 = 4$. Similarly, when $x = 2$, $u = 0$. We can rewrite the integrals in terms of the variable u now. The expression $x\sqrt{4-x^2}\, dx$ now becomes $(-1/2)\sqrt{u}\, du$. Let us see whether that is correct. As $u = 4 - x^2$, $du = -2x\, dx$. So, $(-1/2)\sqrt{u}\, du = (-1/2)\sqrt{4-x^2}(-2x)\, dx = x\sqrt{4-x^2}$ as required.

Then, the first integral is $\int_0^4 (1/2)\sqrt{u}\, du$ and the second integral is $\int_4^0 (-1/2)\sqrt{u}\, du$, which is equal to $\int_0^4 (1/2)\sqrt{u}\, du$. Their sum is the integral $\int_0^4 \sqrt{u}\, du$. The integral of $u^{1/2}$ is $u^{3/2}/(3/2)$, which is $(2/3)u^{3/2}$. This is to be evaluated at 4 and 0 and subtracted. And, that gives $16/3$.

So, this is how we are going to evaluate the areas, by breaking it whenever we need. But the main thing is that we have to express the region in terms of two functions, $y = f(x)$ and $y = g(x)$, where the region should be bounded by those two curves. That is important. That leads to breaking the region into two parts this way.