Basic Calculus - 1
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Lecture 28 - Part 1
Applications of Fundamental Theorem of Calculus - Part 1

This is lecture 28 of basic calculus 1. If you remember, we had introduced the notion of definite integral, then discussed about the fundamental theorems of calculus. The fundamental theorem of calculus had two parts. First, the integral of a derivative is the original function, that is, the integral of $f(x)$ is $f(x)$. We write it in the form of an indefinite integral. But you can also express it in the form of a definite integral. Second, the derivative of the integral is the function itself; that is, the derivative of $\int f(x)\,dx$ is $f(x)$. These are true under some mild conditions like continuity. Using the fundamental theorem of calculus on definite integrals and indefinite integrals we will try to solve some problems today.

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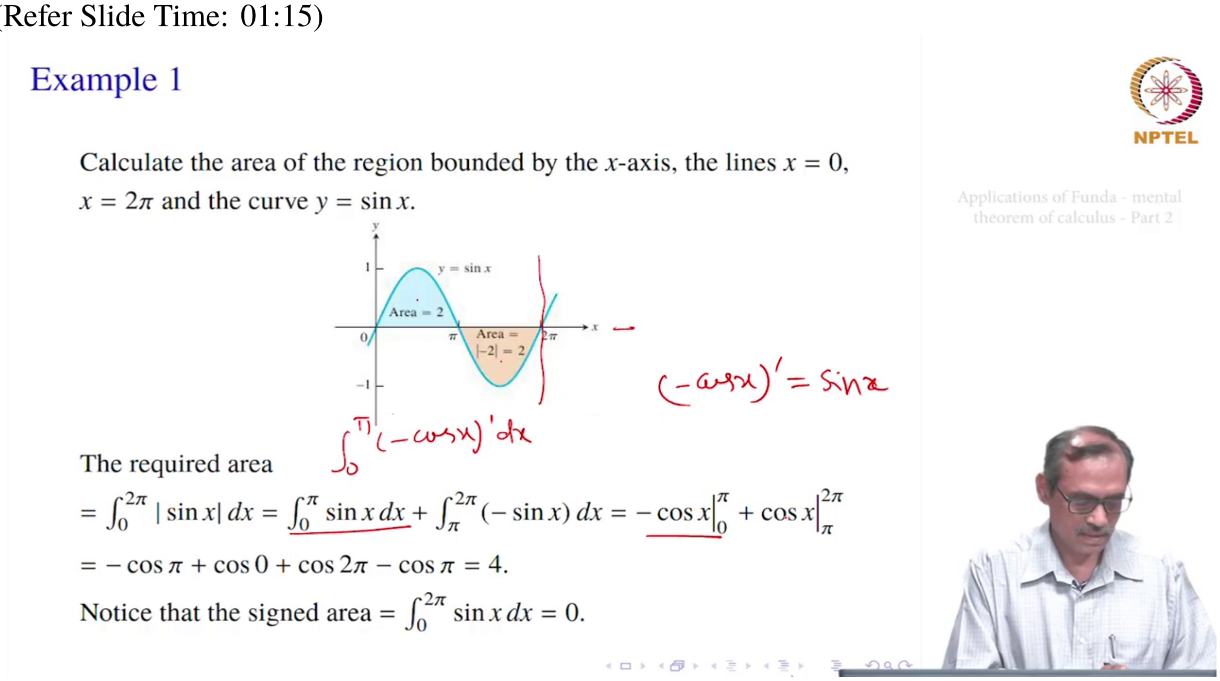

Let us try our First example. Here we are asked to find the area of the region bounded by the x-axis, the lines $x = 0$ and $x = 2\pi$ and the curve $y = \sin x$. We plot $y = \sin x$ this way. We want to compute the area of this region which is in blue and also yellow; it is the sum of these two areas.

We proceed to find the area. Since it is area and not the signed area, we should write it as $\int_0^{2\pi}$ of the modulus of the given function. As the function si $\sin x$, we have the area as $\int_0^{2\pi} \sin x\,dx$.

At $x = \pi$, the function is changing its sign. From 0 to π, $\sin x$ remains positive and from π to 2π, it becomes negative. So, $|\sin x|$ becomes $\sin x$ from 0 to π and $-\sin x$ from π to 2π. Accordingly, we break that into two integrals: one is from 0 to π and another from π to 2π of $|\sin x|$. Now, $|\sin x|$ is $\sin x$ for $x \in [0, \pi]$. We than get the two integrals; their sum is the required area.

Now, we integrate them using the fundamental theorem. We know that the derivative of $-\cos x$ is equal to sin x. So, the integral here is equal to $\int_0^{\pi} (-\cos x)'\, dx$. By the fundamental theorem that should give $-\cos x$ evaluated at π minus $-\cos x$ evaluated at 0, which we write this way $(-\cos x)\big|_0^{\pi}$, with a vertical line 0 to pi.

Similarly, the other one is $\cos x\big|_{\pi}^{2\pi}$ because already there is $-\sin x$, which is the derivative of $\cos x$. Now, we write them in the expanded form that gives $-\cos\pi - (-\cos 0) + \cos(2\pi) - \cos\pi$. As $-\cos\pi$ gives 1, $\cos 0$ gives 1, $\cos(2\pi)$ is 1 and $-\cos\pi$ is again 1, you get answer as 4.

That is the way we use the fundamental theorem of calculus: $\int f'(x)\, dx = f(x)$. But suppose you take the signed area then you would have taken $\int_0^{2\pi} \sin x\, dx$ and that would have given you 0. Since it is signed area, this would have given you $-\cos(2\pi) - (-\cos 0) = 0$. Also geometrically, you can see this area equal to 2. For the signed area, this area is equal to the the integral from π to 2π of $\sin x$. Each of them gives -2, and they cancel to give you 0. In the signed area, you can revert back by telling that the area of this part (yellow one) is $|-2| = 2$, and similarly, the other one gives 2 so that the actual area becomes 4.

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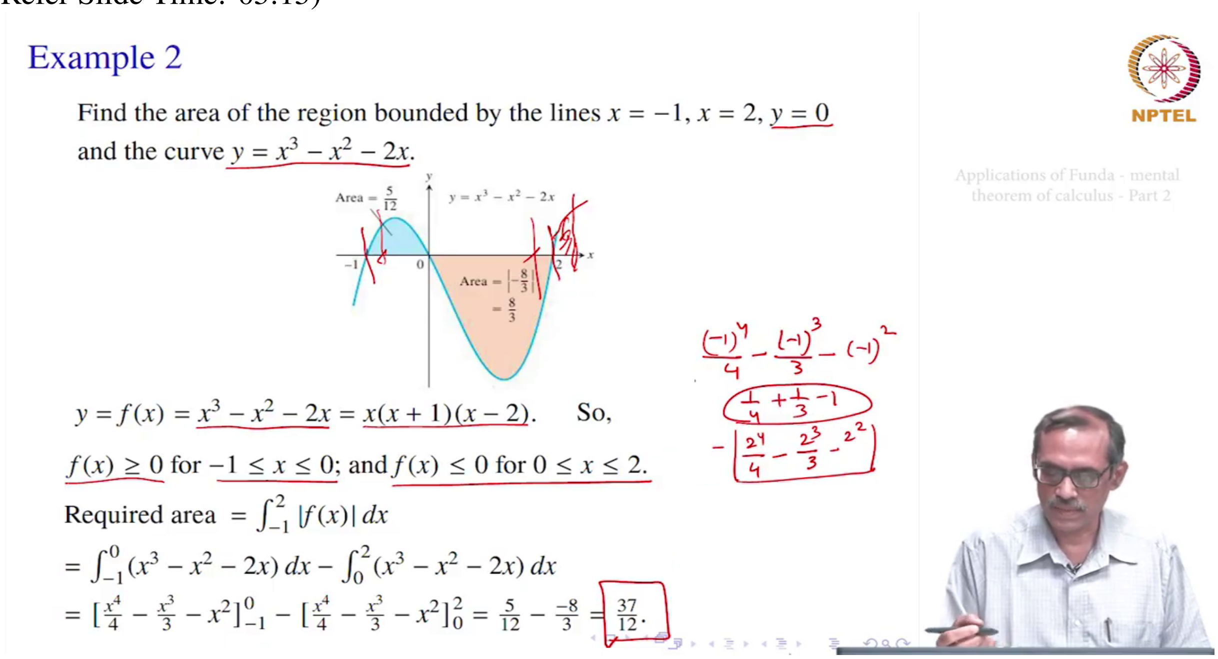

Let us go to the second example. Here, it is a slightly different from the earlier one. It is not exactly $\sin x$ but something similar. The curve is $y = x^3 - x^2 - 2x$. We want to find the region bounded by this curve, the x-axis which is $y = 0$ and the lines $x = -1$ and $x = 2$. They are the points where the curve really crosses the x-axis; so that makes it easier.

But even if it does not cross the x-axis, it will be something like this; and you can take integral up to those points; or, if it goes beyond 2, then you may have to take this area also into consideration.

Now, this is our area to be found. What do we do? We take the integral of mod of the function. So, we must find out what is the sign of the function when x varies from -1 to 2. These are the two lines $x = -1$ and $x = 2$. We can factorize $x^3 - x^2 - 2x$. Take x as common; it is $x(x^2 - x - 2)$, which is $x(x+1)(x-2)$. Multiply and check. This is the correct factorization. We want to see where it is positive or negative. Let us think about -1 to 2. Obviously, we break at 0; that is what the geometry suggests, that is, we break the interval $[-1, 2]$ to $(-1, 0)$ and $(0, 2)$ as the function is 0 at $x = -1, 0, 2$.

Take the first interval $(-1, 0)$. Say, $x \in (-1, 0)$. Then x is somewhere here. Then $x + 1$ is positive and $x - 2$ is negative, because 2 is bigger than x and x is already negative. So, their product will be positive, that is, $f(x) > 0$. So, $f(x) \geq 0$ for $-1 \leq x \leq 0$. That is what the graph also says.

Next, if you go to any point between 0 and 2. Here, x is positive, $x - 2$ is negative, $x + 1$ is positive, so $f(x) \leq 0$ for $x \leq 2$; and that is also shown correctly in the picture.

Then the area will be the integral of modulus of this from -1 to 2. We can now break $|f(x)|$. We know it is equal to $f(x)$ when it is from -1 to 0, and it is $-f(x)$ when x is in $[0, 2]$. So, you would write that as two integrals. It is

$$\int_{-1}^{0} (x^3 - x^2 - 2x)\, dx - \int_{0}^{2} (x^3 - x^2 - 2x)\, dx.$$

Then we integrate using the fundamental theorem. The integral of x^3 is $x^4/4$; the integral of x^2 is $x^3/3$ and that of $2x$ is x^2. This is so, because if you differentiate $x^4/4$ you get back x^3 and so on. Now, this expression $x^4/4 - x^3/3 - x^2$ is to be evaluated at $-1,\ 0,\ 2$ and then subtracted properly.

Look at this notation. We just omit this vertical bar sometimes when there is a bracket already. It is a simplification of the notation. So, we get

$$\left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} - \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2}.$$

At 0 this becomes 0; at -1, this is $(-1)^4/4 - (-1)^3/3 - (-1)^2$. That gives you $1/4 + 1/3 - 1 = -5/12$. At 2, it is $2^4/4 - 2^3/3 - 2^2 = -8/3$. That means the blue area is $5/12$ and the brown one is $8/3$. If we take the signed areas, then they are $5/12$ and $-8/3$. But automatically this minus sign goes since we have computed with $|f(x)|$. So, this becomes $5/12 + 8/3 = 37/12$. That is the area.

Again because throughout it is negative, we could have taken its signed area and then taken the modulus to get the area; and add them. That is same thing as whatever calculation has been done here.

We go to the next example. Here, we are not computing the area; but something else is asked. The function $f(x) = x^{-2}$ is non-negative on $[-1, 1]$; that is clear. We are are thinking of the argument that if the integrand function is non-negative, then the integral should also be non-negative. But if you calculate exactly the integral of x^{-2} from -1 to 1, then it will give $-x^{-1}$ evaluated at 1 and -1 and subtracted. That gives $-(1)^{-1} - (-1)^{-1} = -1 - 1 = -2$. Why is it happening? We first thought that our integral must be non-negative, but if you compute actually, then are getting negative. What is the reason?

Well the reason is not difficult to give. Look at the first line itself. The integrand is non-negative; but then there is a caveat there. We see that it is not defined at $x = 0$; it blows up near 0. Since $f(x)$ is not defined on $-1, 1]$, it is not a function over $[-1, 1]$. Of course, it is defined on $[-1, 0)$ and also on $(0, 1]$. So, you could have broken that into sum of two integrals using our convention.

If at one point it is not defined, then you can break it. Of course, it will be different problem. It will be something like $\int_{-1}^{0} x^{-2}\,dx + \int_{0}^{1} x^{-2}\,dx$.

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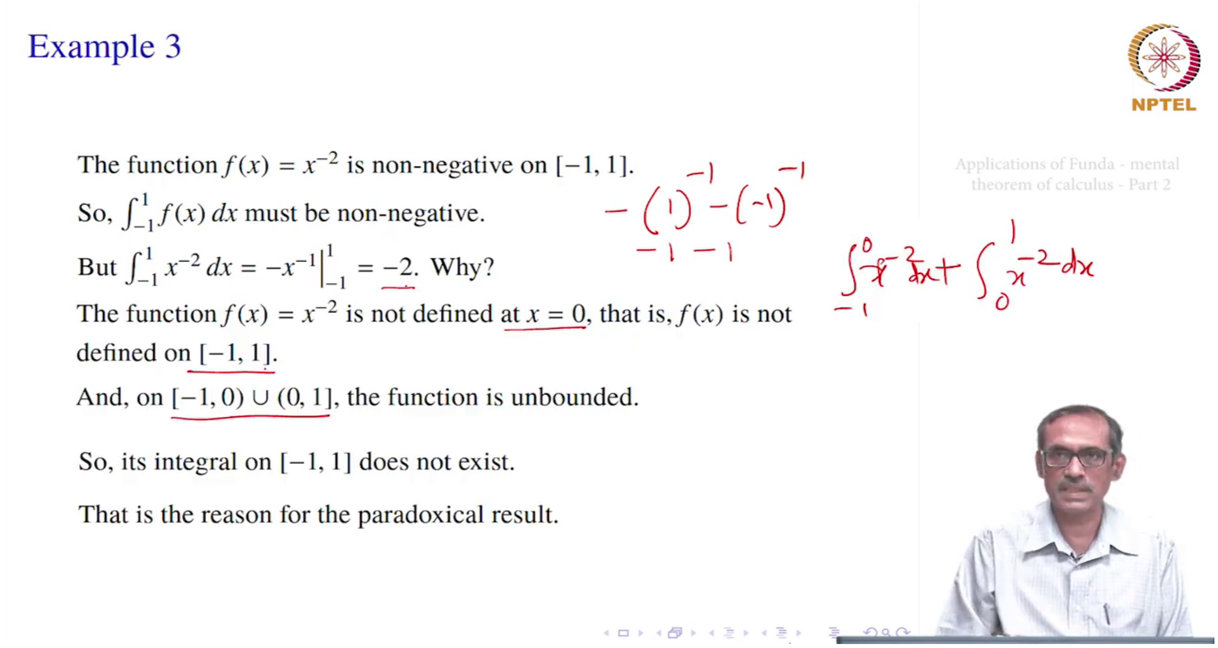

What about this integral even if it is not defined at 0? And, what happens near 0? When x is near 0, x^{-2} becomes very large. That means, this function is really unbounded. Since it is unbounded it might become ∞ or $-\infty$. However, there is no minus here since it is non negative. So, it becomes ∞. That means its integral does not exist. Recall that we have defined integrals for bounded functions only. Since the integrand is unbounded this integral does not exist. That is the reason for this paradoxical result. In fact, you can manipulate in many ways to get different results also, because the integral does not exist.

That is a quiz type of problem. We have to be really careful. We have see whether the function is defined or not, and whether it is becoming bounded or unbounded in its domain. Those have to be seen before applying our fundamental theorem of calculus. Because that assumes all these things that the integrals exist, in fact, that assumes that the integrand should be continuous also.

Let us go to the next problem. We want to find an anti-derivative of this function, which is given as $(1/3)x^{-2/3} + 4\sec(3x)\tan(3x)$. An anti-derivative means whose derivative will be the given function. So, we go straight forward, because we can use our limit properties like the integral of $f + g$ will be the integral of f plus the integral of g, provided they exist. We go for the indefinite integrals.

Now, $(1/3)x^{-2/3}$ would give us this $(1/3)x^{1/3}/(1/3) + C_1 = x^{1/3} + C_1$, where C_1 is some constant. That is, the general anti-derivative of $(1/3)x^{-2/3}$ is $x^{1/3} + C_1$. Similarly, the integral of $\sec(3x)\tan(3x)$ is $(1/3)\sec(3x)$ because if you differentiate $\sec(3x)$, it would give $\sec(3x)\tan(3x)$ into 3. So, this integral gives you $(1/3)\sec(3x) + C_2$ for some constant C_2. So, the anti-derivative will be sum of these two. But be careful to see where it is defined or where it is not defined. This will be the anti-derivative once meaningful. At $x = 0$ of course, the function itself is not defined.

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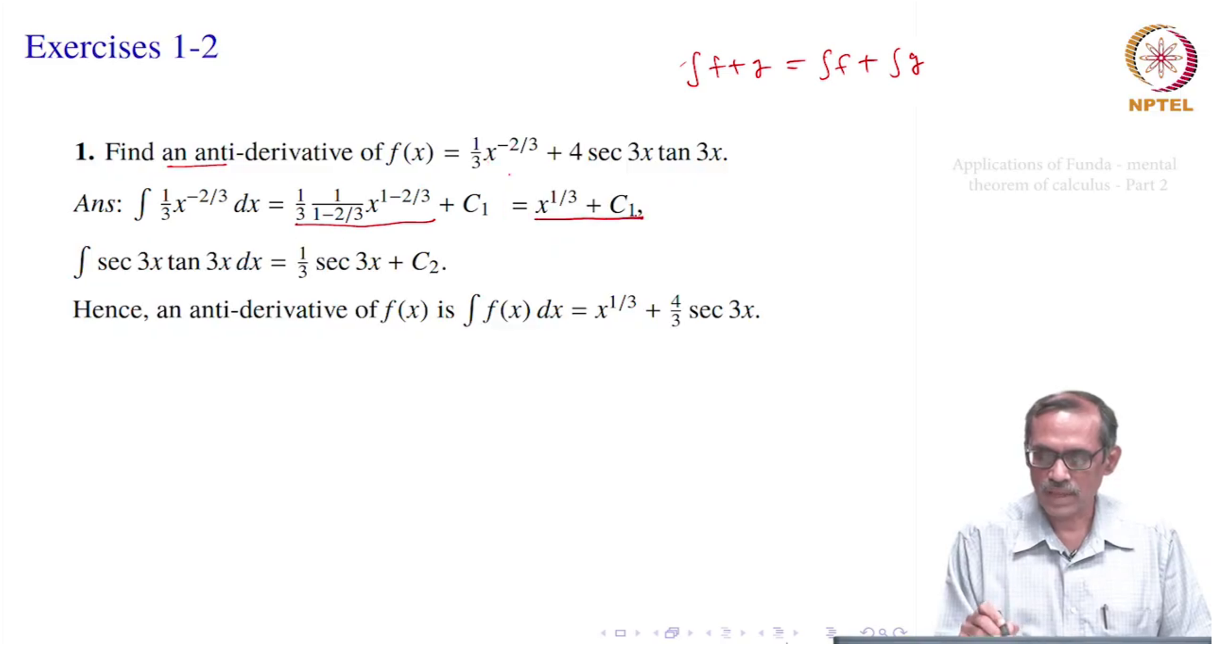

We have used the properties of integrals and of course the fundamental theorem of calculus, which is always used implicitly.

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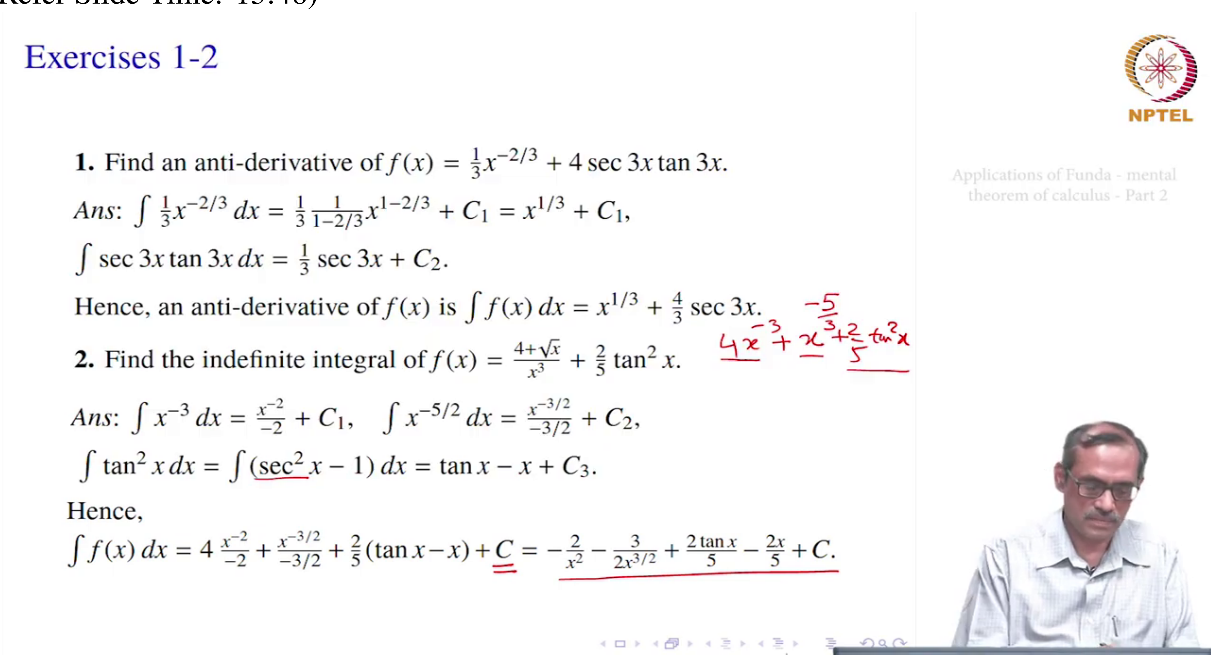
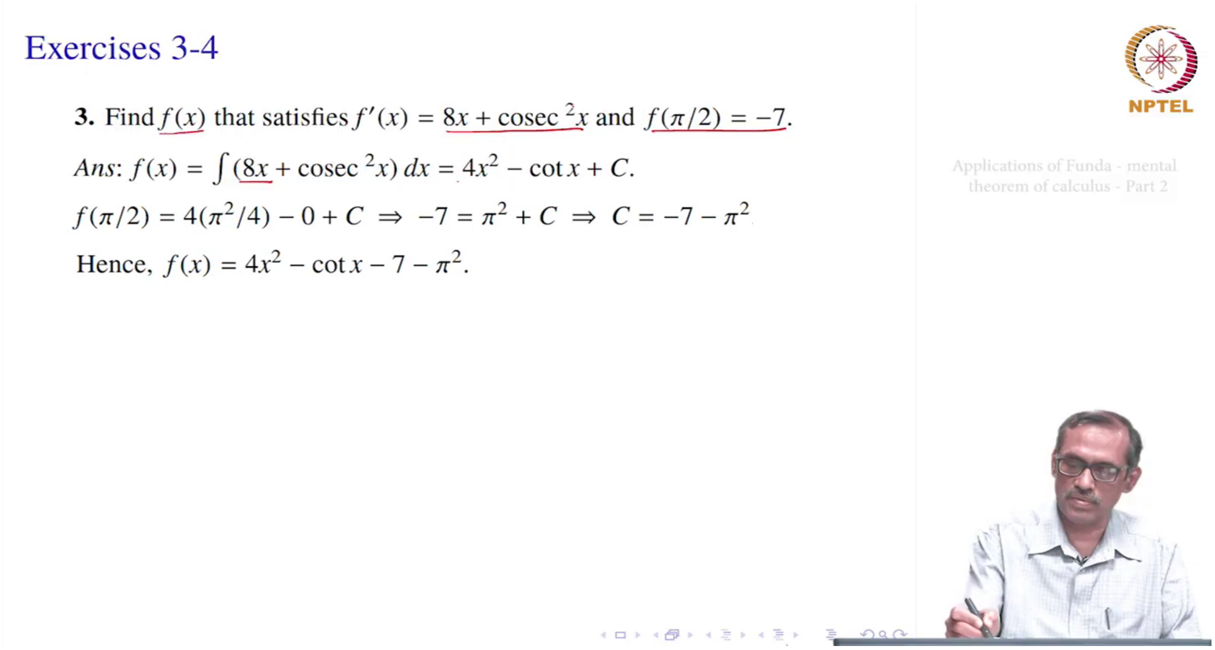


In this exercise, you want to find the indefinite integral of the function $f(x) = (4+\sqrt{x})/x^3 + (2/5)\tan^2 x$. Sometimes it is better to expand it and write separately as the sum of some things so that we can use our earlier theorem. Here, it is $4x^{-3} + x^{-5/2} + (2/5)\tan^2 x$. That is how it looks.

We have to find indefinite integral of this, indefinite integral of this, indefinite integral of this and then add them up. First, for x^{-3}. We use x^m giving $x^{m+1}/(m+1)$ for $m \neq 1$. Here, we get $x^{-3+1}/(-3+1) = -(1/2)x^{-2} + C_1$. Similarly, $x^{-5/2}$ gives $x^{-5/2+1}/(-5/2+1) = -(2/3)x^{-3/2} + C_2$. For $\tan^2 x$, we see that it is $\sec^2 x - 1$. We get $\sec^2 x$ by differentiating $\tan x$ and we get 1 by differentiating x. So, its integral is $\tan x - x + C_3$.

Then we multiply with the given constants and get back our indefinite integral. It is 4 times the first integral plus the second integral plus 2 by 5 times the third integral. In these expressions we also get $4C_1 + 4C_2$ plus some constant times C3; they are all accommodated by writing plus C, where C is a constant. So, we just go for their parts, where constants are not there. The first one is $-x^{-2}/2$; the second one is $-(2/3)x^{-3/2}$ and the third one is $(2/5)(\tan x - x)$ and then we add a constant C. It simplifies to this expression, which is the indefinite integral.

We are discussing every detail now. After some time you will do it quickly and get used to it. (Refer Slide Time: 18:43)



Let us, get the third problem. We want to find one function $f(x)$ that satisfies two conditions. Its derivative is $8x + \operatorname{cosec}^2 x$ and when it is evaluated at $\pi/2$, we should get -7. That means our constant (of integration) will be accommodated in such a way that $f(\pi/2) = -7$. First thing is, we should find one indefinite integral. Here, this $f'(x)$ and $f(x)$ will be the anti-derivative of $f'(x)$. So, we should compute first the anti-derivative of $8x + \operatorname{cosec}^2 x$.

And that is easy, because integral of $8x$ will give $4x^2$ because if you differentiate $4x^2$, you get back $8x$. And, if you differentiate $\cot x$, you get $-\operatorname{cosec}^2 x$. So, integral of $\operatorname{cosec}^2 x$ is $-\cot x$. The indefinite integral is $f(x) = 4x^2 - \cot x + C$ for some constant C. Then, we go for finding out this C, so that the other condition is met. The other condition is $f(\pi/2) = -7$. Evaluating $f(\pi/2)$, the left side gives $4(\pi^2/4) - \cot(\pi/2) + C$, which is $\pi^2 + C$. The right side is -7. Hence, $C = -7 - \pi^2$. Then we plug it into whatever we have got earlier. That is, $f(x) = 4x^2 - \cot x - 7 - \pi^2$. That is quite straightforward.

We go to next problem. We want to evaluate this definite integral, where the limits are from 0 to π and the integrand is $2\cos^(x/2) + (1/2)(\cos x + |\cos x|)$. Look at $|\cos x|$. We must find out when is $\cos x$ positive and when it is negative when x varies in the interval $[0, \pi]$.

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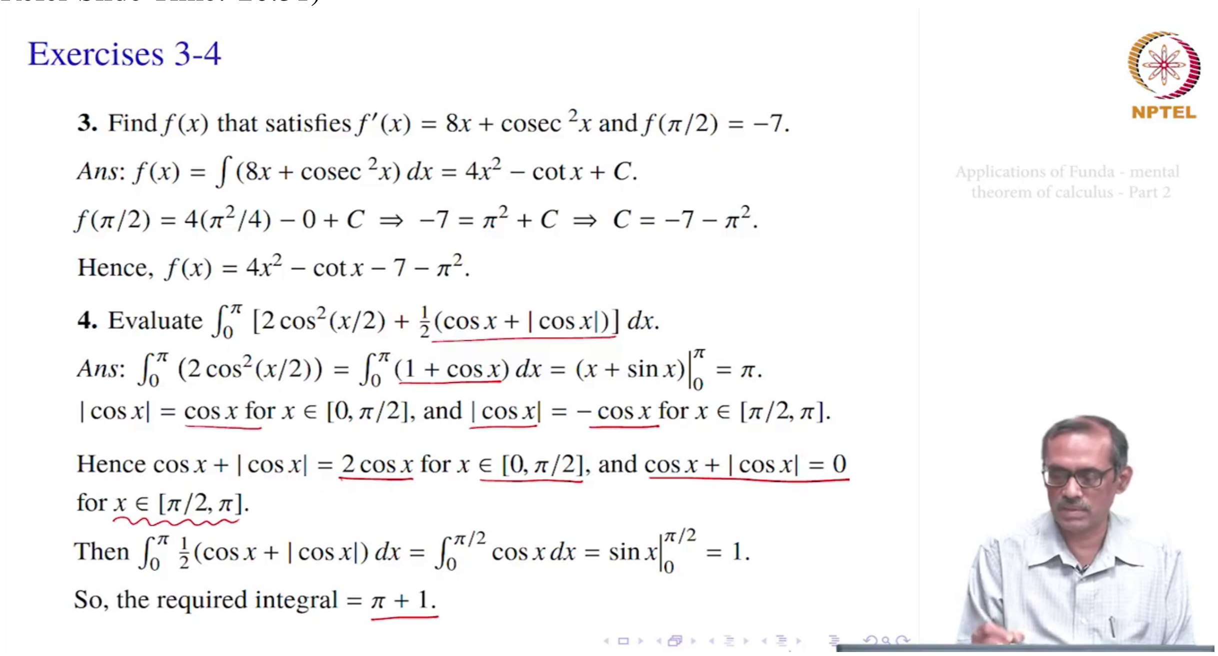

First, when you take $x \in [0, \pi/2)$, $\cos x$ remains positive. When $x \in (\pi/2, \pi]$, $\cos x$ becomes negative. So, we should break this interval $[0, \pi]$ to two different intervals with break point as $\pi/2$.

Notice that the first expression poses no problem, since $\cos^2(x/2)$ is always positive, or rather, non-negative. We go straight forward for the first integral. We write $2\cos^2(x/2)$ as $1 + \cos x$ using $\cos(2A)$ formula. There, 1 has the integral x and $\cos x$ has the integral as $\sin x$. So, we get $x + \sin x$. This is to be evaluated at 0 and π and subtracted. We get $\pi + \sin \pi = \pi$.

And, for the second one, $|\cos x| = \cos x$ for $x \in [0, \pi/2]$ and $|\cos x| = -\cos x$ for $x \in [\pi/2, \pi]$, we write that integral as one from 0 to $\pi/2$ plus the other from $\pi/2$ to π. For $x \in [0, \pi/2]$, we go for this expression first. This is $(1/2)(\cos x + \cos x) = \cos x$ in $[0, \pi/2]$ And, the other one is $(1/2)(\cos x - \cos x) = 0$ for $x \in [\pi/2, \pi]$. Therefore, the integral from 0 to pi is really the integral of $\cos x$ from 0 to pi by 2. Then, you get $\int_0^{\pi/2} \cos x\, dx$. This is equal to $\sin x$ evaluated at $\pi/2$ and 0 and then subtracted. This simplifies to 1.

So, we have to be concerned about where the function becomes positive or negative, because it was mod. That is the only thing done in this problem; otherwise it is straightforward application of the fundamental theorem.