Basic Calculus - 1
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Lecture 29 - Part 2
Rule of Substitution - Part 2

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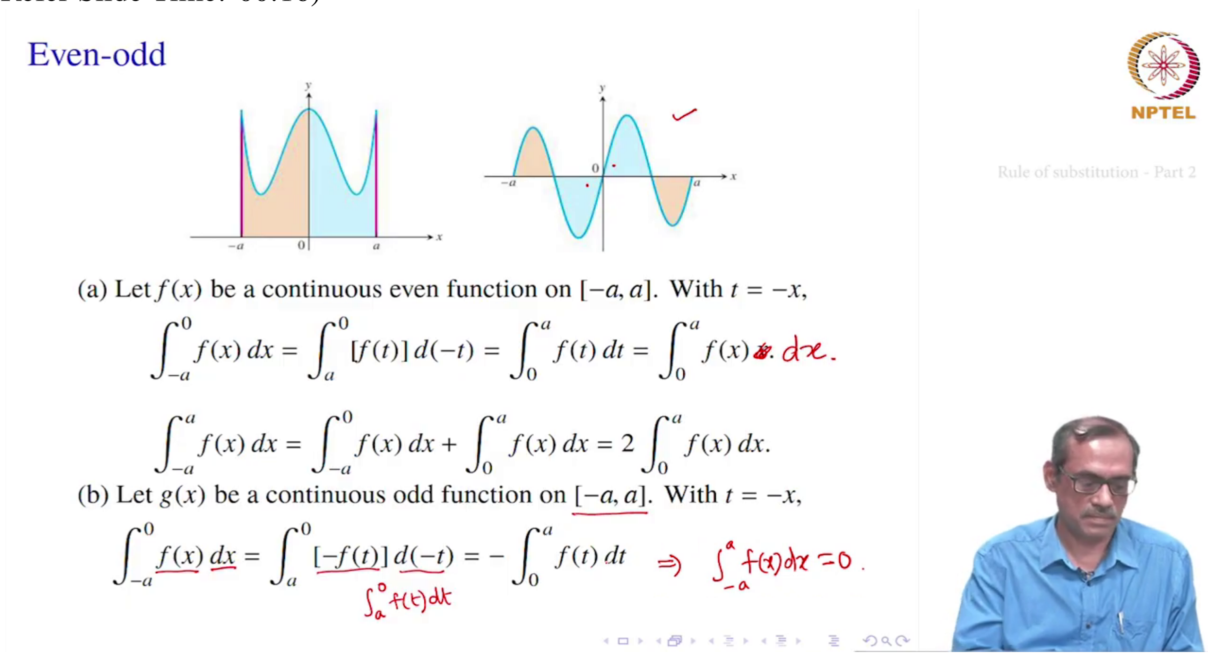

It has to do something with even odd functions. They will be helpful in the substitution while getting the integrals. Recall what is an even function: $f(-x) = f(x)$ for each x; that is what even functions are. For an odd function we have $f(-x) = -f(x)$ for every x.

Suppose we have an even function. That means, it is the first one, $f(x) = f(-x)$. Assume that we have the symmetric interval: $[-a, a]$. It is a very restricted case we are considering. We have an even function and our limits are $-a$ to a. It will be really a sum of two integrals $\int_{-a}^{0} + \int_{0}^{a}$. Consider the first integral. We want to use evenness. If you take $t = -x$, then we would get the limits for t as a to 0. This is so because when $x = -a$, $t = -(-a) = a$ and when $x = 0$, $t = -0 = 0$. The integral is $\int_{a}^{0} f(-t)\,d(-t)$. As it is an even function $f(-t) = f(t)$, and $d(-t) = -dt$. The integral is from a to 0. Accommodating the minus sign, we obtain $\int_{0}^{a} f(t)\,dt$. It does not matter whether we change this variable. We can write this also as $\int_{0}^{a} f(x)\,dx$. The variable does not matter here; it is definite integral, and any variable you could have taken.

So, we can write $\int_{-a}^{a} f(x)\,dx$ as the sum of this with the other one, which was $\int_{0}^{a} f(x)\,dx$. We obtain

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \quad \text{when} \quad f(x) \;\text{ is even.}$$

Sometimes it becomes easier to apply this, because you have to evaluate at 0 instead of at $-a$.

Now, we go for the next one, which is an odd function. So, it looks something like this: $f(-x) = -f(x)$. Suppose $g(x)$ is a continuous odd function on the symmetric interval $[-a, a]$. With the same substitution $t = -x$, we find that in $\int_{-a}^{0} f(x)\,dx$ the limits for t are a and 0, and the integrand is $f(-t) = -f(t)$, and $dx = -dt$. That gives the integral $\int_{-a}^{0} f(x)\,dx$ equal to $\int_{a}^{0} -f(t)(-dt)$. This is equal to $-\int_{0}^{a} f(t)\,dt$. Again, writing x in place of t, we get $-\int_{0}^{a} f(x)\,dx$.

When you take the sum, they cancel out and give 0. So, what do we get? This implies that

$$\int_{-a}^{a} f(x)\,dx = 0 \quad \text{when } f(x) \text{ is odd.}$$

F or an odd function, you can see also geometrically how this happens. This integral is a signed area; so this area and this area really cancel out. Similarly, this area and this area also cancel out. In case of an even function, the areas have the same sign. So, these two are same. Then, the total becomes twice the area of this region.

That is what we say, if it is even, then it is twice; and if it is odd, then it becomes 0; the integral becomes 0. Sometimes it helps to simplify some integrals.

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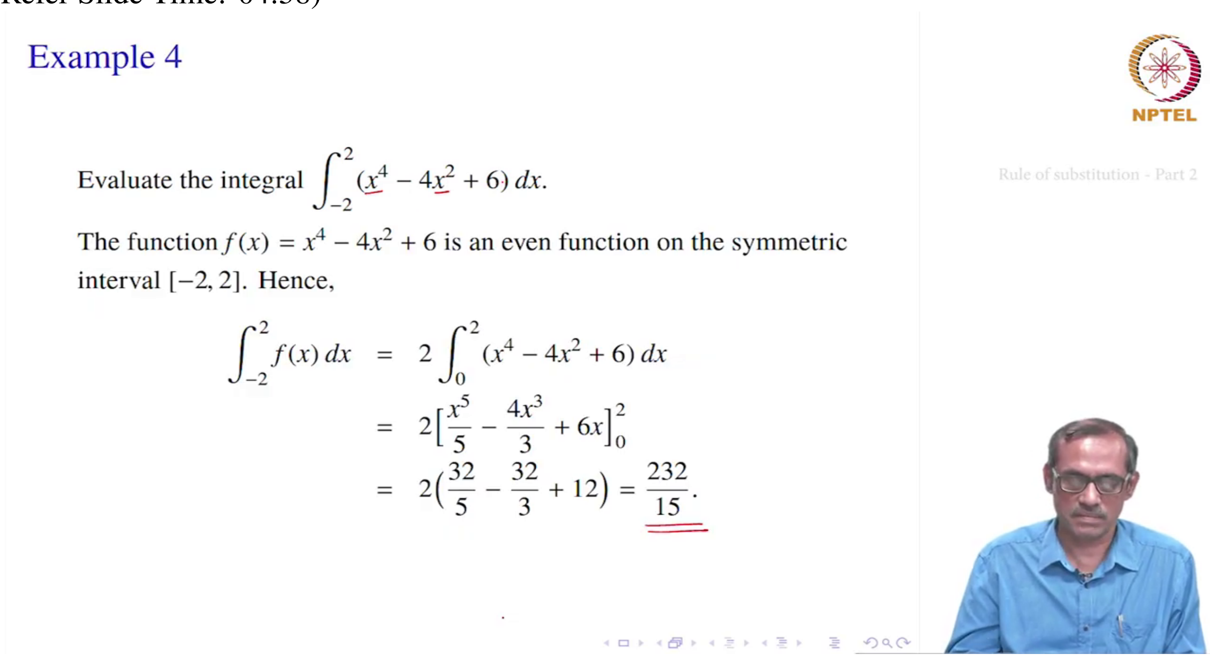

Let us see how it is applied. We want to evaluate the integral $\int_{-2}^{2}(x^4 - 4x^2 + 6)\,dx$. Is it even odd? Well, $[-2, 2]$ is a symmetric interval; we know that x^4 is even, x^2 is even, and 1 is also even. So, this is really an even function. We can straight forward write that $\int_{-2}^{2}$ is twice of $\int_{0}^{2}$ of the same function. The integral is equal to $2\int_{0}^{2}(x^4 - 4x^2 + 6)\,dx$, which can be integrated now.

Upon integration, x^4 gives $x^5/5$, $-4x^2$ gives $-4x^3/3$ and 6 gives $6x$. It has to be evaluated at 0 and 2, and then subtracted. When you simplify, you would get the answer as $232/15$.

So, instead of evaluating this whole expression again at -2, we just evaluated at 0; that became easier. That is how evaluation of definite integrals becomes easier by using even or odd functions.

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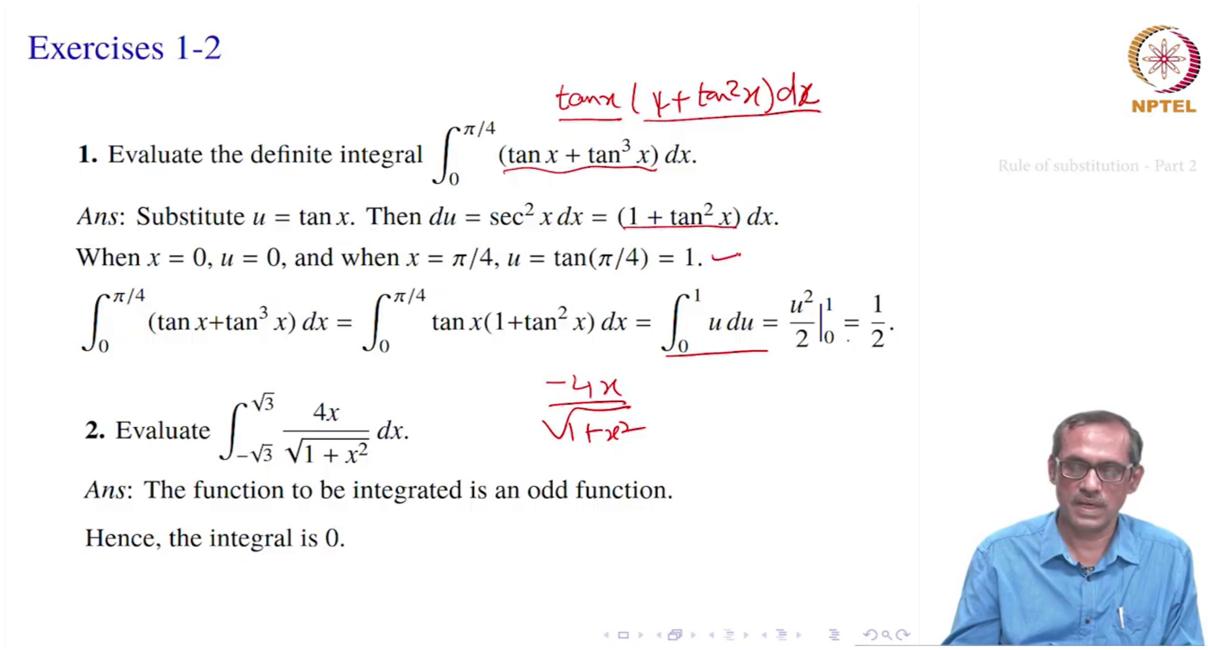

Let us see some more problems. We want to evaluate the definite integral again. It is $\int_0^{\pi} (\tan x + \tan^3 x)\,dx$. As we have decided, take some time, pause the video here, and then come back to this after you have proceeded up to some steps.

Well, how do we go about it? Of course, this can be integrated, but using logarithmic functions, which we have not introduced till now. Well, we substitute $u = \tan x$. That $\tan x$ is creating problems; so, let us take $u = \tan x$. Then, du becomes $\sec^2 x\,dx$, which is $(1 + \tan^2 x)\,dx$. Now you can see that this substitution is helping us; this integrand is really $\tan x(1 + \tan^2 x)$. So, this is your u and this is du. You have to find, of course, the limits. When $x = 0$, $u = \tan 0 = 0$, and when $x = \pi/4$, $u = \tan(\pi/4) = 1$. So, the limits are from 0 to 1; the integral is $\int_0^1 u\,du$. That gives $u^2/2$ evaluated at the limits 0 and 1, and subtracted. At 0, it is 0; and that gives $1/2$.

That is how it will go. But we will be lucky if it comes in that form. If it does not, then of course, we will not be able to integrate now, something else may be required.

Let us go to next problem. Evaluate the integral $\int_{-\sqrt{3}}^{\sqrt{3}} 4x/\sqrt{1+x^2}\,dx$. Since the interval is symmetric, let us find whether the integrand is even or odd. In $4x/\sqrt{1+x^2}$, if you substitute $-x$, then you would get $-4x/\sqrt{1+(-x)^2} = -4x/\sqrt{1+x^2}$. Hence, it is an odd function. Hence, the integral is 0. We do not have to really evaluate it, because it is an odd function.

Let us take the next problem. We want to evaluate the integral $\int_0^1 10\sqrt{x}/(1+x^{3/2})^2\,dx$. The problem is this $x^{3/2}$ and also $\sqrt{x}$. We know that the derivative of $x^{3/2}$ is $(3/2)\sqrt{x}$. So, we can think of the differential of $x^{3/2}$ as $\sqrt{x}\,dx$ with some constant multiplied. But the derivative of 1 is 0; so we can think of $1 + x^{3/2}$ instead of $x^{3/2}$. That is how we are going to substitute and get the result.

So, we substitute $u = 1 + x^{3/2}$. Then du becomes $(3/2)\sqrt{x}\,dx$. We go for the limits. When $x = 0$, $u = 1 + 0^{3/2} = 1$, and when $x = 1$, $u = 1 + 1^{3/2} = 2$. Therefore, the integral can be written as $\int_1^2 10(2/3)(1/u^2)\,du$.

Notice that $du = (3/2)\sqrt{x}\,dx$. So, in place of $\sqrt{x}\,dx$ we can substitute $(2/3)\,du$. And $(1+x^{3/2})^2$ now becomes u^2. The limits 0 to 1 become 1 to 2. That is how we get the equality there.

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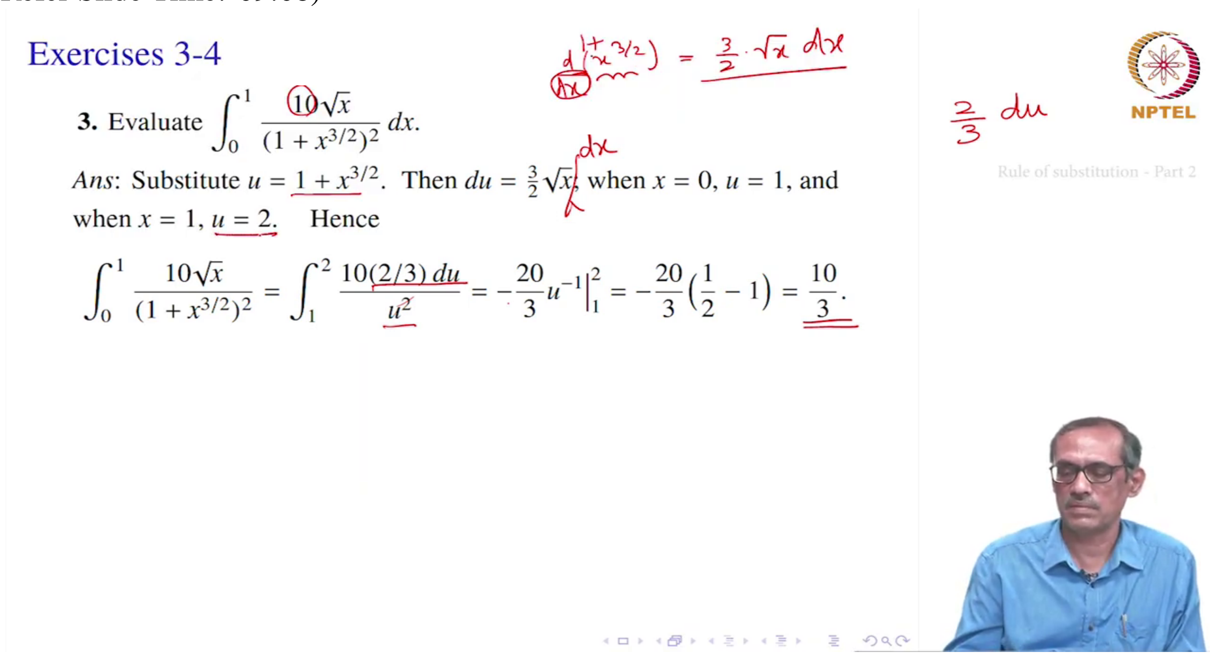

Now, this integral is is really $20/3$ multiplied by integral of $1/u^2$, which is $-1/u$. This is so because the integral of u^{-2} is $u^{-2+1}/(-2+1) = -u^{-1}$. So this $-(20/3)u^{-1}$ is to be evaluated at 1 and 2, and subtracted. That gives $-20/3$ into half, as $u = 2$, and -1 when $u = 1$. That simplifies to $10/3$.

Notice that our observation that something appears here whose derivative is multiplied there helped us evaluate the integral. That is the main thing we have to see, and then take that as the new variable. That is what the rule of substitution allows us to do. But observing these these are yours. Rule of substitution does not say which one to substitute; you have to get it from the problem.

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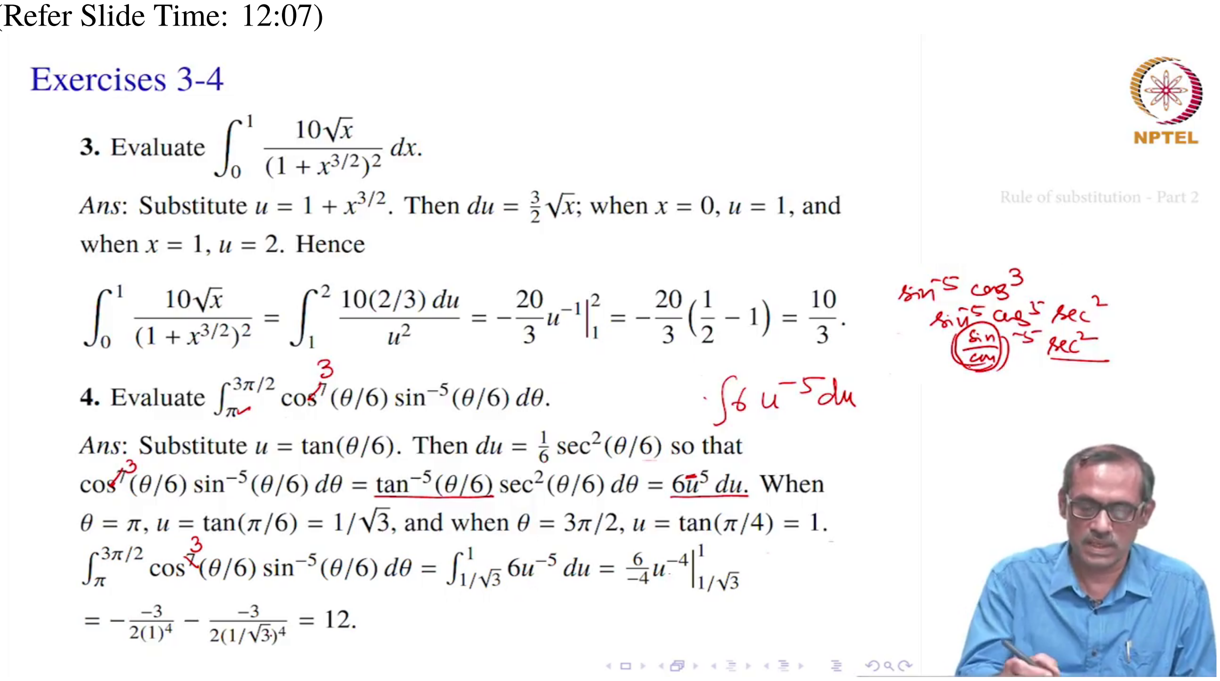

Let us take the next problem. We want to evaluate the integral $\int_\pi^{3\pi/2} 2\cos^7(\theta/6)\sin(-5\theta/6)\,d\theta$.

Well, there is a difference of 2, we correct it to $\int_{\pi}^{3\pi/2} \cos^3(\theta/6)\sin^{-5}(\theta/6)\,d\theta$. Because is is $\cos^3$, it will be easier to do with $\tan x$. We can write $\sin^{-5}$ and $\cos^3$ as $\tan^{-5}\sec^2$. (Verify: $(\sin/\cos)^{-5}$ times $\sec^2$ is same as $\sin^{-5}\cos^2$.) The derivative of this tan is really $\sec^2$. So, you can think of $\tan(\theta/6)$ as u. That will make it easier.

Let us substitute $u = \tan(\theta/6)4$. Then du is equal to the derivative of $\tan(\theta/6)$, which gives $\sec^2(\theta/6) \times (1/6)$. What about the other function? It is $\tan^{-5}(\theta/6)$, which is equal to u^{-5}. So, the integral is $\int 6u^{-5}\,du$. Because $du = (1/6)\sec^2(\theta/6)\,d\theta$. So, So, that is how the integral now looks like, it will be $\int 6u^{-5}\,du$. We have to take the limits, of course. When $\theta = \pi$, $u = \tan(\theta/6)$ becomes $\tan(\pi/6) = 1/\sqrt{3}$; and when $\theta = 3\pi/2$, $u = \tan(3\pi/12) = tan(\pi/4) = 1$. So, we can write

$$\int_{\pi}^{3\pi/2} 2\cos^7(\theta/6)\sin(-5\theta/6)\,d\theta = \int_{1/\sqrt{3}}^{1} 6u^{-5}\,du.$$

When you integrate, u^{-5} gives $u^{-4}/(-4)$. Then, the answer is 6 times this evaluated as the limits and then subtracted. That gives $-6u^{-4}/4\big|_{1/\sqrt{3}}^{1}$. It simplifies to 12. That is how it looks.

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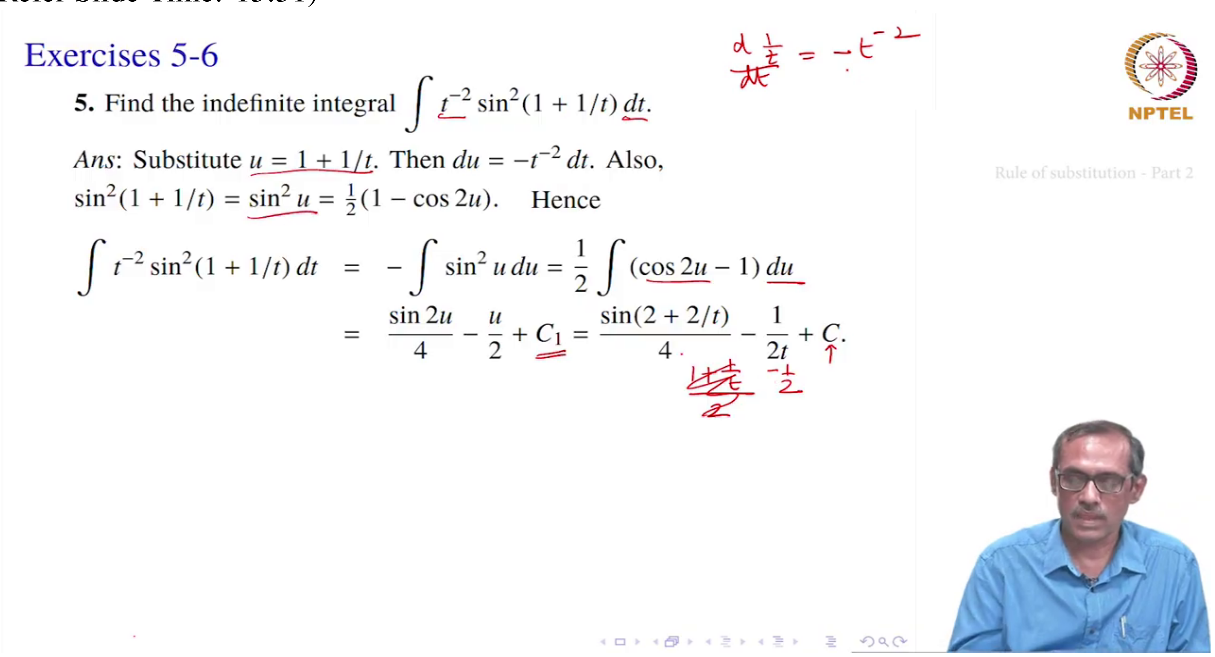

Let us go to next problem. Find this indefinite integral. Now, it is not a definite integral. The indefinite integral is $\int t^{-2}\sin^2(1+1/t)\,dt$. Here, we have t^{-2} and we have $1/t$ here. We know that the derivative of $1/t$, that is, the derivative of t^{-1} is $-t^{-2}$. If you add 1 to that, of course, you would get the same derivative. It thus suggests that we substitute a new variable for $1 + 1/t$.

That is what we do. Let us write $u = 1 + 1/t$. Then, $du = -t^{-2}\,dt$. That accounts for this differential. And $\sin^(1 + 1/t)$ becomes $\sin^2 u$. But we do not have a formula to integrate $\sin^2$ directly. We rewrite it as $(1/2)(1 - \cos(2u))$. Then the indefinite integral becomes

$$\int t^{-2}\sin^2(1+1/t)\,dt = \int -\sin^2 u\,du = \int \frac{1}{2}(\cos(2u) - 1)\,du.$$

Now $\cos(2u)\,du$ gives you $\sin(2u)/2$ and one half is here. So, it is $\sin(2u)/4$. And -1 here gives

$-u$, half is there, and then plus C_1. That is the indefinite integral. It is equal to $\sin(2u)/4 - u/2 + C_1$. We then simplify after putting in the substitution. It is in u now; it has to be transferred to t. You substitute back. It is $\sin(2+2/t)/4-(1+1/t)/2+C1$. This give $\sin(1/2+1/(2t))-1/(2t)-1/2+C_1$. Now C_1 is an arbitrary constant. So, $C_1 - 1/2$ can be written as a new constant C. That is how the indefinite integral will look like now, for any arbitrary constant C.

See, how we accommodated some particular constant with the arbitrary constant. In an indefinite integral, there is no need to keep this minus half. Also we could have kept this minus half plus C_1 as it was. That is also correct.

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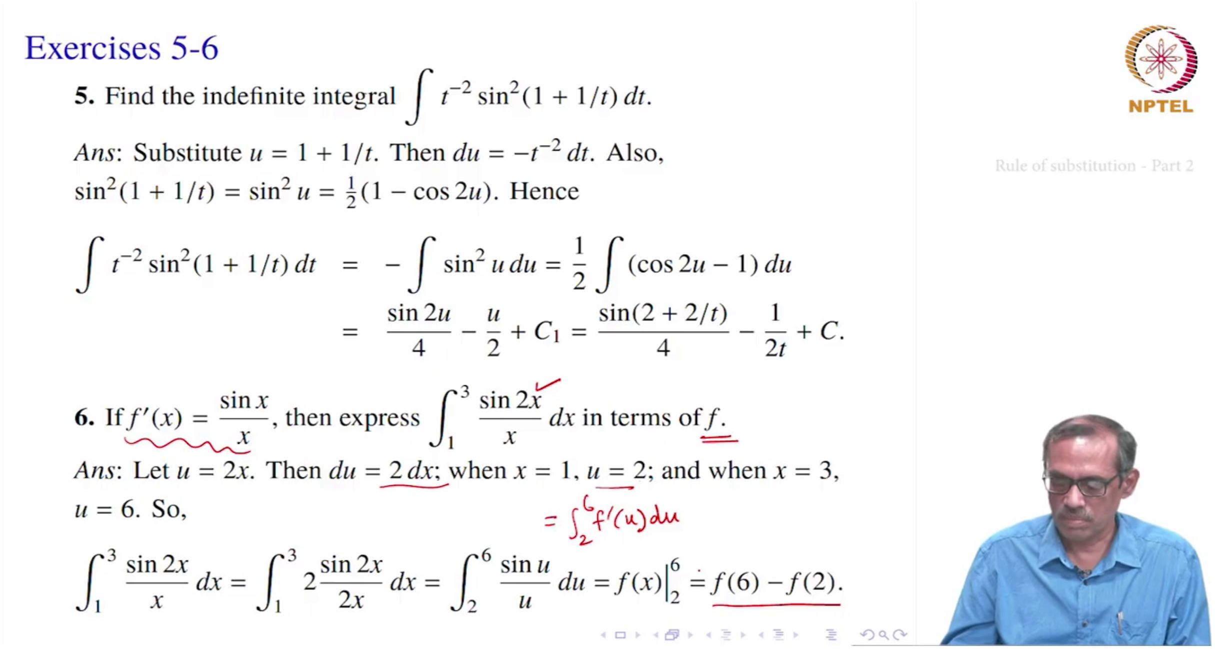

Let us go to the next problem. If $f'(x) = \sin(x)/x$, then we want to express the integral $\int_1^3 \sin(2x)/x\,dx$ in terms of f. It is a complicated one. We want to evaluate this integral or express this integral in terms of a function whose derivative is $\sin(x)/x$. We do not know exactly what is the function whose derivative is $\sin(x)/x$. It looks that the rule of substitution would help.

Now you want to express it in terms of the function whose derivative is $\sin(x)/x$. We should have $\sin(x)/x$ here first. Then, we can think what to do. Let us consider $\sin(2x)$, say, we take $u = 2x$. With that, we have $du = 2\,dx$; and when $x = 1$, $u = 2$; when $x = 3$, $u = 6$. It is a twice of that. So, $\int_1^3 \sin(2x)/x\,dx$ can be written as $\int_2^6 \sin(u)/u\,du$.

Now, $\sin(x)/x$ is given to be $f'(x)$ for some function $f(x)$. So, $\sin(u)/u = f'(u)$. Then the integral is equal to $\int_2^6 f'(u)\,du$. By the fundamental theorem of calculus, this integral is equal to $f(6) - f(2)$.

Now, we are able to express the integral in terms of f. And in the abstract, the indefinite integral will be equal to $f(u) + C$. When you differentiate it, you would get back the integrand. Let us stop here.