Basic Calculus - 1
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Lecture 29 - Part 1
Rule of Substitution - Part 1

Well, this is lecture 29 of Basic Calculus 1. In the last one or two lectures we had covered the fundamental theorem of integral calculus or we just said fundamental theorem of calculus, and then discussed some of its applications to problems. Today we will be discussing the rule of substitution, which greatly simplifies our work while doing integration of some functions. We will see how does it work.

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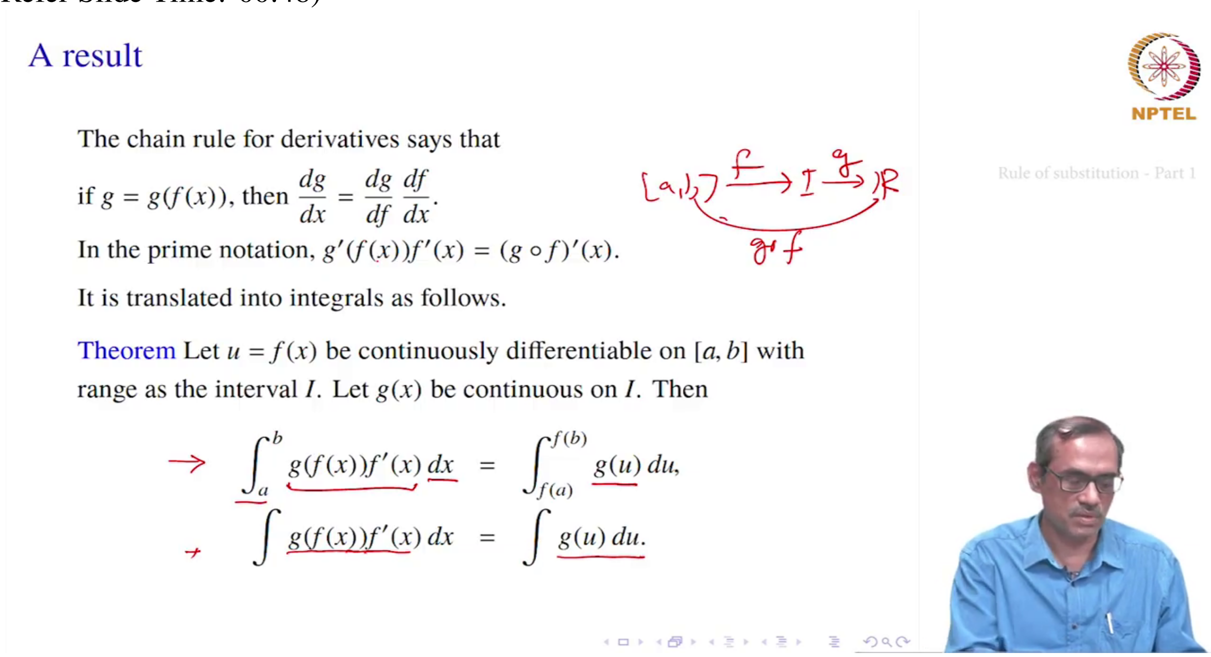

The first thing is that the rule of substitution does not come out of the blue. It comes from the chain rule for derivatives. Recall that the chain rule says that if you have a composite function like $g(f(x))$, write that as G for simplification, then dG/dx will be equal to dG/df times df/dx. That is what the chain rule for derivatives says. Now, if you write in the prime notation it will be $(g \circ f)'(x) = g'(f(x))\, f'(x)$.

We translate that into integrals. The function to be integrated is sometimes called the integand. Suppose the integral is in this form $\int_a^b g(f(x))\, f'(x)\, dx$. The integrand here is $g(f(x)f'(x)$. The function to be integrated is $g(f(x))\, f'(x)$. It looks like the earlier chain rule, which looks like $dg/df \times df/dx$. Suppose we put $u = f(x)$. Then, the integrand now is $g(u)u'$.

These are all necessary conditions we need to put so that this composition would work. Let $u = f(x)$ be continuously differentiable on the closed interval $[a, b]$. Since it is a continuously differentiable function, it is continuous. Also, its range is assumed to be a closed bounded interval I. Suppose g is also continuous on I. That means you have f that takes $[a, b]$ to I, and then g

takes I to $\mathbb{R}$. The composite function $g \circ f$ is defined from $[a, b]$ to $\mathbb{R}$. You can write $g(f(x))$ as $g(u)$ as that may be easier to handle. Let us look at the second one. The indefinite integral $\int g(f(x))\, f'(x)\, dx$ is equal to the integral $\int g(u)\, du$. This u is an abbreviation of $f(x)$.

In terms of the definite integral, when you have the limits, the first left side has the limits for x as a to b, but on the right side you have the limits for u, which is $f(x)$, as $f(a)$ and $f(b)$. That is how they are related to the general. We will see in the proof how the chain rule is used to derive these. It will be obvious from the proof.

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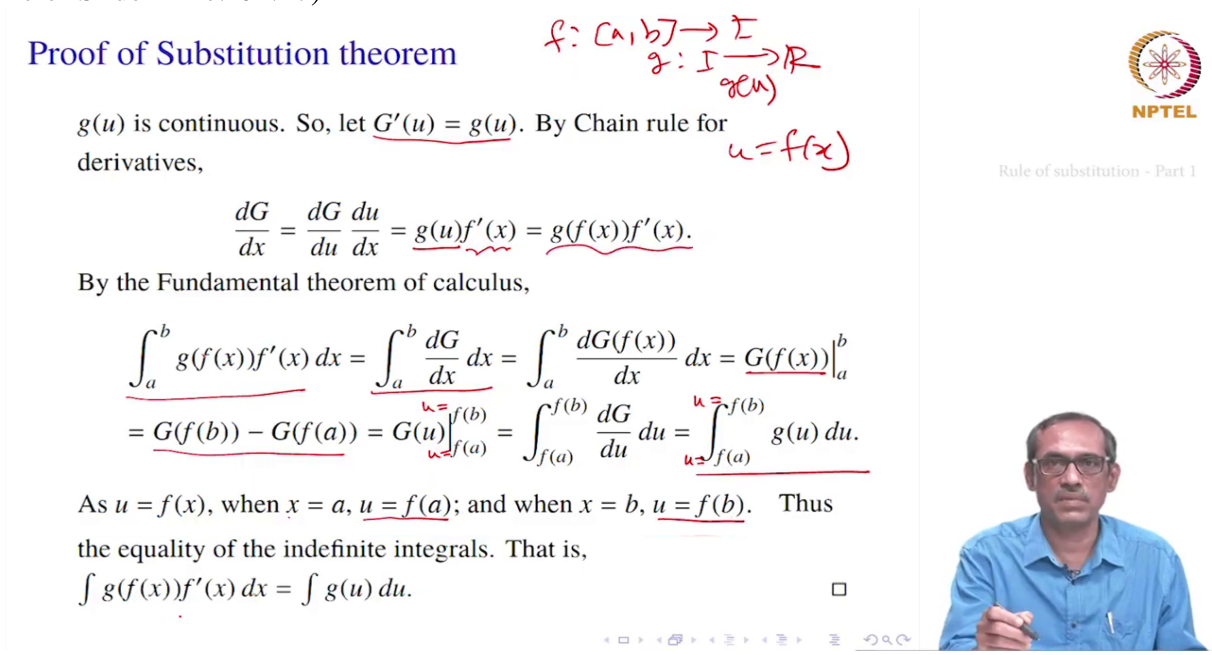

So, let us try to prove this. We have $g(u)$; remember that $g : I \to \mathbb{R}$ and we are writing $g(u)$ for $u \in I$. We have $f : [a, b] \to I$; that is how it is. Now $g(u)$ is continuous; so it can be integrated. As it is continuous on a closed bounded interval, we can write it as $G'(u)$ for some function G. This function G is a primitive or an anti-derivative of g. We have $G'(u) = g(u)$.

Now, the chain rule gives $dG/dx = dG/du \times du/dx$. And $dG/du = G'(u) = g(u)$. Also, $du/dx = f'(x)$ since $u = f(x)$. So, $dG/dx = g(f(x))\, f'(x)$. We then use the fundamental theorem of calculus to obtain $\int_a^b g(f(x)\, f'(x)\, dx = \int_a^b dG/dx\, dx$. As G is a function of $f(x)$, this is really $\int_a^b dG(f(x))/dx\, dx$. Since its integral is $G(f(x))\big|_a^b$, we get $F(f(b)) - G(f(a))$. Using $u = f(x)$ we could have written it as $\int_{f(a)}^{f(b)} dG/du\, du$, which is equal to $G(u)$ from $f(a)$ to $f(b)$. However, $dG/du = g(u)$. So, it can be written as $\int_{u=f(a)}^{u=f(b)} g(u)\, du$. That is what our first thing to be proved:

$$\int_a^b g(f(x))\, f'(x)\, dx = \int_{u=f(a)}^{u=f(b)} g(u)\, du.$$

This is in the in terms of definite integrals. A similar thing happens for indefinite integrals. Let us see. We can see how the limits are coming in an alternative way. As $u = f(x)$, when $x = a$, $u = f(a)$; and when $x = b$, $u = f(b)$. So, you can see directly that by taking the indefinite integral.

We thus get

$$\int g(f(x))\, f'(x)\, dx = \int g(u)\, du.$$

But we can really take it as a rule and make a simplification instead of going through this composition. We can look at that in a different way, that is, by considering the differential.

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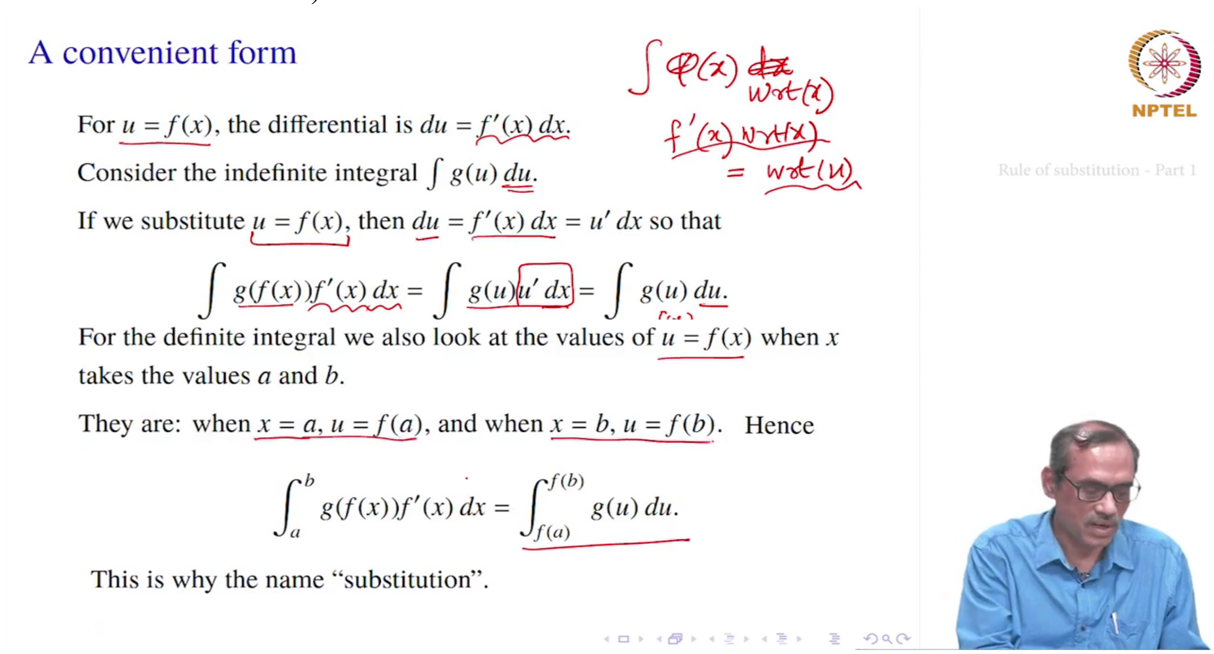

Suppose $u = f(x)$; that is what we have taken there. Now its differential is $du = f'(x)\, dx$. Then in the indefinite integral $\int g(u)\, du$, you think of $g(u)$ as $g(f(x))$. We just substitute $du = f'(x)\, dx$ here. You then obtain the formula $\int g(u)\, du = \int g(f(x)\, f'(x)\, dx$.

Everything matches because of the differential notation. In fact, it looks trivial. However, $\int \phi(x)\, dx$ means it is the integration of $\phi 9x)$ with respect to x. Had we given a notation for "with respect to x" as something other than dx, then it would not have appeared so obvious. Thus, it requires a proof. But after proving it we find that if we consider the differential, then this looks obvious. We then find that instead of a notation for "with respect to x", dx is really the differential. Once we accept this, then du becomes equal to $f'(x)\, dx$. Then, we can really substitute du for $f'(x)\, dx$ in the integral. In that case, $g(u)$ becomes $g(f(u))$ and we you get directly the formula that we have proved in the theorem.

See, our notation in writing "with respect to x" in an integral as dx now helps us. And, this notation is consistent with the chain rule or the rule of substitution. That is why this is called the rule of substitution, which is the chain rule for differentiation in disguise. And this enables us to treat the notation dx as the actual differential.

Since we are able to consider that as the differential, in $g(u)\, du$, if you substitute $u = f(x)$, then that gives rise to the left hand side. Reason: the differential du becomes $f'(x)\, dx$. Of course, this is the reason why this notation was adopted. These things are consistently going through when dx in an integral is considered as the differentials. We thus make it a rule and a mnemonic. Instead of using the theorem in that form always this will be a convenient form to use. We call this the rule of substitution.

So, what do we have to do really? If you find somehow that the function in the integrand will be simplified if you substitute $u = f(x)$, then you substitute and see that for the limit of integration, when $x = a$, $u = f(a)$ and when $x = b$, $u = f(b)$, and in that case $du = f'(x)\,dx$. Then, convert the integral ${}_a^b g(f(x))f'(x)\,dx$ to $\int_{f(a)}^{f(b)} g(u)\,du$ and integrate to get the result directly. This is just a magic of the notation, so to say, but we can use this as a mnemonic and continue with our integration.

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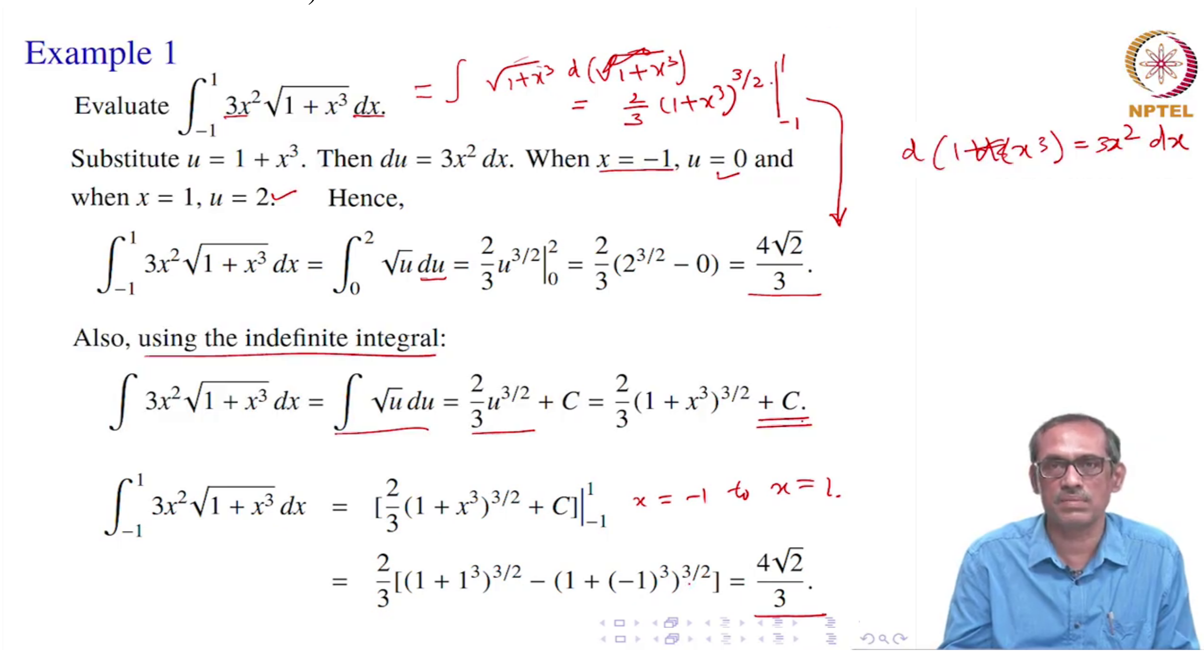

Let us take an example to see how the rule of substitution is applied. Here, we have to evaluate $\int_{-1}^{1} 3x^2\sqrt{1+x^3}\,dx$. It looks difficult to do because we know how to integrate polynomials if there are x to the power something and so on, but nothing more. We observe that $3x^2$ is the derivative of x^3, or, the differential of x^3 is $3x^2\,dx$. Also it is square root of $1 + x^3$, where the derivative of 1 is 0. So, instead of x^3, we could have considered the differential of $1 + x^3$, which is again $3x^2\,dx$. It looks if we substitute $u = 1 + x^3$ and apply the rule of substitution, then that would help us to integrate. So, let us try that.

Substitute $u = 1 + x^3$. With that substitution the differential du becomes $3x^2\,dx$. We have to be concerned about the limits. When $x = -1$, $u = 1 + (-1)^3 = 0$ and when $x = 1$, $u = a + 1^3 = 2$. So, the integral $\int_{-1}^{1} 3x^2\sqrt{1+x^3}\,dx$ can be written as $\int_0^2 \sqrt{u}\,du$. This will be easy to integrate. We know its integration will be $u^{1/2+1}/(1/2+1)$, which is $(2/3)u^{3/2}$. This is to be evaluated at 2 and 0 and subtracted. It simplifies to $4\sqrt{2}/3$.

This is how the rule of substitution is used. Sometimes when you get experienced, we will simply start as follows. The term $3x^2\,dx$ is the differential of $1+x^3$, so the integral is $\int_{-1}^{1} \sqrt{1+x^3}\,d(1+x^3)$. We look at the indefinite integral there and find that the integral is $(2/3)(1+x^3)^{3/2}$. Since the terms involve x, we use the same limits of integration and write that the integral is equal to

$(2/3)(1+x^3)^{3/2}\Big|_{-1}^{1}$. And then that will simplify to $4\sqrt{2}/3$. But follow this method after some time, not now. For now, substituting will be easier instead of going directly through that.

Look at the corresponding indefinite integral. The indefinite integral will not have any limits. It will be simply $\int 3x^2\sqrt{1+x^3}\,dx$. When substitution is used it leads to $\int \sqrt{u}\,du + C$, because indefinite integrals will introduce a constant. When substituted back, that will give $(2/3)(1+x^3)^{3/2}+C$, as we have seen. We have to remember that in the indefinite integral there can be an arbitrary constant. If you know the indefinite integral, then using the limits of integration -1 to 1 can be used to evaluate it. And this evaluation is in terms of x and not in terms of u. So, this is after a bit of experience. If you can see the substitution, then you do not need to put that $u = 1 + x^3$, but directly do it. As we know, finally it simplifies to $4\sqrt{2}/3$. Either way you can do it: by changing the limits for u or by finding the integral in terms of x and use the limits 1 and -1.

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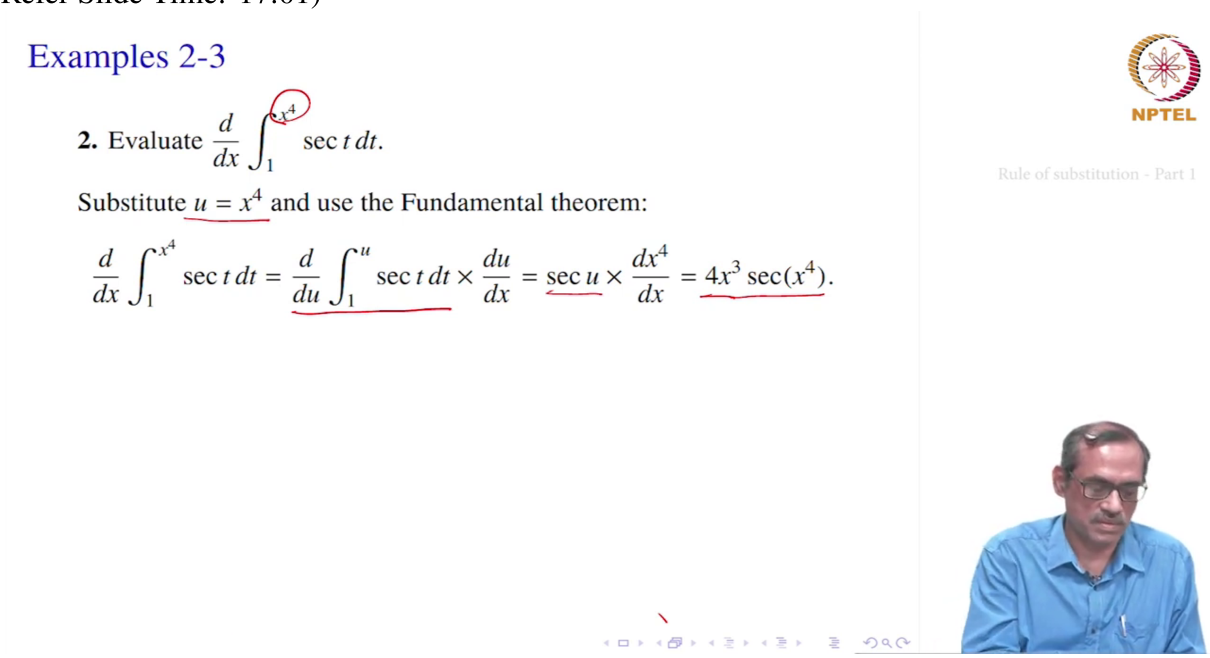

Let us take another example. Evaluate the derivative with respect to x of the integral $\int_1^{x^4} \sec t\,dt$. We do not have a direct formula for integrating $\sec t$. Had it been $\sec^2 t$, we could have got $\tan t$. But it is $\sec t\,dt$. Maybe, we have to use the fundamental theorem; but the fundamental theorem says that derivative, that is, $\frac{d}{dx}\int_a^x$, where as here we have the upper limit as x^4. So, what do we do?

We can use the chain rule. If the integral is $F(x)$, then $dF/dx = dF/dx^4 \times dx^4/dx$. Essentially, we substitute $u = x^4$, because that is where the trouble is and use the fundamental theorem. Now, $\frac{d}{dx}\int_1^{x^4} \sec t\,dt = \frac{d}{dx^4}\int_1^{x^4} \sec t\,dt \times dx^4/dx$. That is how it looks now. With our substitution, that is equal to $\frac{d}{du}\int_1^{u} \sec t\,dt \times dx^4/dx$. We can use the fundamental theorem, to get $\sec u$ times dx^4/dx. That is equal to $\sec(x^4)(4x^3)$. So, you see we did not have to worry about what will be the this integral; but since it was the differentiation of an integral we could do it with the rule of substitution.

Let us go to next example. Here, we have to evaluate $\int_{\pi/4}^{\pi/2} \cot\theta \operatorname{cosec}^2\theta\,d\theta$. It should strike something. We have $\cot\theta$, we have $\operatorname{cosec}^2\theta$; we know that the derivative of $\cot\theta$ is $-\operatorname{cosec}^2\theta$. So, we can introduce a minus sign and substitute u for $\cot\theta$ so that the differential $\operatorname{cosec}^2\theta\,d\theta$ will be

$-du$, if u is equal to $\cot\theta$.

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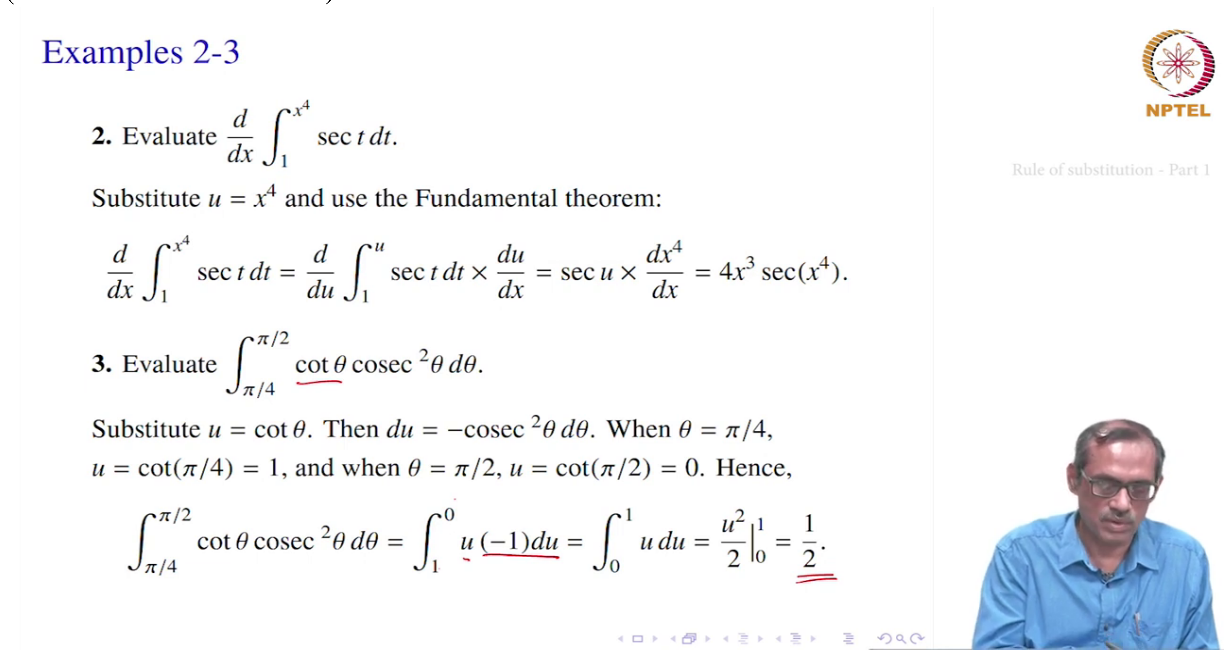

Let us try. Suppose we substitute $u = \cot\theta$. Then you get the differential $du = -\operatorname{cosec}^{\theta} d\theta$. And look at the limits. When $\theta = \pi/4$, $u = \cot(\pi/4) = 1$ and when $\theta = \pi/2$, $u = \cot(\pi/2) = 0$. So our limits will be from 1 to 0; that is, the integral is written as $\int_1^0 u\,(-du)$. As the limits of integration go from 1 to 0, you can change that with this negative sign. That will be $\int_0^1 u\,du$. Now, can integrate this. It is $u^2/2$ evaluated at 1 and 0, then subtracted. It gives the answer as $1/2$.

This is how we will be using the rule of substitution. That will be more convenient than directly doing. Sometimes if it is a bigger expression and we can see directly the substitution, then we substitute rather than doing directly. That will minimize mistakes.