Basic Calculus - 1
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Lecture 27 - Part 2
Fundamental theorem of calculus - Part 1

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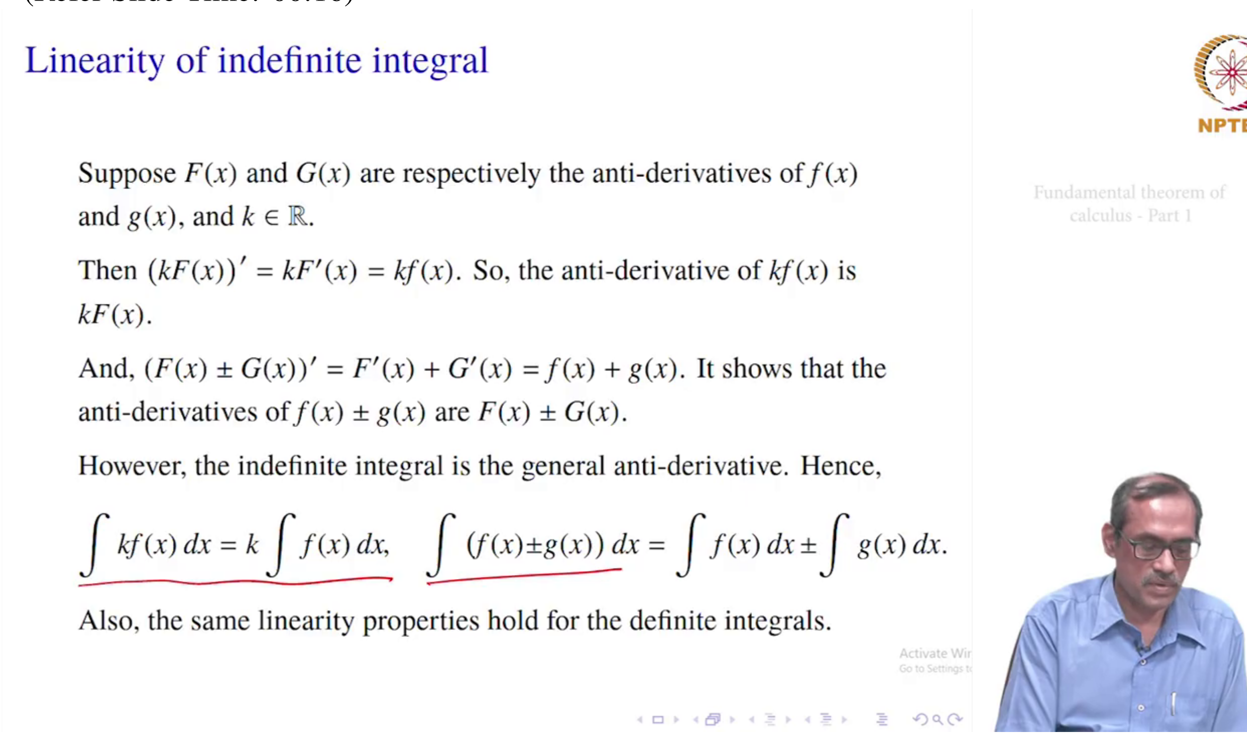

This is how we obtain the integral of $\sec^2 x$ as $\tan x$. Of course, a "plus C" is there to get all the antiderivatives. It will be helpful to know the derivatives of elementary functions and the properties of integrals like the integral of a sum is the sum of integrals, and multiplication by a constant k etc. These properties are easy to remember since they come of the properties of the derivatives.

If you take the derivative of $kf(x)$, then it will be equal to $kf'(x)$. Now, if $F(x)$ is an antiderivative of $f(x)$, then the antiderivative of $kf(x)$ will be $kF(x)$. So, that is behaving the same way with respect to multiplying a constant. Similar thing happens for addition and subtraction. Suppose $G(x)$ is an antiderivative of $g(x)$. Then, the derivative of $F(x)$ is $f(x)$ and the derivative of $G(x)$ is $g(x)$. As we know the derivative of $F(x) + G(x)$ is $f(x) + g(x)$. It thus follows that the antiderivative of $f(x) + g(x)$ is $F(x) + G(x)$.

Together these two properties are called linear properties of the antiderivative. Remember that the indefinite integral is the general antiderivative plus some constant C. In that case, we just write it this way:

$$\int kf(x)\,dx = k \int f(x)\,dx, \quad \int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx.$$

We know that similar properties also hold for the definite integrals. Suppose you are able to find the indefinite integral, then you can get the definite integral by evaluating that at b, at a, and then subtract them. That is how we will be obtaining the value of the definite integral.
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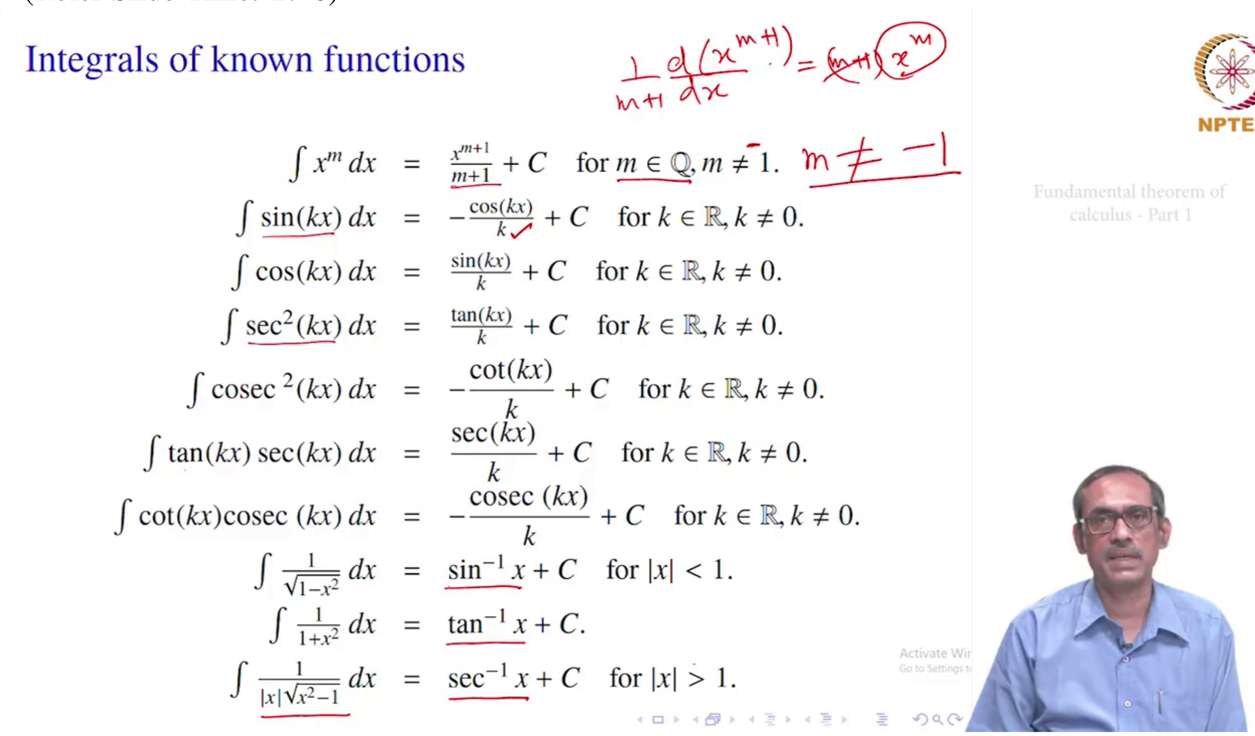

Of course, we need some repository of "which functions give which integrals". We know that the derivative of x^{m+1} with respect to x is equal to $(m+1)x^m$. We write that in terms of integral by bringing that $(m+1)$ down. We write that this as $\int x^m\,dx = (1/(m+1))x^{m+1}$. That is how this first formula about the derivative is related to this indefinite integral. Of course, we should add a constant. The formula so obtained is

$$\int x^m\,dx = (1/(m+1))x^{m+1} + C \quad \text{for some constant } C.$$

Recall that we have discussed this power function x^m only for rationals m. So, in the above formula, m is a rational number, but $m \neq -1$, because $m+1$ will then become 0. For $m = -1$, what will be the answer? That will come from the logarithm function, because derivative of $\log x$ will be equal to $1/x$. From there it should come. But we have not discussed that yet. So, we do not discuss the case $m = -1$ now. All that we remember is that for $m \neq -1$, we can apply the above formula.

Similarly, if you differentiate $\sin(kx)$, then you would get $-\cos(kx)$ times the derivative of kx with respect to x, which is k. Then, $\int \sin(kx)\,dx = -(1/k)\cos(kx) + C$. To see whether it is all right, all that you have to do is differentiate the right side and see that it matches with the function to be integrated. Here, if you differentiate $-(1/k)\cos(kx) + C$, you get $\sin(kx)$. So, we can write $\int \sin(kx)\,dx = -(1/k)\cos(kx) + C$. That is the indefinite integral. Obviously, k should not be equal to 0 here. If $k = 0$, you just get C; the other term is gone.

Similarly, $\int \cos(kx) = (1/k)\sin(kx) + C$ because the derivative of sine gives you cosine. And, the derivative of $\tan(kx)$ is $\sec^2(kx)$ times k. So, we divide by k and find that the derivative of

$(1/k)\tan(kx)$ is $\sec^2(kx)$. So, $\int \sec^2(kx)\,dx = (1/k)\tan(kx) + C$. Again, k should not be 0. Similarly, $\int \operatorname{cosec}^2(kx)\,dx = -(1/k)\cot(kx) + C$.

Since we know all these formulas for the derivatives, we are rewriting them in terms of indefinite integrals.

Since the derivative of $\sec(kx)$ is equal to $k\tan(kx)\sec(kx)$, we have $\int \tan(kx)\sec(kx)\,dx = (1/k)\sec(kx) + C$. Similarly, the formula for the derivative of $\operatorname{cosec} x$ gives the integral: $\int \cot(kx)\operatorname{cosec}(kx)\,dx = -(1/k)\operatorname{cosec}(kx) + C$.

We know that the derivative of $\sin^{-1} x$ is equal to $1/\sqrt{1-x^2}$ for $|x| < 1$. Recall that $\sin^{-1} x$ is defined for $|x| \le 1$. Then, we write that $\int 1/\sqrt{1-x^2}\,dx = \sin^{-1} x + C$ for $|x| < 1$. And at $x = 1$ the function $1/\sqrt{1-x^2}$ is not defined. As $1/\sqrt{1-x^2}$ is defined for $|x| < 1$, we have the above formula only for $|x| < 1$.

Similarly, $\tan^{-1} x$ has the derivative $1/(1+x^2)$. So, we write $\int 1/(1+x^2)\,dx = \tan^{-1} x + C$. As $\sec^{-1} x$ has the derivative as $|x|\sqrt{x^2-1}$, we have $\int |x|\sqrt{x^2-1}\,dx = \sec^{-1} x + C$. And this formula is valid for $|x| > 1$ since $x^2 - 1$ must be greater than or equal to 0.

These are the formulas, which will be helpful for us while evaluating the integrals. If you do not remember some of them, you go back to the derivatives, see what their derivatives are, and then you can just rewrite those in terms of integrals.

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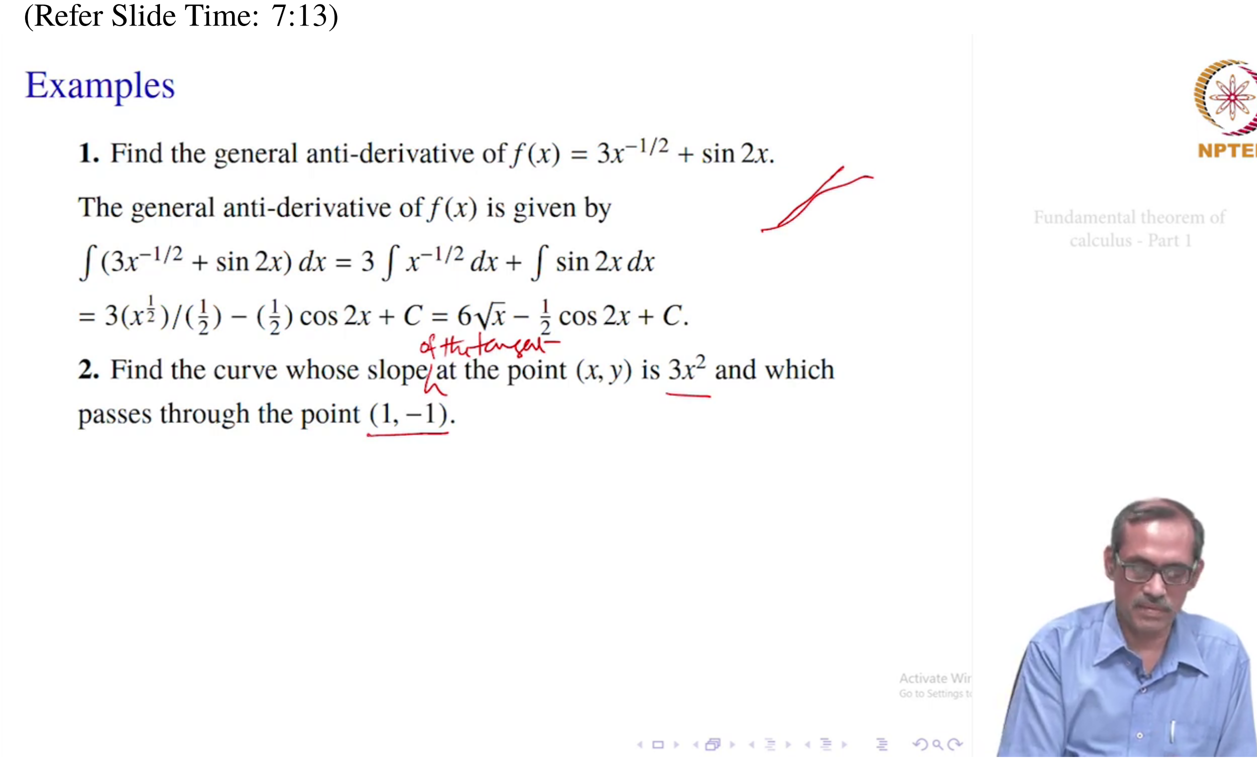

Let us solve some examples basing on this notion. In the first example, we want to find the general antiderivative of $f(x)$; that means the indefinite integral of $f(x) = 3x^{-1/2} + \sin(2x)$.It is really $3/\sqrt{x} + \sin(2x)$. What will be its general antiderivative? We go back to the property of linearity and so on. You have the indefinite integral equal to $3\int x^{-1/2}\,dx + \int \sin(2x)\,dx$. For $x^{-1/2}$, you apply the first formula x^m, that is, $\int x^m\,dx = (1/(m+1))x^{m+1}$. This gives the integral of $x^{-1/2}$ equal to $[1/(-1/2+1)]x^{-1/2+1} = 2x^{1/2}$. Then, you find the function whose derivative is $\sin(2x)$. You also know that the integral of $\sin(kx)$ is $-(1/k)\cos(kx)$. So, the integral of $\sin(2x)$ is $-(1/2)\cos(2x)$.

Hence, the integral of $3x^{-1/2} + \sin(2x)$ is $3 \times 2x^{1/2} - (1/2)\cos(2x)$. Of course, some arbitrary constant might be there because it is indefinite integral. We simplify. This gives you $6\sqrt{x} - (1/2)\cos(2x) + C$. That is the indefinite integral or the general antiderivative of this function.

This is how our formulas will be helpful, the formulas and the linearity property.

Let us go to the second one. Here, we want to find a curve (or the curve, since there exists only one, that is what the question says) with slope of the tangent at the point (x, y) as $3x^2$. That means, some curve is there, and you take any point x; then the slope of the tangent to the curve at the point (x, y) is $3x^2$. Of course, as x varies the slope will be different. We need to find out such a curve which also passes through the point $(1, -1)$. So, we are going to solve a differential equation. But this can be tackled through the indefinite and definite integrals.

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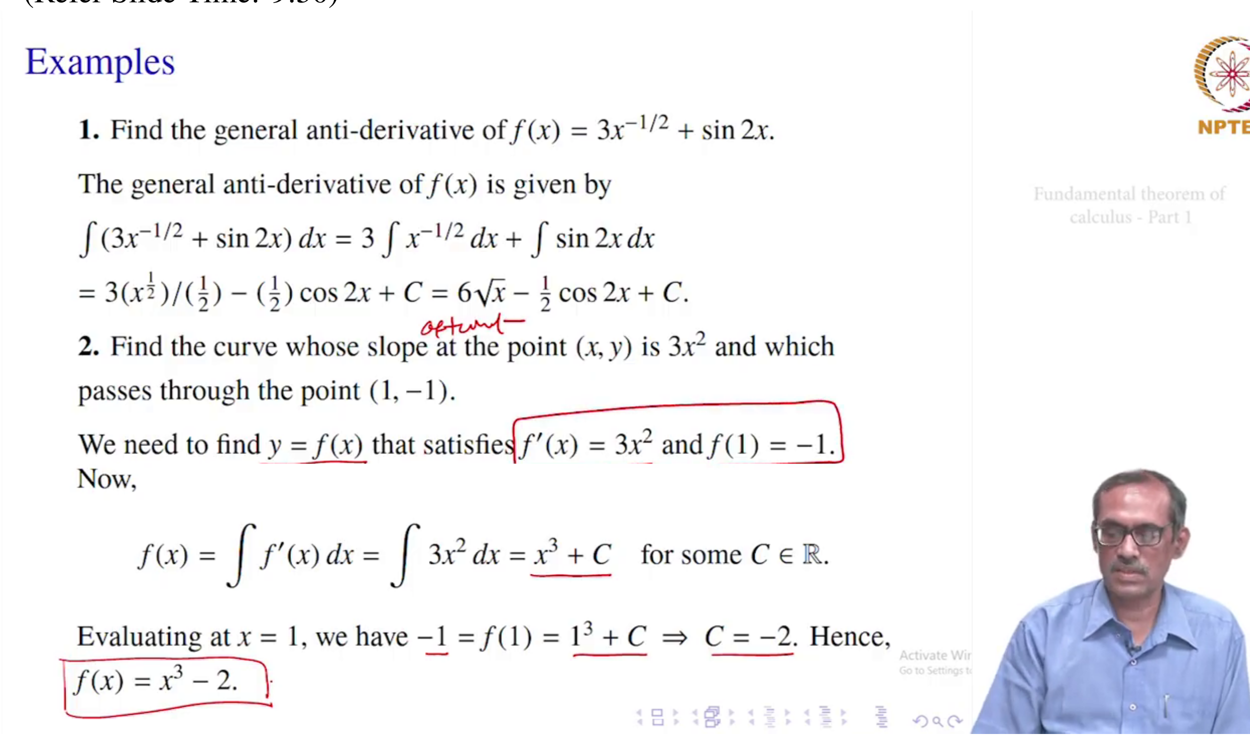

What does it ask? Let us reformulate it. We want to find $y = f(x)$ Its slope of the tangent at the point (x, y), which is $f'(x)$, is given to be equal to $3x^2$. And it passes through the point $(1, -1)$, which means $f(1) = -1$. So, you have $f'(x) = 3x^2$ and $f(1) = -1$. We have to solve for $y = f(x)$ using these two things.

Since $f'(x) = 3x^2$, by the second fundamental theorem, $\int x^m\,dx = (1/(m+1))x^{m+1} = \int f'(x)\,dx$, which is equal to $\int 3x^2\,dx$. We know that the derivative of x^3 is $3x^2$. So, this integral will be equal to $x^3 + C$. for some constant C, some real number C. Now, $f(x) = x^3 + C$ for an arbitrary constant C. We have to evaluate the arbitrary constant by using the constraint $f(1) = -1$. When you take $f(1)$, this gives on the left side, -1 and on the right side you get $1^3 + C$. So, $c = -2$.

Then, $f(x) = x^3 - 2$.

That is how simple differential equations whose derivative is known and a point where the curve passes, can be solved.

Let us go to another example. Here we are asked to calculate the area bounded by the curve $y = 6 - x - x^2$ and the x-axis. The picture is absent in the problem. We have to get that from it. Sometimes it is helpful to see what is the curve. But, even if you do not have the curve, you can still proceed analytically. We know that it is a parabola, an inverted parabola. It looks like this $y = 6 - x - x^2$. We want to find the area bounded by this curve and the x axis.

The curve intersects the x-axis at at least two points in order that an area is formed, and then nothing else is required to get a region which is bounded by these two. We do not have the lines $x = a$ and $x = b$ here. That means, wherever it crosses the x-axis, the area should come from there. Of course, if you look at the figure, it is making clear that this curve is intersecting the x-axis at two points. And, we want to find this area which is colored blue. In order to see what are those points, we must find out first where does this curve crosses the x-axis.

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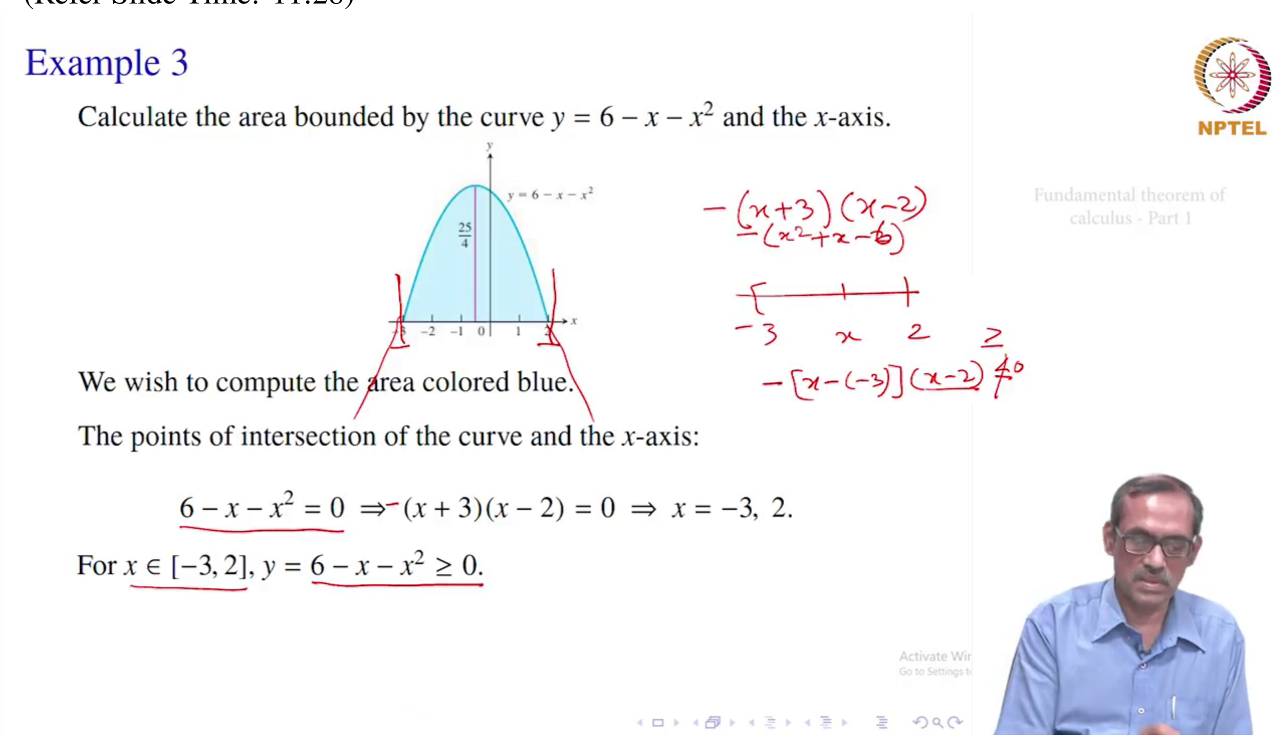

When it crosses the x-axis, we have $y = 0$. So, when $6 - x - x^2$ is equal to 0? That is our first question. In order to compute the area, we need to find the points of intersection; we want to find x such that $6 - x - x^2 = 0$. This can be factored as $-(x+3)(x-2)$. So, $(x+3)(x-2) = 0$. That gives you the two points where the curve crosses the x-axis. They are the points $(-3, 0)$ and $(2, 0)$. These two points are really our lines $x = -3$ and $x = 2$ if you go back to the general setup. We want to find the area bounded by this curve, the x-axis and the lines $x = -3$ and $x = 2$.

And you see this is a bounded area, the other one will go unbounded, right? It will go this way; so, that is the unbounded area, and this is the only bounded area. We want to find this blue colored area. Then it will be the integral from -3 to 2 as earlier of the given $f(x)$; and that is the area.

We have to see that it is the area; an area should be always non-negative. So, this curve in the picture lies above the x-axis. It has to be non-negative. But then we have to see it analytically also. Let us take any point x between -3 and 2. We see that $(x-(-3))$ is positive and $x-2$ is negative. That means the $(x+3)(x-2)$ is negative. It maybe equal to 0; it is so at these two points. Our curve is minus of this thing; so that should be greater than or equal to 0. That means for any x inside the $[-3,2]$, $6-x-x^2 \geq 0$. Of course, it is greater than 0, if $x \in (-3,2)$; it is so except the points -3 and 2. If x is in between -3 and 2, but not equal to them, then it is positive, and at these points, it is equal to 0. So, we write $6-x-x^2 \geq 0$. for any $x \in [-3,2]$. That means the curve lies above the x-axis. And when you take the area bounded by this and the x-axis, automatically that area becomes positive, even if you cannot compute the definite integral.

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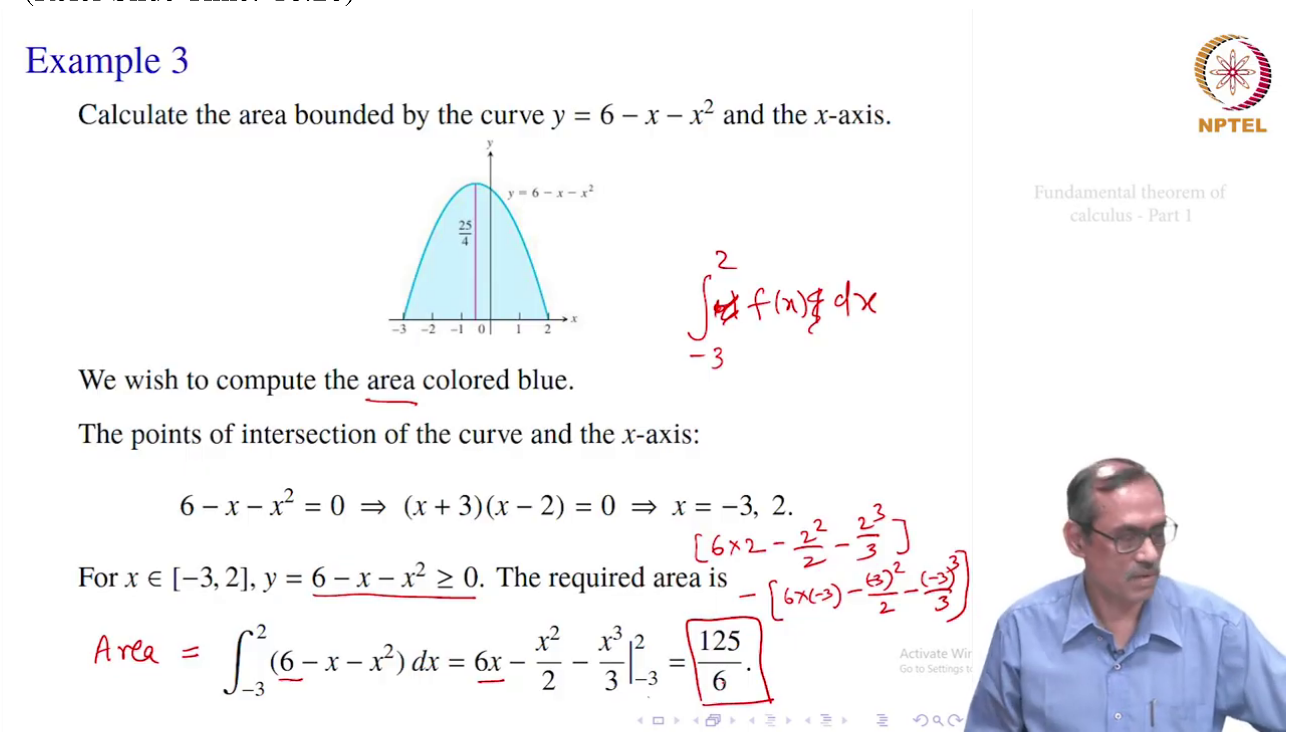

Now, the area will be equal to the definite integral. The required are is equal to $\int_{-3}^{2}(6-x-x^2)\,dx$. It is not from 2 to -3; that will be negative of that, it is from -3 to 2. In fact, our area should be coming like $\int_{-3}^{2}|6-x-x^2|\,dx$. If it is from 2 to -3, then the function under the integral sign should be $-|6-x-x^2|$. Now, $|6-x-x^2| = 6-x-x^2$; so that is the area.

We need to evaluate this integral. Notice that if you differentiate $6x$, you get 6. So, the integral of 6 should be equal to $6x$. And when you differentiate $x^2/2$, you get x. So, the integral of x is $x^2/2$. Similarly, if you differentiate $x^3/3$, you get x^2. So, the integral of x^2 is $x^3/3$. Using the linearity property, we obtain the integral as $6x - x^2/2 - x^3/3$.

And the connection between definite and indefinite integral is, whatever indefinite integral you get it should be evaluated at -3 and 2 and subtracted. In fact, you should get plus C in the indefinite integral, but that C will get canceled, because when you evaluate at -3, you have one

C, and when you evaluate at 2, that C gets canceled. So, we need to evaluate this expression $F(x) = 6x - x^2/2 - x^3/3$ at 2 and subtract from it the evaluation of $F(x)$ at -3. So, we obtain

$$F(2) - F(-3) = 6(2) - 2^2/2 - 2^3/3 - [6(-3) - (-3)^2/2 - (-3)^3/3.$$

If you simplify it, it will turn out to be $125/6$. Here it so happens that the curve is equal to $|f(x)|$. So, it is the actual area. In general, $\int f(x)\,dx$ gives the signed area and the actual area is $\int |f(x)\,dx$. These two happen to be same here because the function itself is lying above the x-axis.

This is how we will be computing signed areas and actual areas. Let us stop here.