Basic Calculus - 1
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Lecture 27 - Part 1
Fundamental theorem of calculus - Part 1

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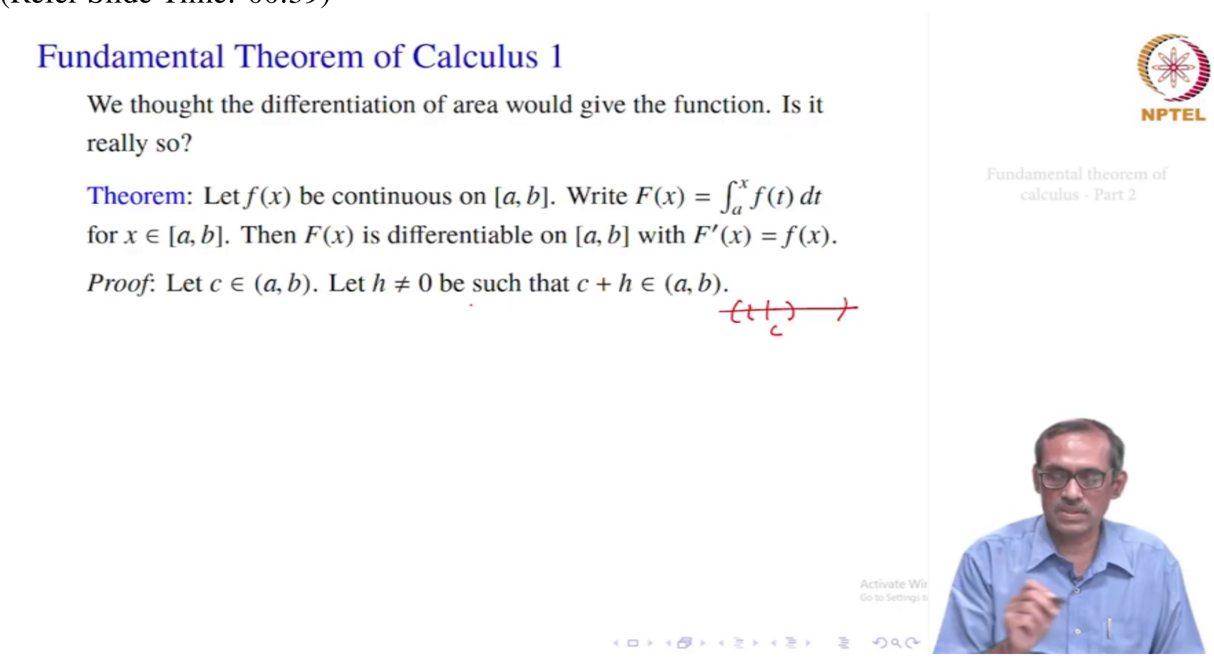

Well, this is lecture 27 of Basic calculus 1. In the last two lectures, we had introduced the notion of definite integral. Remember, our quest was to find a function whose derivative will be the given function. And we came to introduce the notion of area, that particular area below this curve, bounded by that curve, the x-axis and the two lines $x = a$ and $x = b$. We thought that the differentiation of this area would give us the function; this is what we thought and that is why we introduced the area.

The time has come to show that it is really working that way. That means, we need to prove some result like this: "if $f(x)$ is a continuous function on the closed interval $[a, b]$, then you write $F(x) = \int_a^x f(t)\,dt$ for any x inside this interval $[a, b]$. Then, $F(x)$ is differentiable on $[a, b]$ and $f'(x) = f(x)$." That is what we require here. This means that the derivative of the integral will be the curve or the given function. That is what we want. And then you may say that differentiation is the reverse process of integration. Not only that, but also we need to something else. If you take the integral like this: $F(x) = \int_a^x f(t)\,dt$, then its derivative should be equal to small $f(x)$ and then another thing will be that the integral of the derivative should be equal to that function. These results will come as the Fundamental Theorem of Calculus.

The Fundamental Theorem of Calculus 1 use the condition that $f(x)$ is continuous on $[a, b]$. Then, we know that $\int_a^b f(t)\,dt$ exists. So, we call that as the function $F(x)$. Now all that we have to do is show that this $F(x)$ is differentiable and its derivative is equal to $f(x)$ at any point $x \in [a, b]$.

To show this, let us start with some point c inside the open interval (a, b). We will come to the endpoints later. Suppose $c \in (a, b)$. Let us take any h, which is not equal to 0 such that $c + h \in (a, b)$. Now, we are having the point c and then we are choosing some h so that $c + h$ also belong to this interval. That means h cannot go beyond the $|c - a|$ or $|c - b|$; h cannot be larger than those. Such an h can be chosen.

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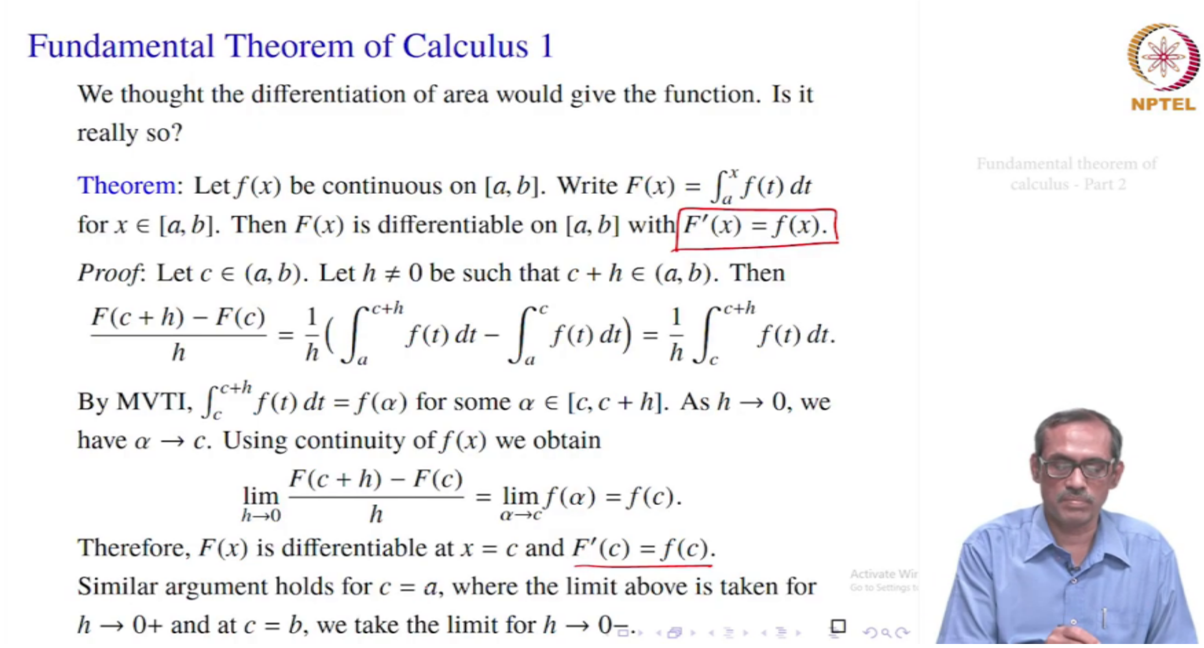

Let us take some h like this so that $c + h \in (a, b)$. Then we formulate this. We want to compute $F'(c)$. By definition, that will be the limit as $h \to 0$ of $[F(c + h) - F(c)]/h$. We try to compute this. What doe this exactly mean in terms of the integrals? $F(c + h) = \int_a^{c+h} f(t)\,dt$ and $F(c) = \int_a^c f(t)\,dt$. So, we subtract the latter from the former and divide by h. Due to the property of the integral, since it is from a to $c + h$ and it is from a to c, if you subtract, you get c to $c + h$. That is what we write. It is $(1/h) \int_c^{c+h} f(t)\,dt$.

We are interested in finding out what is the value of this expression $(1/h) \int_c^{c+h} f(t)\,dt$ as $h \to 0$. Let us look at this integral. We can apply the Mean Value Theorem for Integrals on this, since $h = c + h - c$ is the length of the interval on which the function $f(t)$ is integrated. By the Mean Value Theorem for Integrals, this is equal to $f(\alpha)$ for some α between c and $c + h$. Thus,

$$\frac{F(c+h) - F(c)}{h} = f(\alpha) \quad \text{for some } \alpha \in [c, c+h].$$

Now, if h goes to 0, then where does α go? This α is in between c to $c + h$; so, α will go to c. Now, we are interested in finding the limit of this expression. The limit of $[F(c + h) - F(c)]/h$ as $h \to 0$ is same as the limit of $f(\alpha)$ as $h \to 0$. Since $f(x)$ is continuous, the limit of $f(\alpha)$ will be f of the limit of α. The limit of α as $h \to 0$ is c. So, the limit of $f(\alpha)$ as $h \to 0$ is $f(c)$. We wanted really this, because the left side is the derivative of $F(x)$ at c. So, $F'(c) = f(c)$; that is what we have obtained. Moreover, at $x = c$, the function $F(x)$ is also differentiable. This means, at every interior point of the interval $[a, b]$, we see that $F'(c) = f(c)$.

It remains to decide about the end points a and b. At a it will be the limit as h goes to 0+, from the positive side of 0, of $[F(a+h) - F(a)]/h$. This is a similar process. We continue as above to find that the limit is equal to $F'(a)$, which is also equal to $f(a)$. Notice that $F'(a)$ is only a ne-sided limit here. Similarly, at b $F'(b)$ is equal to the limit of $[F(b) - F(b-h)]/h$ as $h \to 0+$. And, this will be equal to $f(b)$.

Then you see that at the endpoints also similar things happen. That is why we say that $F'(x) = f(x)$ for every $x \in [a, b]$. And that gives the proof of the result.

So, this is only one side of the inverse process as we thought of. That is if you have a continuous function $f(t)$, we define this particular $F(x)$ using the integral, then its derivative will equal to $f(x)$. That is, the derivative of the integral is equal to the function; that is what basically it says.
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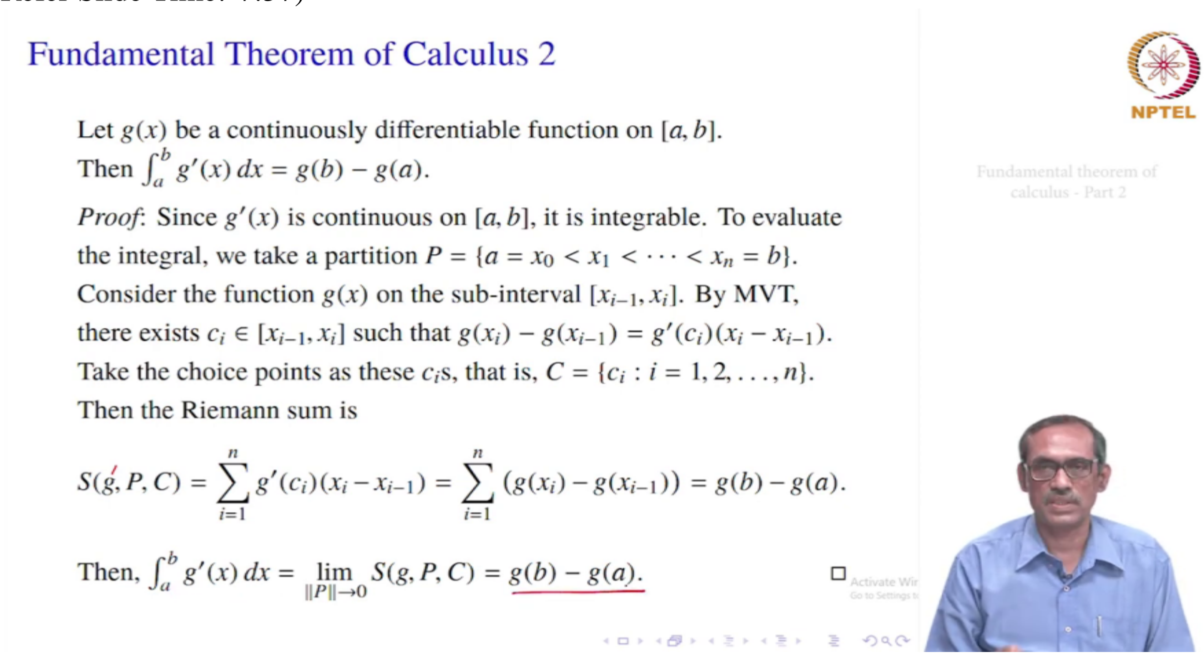

Let us see the other process that the integral of the derivative is also equal to that function also. So, let us formulate it first. Let $g(x)$ be a continuously differentiable function on $[a, b]$. You want the derivative of g to be continuous so that integration will exist. It can exist under some weaker restrictions, but we are happy with continuity itself. So, then what does it say? The result says that $\int_a^b g'(x)\,dx = g(b) - g(a)$. That means if you take any point c instead of b it would give you $g(c) - g(a)$. That is, you get the integral of the derivative equal to that function minus $g(a)$ there. We will see how to reach that.

Let us have a proof of this. The assumption is that $g'(x)$ is continuous on $[a, b]$. Since it is continuous, it is integrable. So it makes sense to write $\int_a^b g'(x)\,dx$. You want to evaluate this and see that it is equal to $g(b) - g(a)$. To evaluate this integral, by definition, we have to take a partition, form the Riemann sum, take the limit as the norm of the partition goes to 0. So, let us start with the partition, say, $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is a partition of the closed interval $[a, b]$. Consider the function $g(x)$ on each sub-interval $[x_{i-1}, x_i]$. It is a continuous function on this closed interval. Of course, it is differentiable because that is assumed. Then by the Mean

Value Theorem for the derivatives, there exists a point c_i inside this interval $[x_{i-1}, x_i]$ such that $g(x_i) - g(x_{i-1}) = g'(c_i)(x_i - x_{i-1})$.

If you remember, we have used this trick in solving one or two problems earlier. In fact, we have borrowed this idea from this proof, and now we have come back to the original proof.

Here, we use this Mean Value Theorem to find a point c_i such that $g'(c_i)$ is equal to $[g(x_i) - g(x_{i-1})]/(x_i - x_{i-1})$. Since these c_is are fixed inside the sub-intervals, we choose our choice set as the set of all these points c_i. Then, the Riemann sum $S(g, P, C)$ will be equal to $\sum_{i=1}^{n} g'(c_i)(x_i - x_{i-1})$. Due to our Mean Value Theorem, this sum is equal to $\sum_{i=1}^{n} g(x_i) - g(x_{i-1})$. This is a telescoping sum. It will simplify to $g(x_n) - g(x_0) = g(b) - g(a)$. Then, you take the limit as $\|P\| \to 0$. Since it is a number, it is a constant, the limit also should be equal to that constant, which is $g(b) - g(a)$. And, that is the proof. So, the proof ends here proving that $\int_a^b g'(x)\,dx = g(b) - g(a)$.

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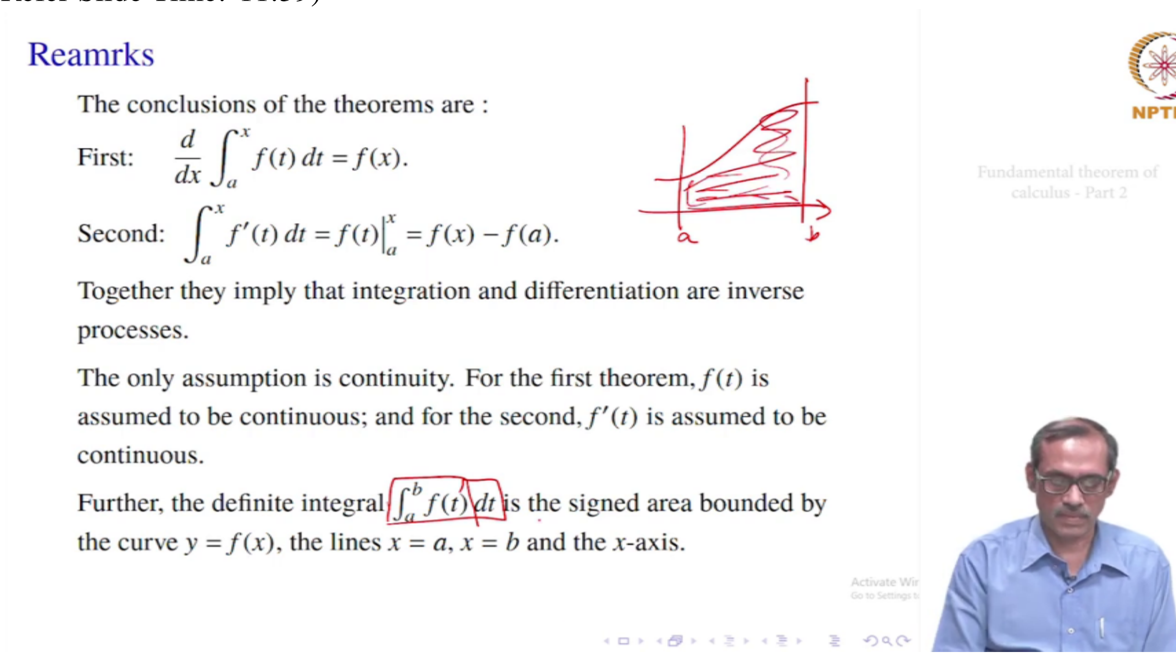

So, let us summarize. The first theorem says that the derivative of $\int_a^x f(t)\,dt$ is equal to $f(x)$. In the second one, let us take the closed interval $[a, x]$ instead of $[a, b]$. And let us write $f'(x)$ instead of $g'(x)$. Then, it states that $\int_a^x f'(t)\,dt = f(x) - f(a)$. We have got these two things from the Fundamental Theorem. Together they imply that integration is really the reverse process of differentiation. You would get integral of the derivative as the function up to some constant, and the derivative of the integral as the function.

This constant cannot be avoided because we know that if derivatives of two functions are same, then those two functions can differ by a constant. Because we take the integral from a to x we have that constant as $f(a)$. We remark that the only assumption we use is continuity. In the first one we have continuity of f and in the second one we have continuity of f'. That is how we get these results.

By definition, the definite integral $\int_a^b f(t)\,dt$, is the signed area of the region bounded by the curve $y = f(x)$ or $y = f(t)$ and the lines $x = a$, $x = b$ and the x-axis. The region looks something like this. This is the signed area which is our integral $\int_a^b f(t)\,dt$. Starting with that we have obtained these two results.

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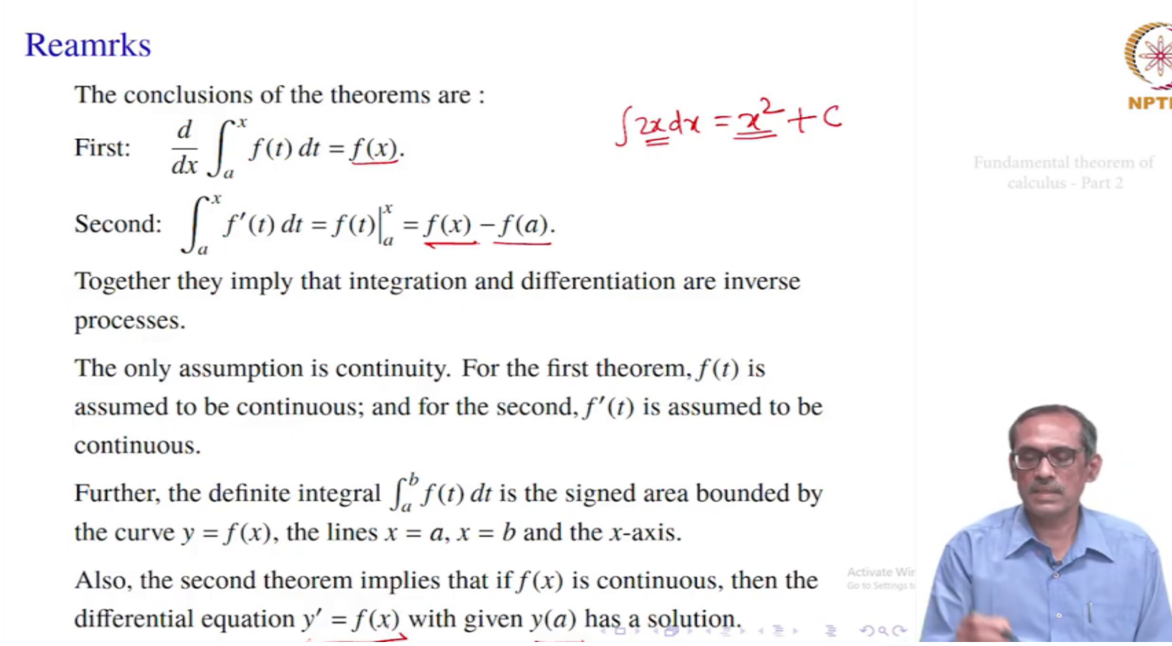

The second theorem implies that if $f(x)$ is continuous, then the differential equation $y' = f(x)$ has a solution. What does that mean? Solving the differential equation means you start with $y' = f(x)$; and you want to find one function y such that its derivative will be equal to $f(x)$. The answer is that such a function y should be the integral. In the integral, you would get a constant. So, once you know $y(a)$, you can completely solve the differential equation. Otherwise, you can always solve this differential equation to get at least one solution where a constant will be present. We will see how it is applied later.

We will introduce something new here. Since in the integral, we have $f(x) - f(a)$, it does not say that the integral of $2x$ is x^2. If it is a reverse process, then you should get the integral of $2x$ as x^2. But we know that it can differ by a constant. So, it needs some more formalization.

We will say that a function $F(x)$ that satisfies $F'(x) = f(x)$ is an antiderivative of $f(x)$. That means you differentiate that you get this function. So, in a way it is an antiderivative of $f(x)$. The first theorem shows that an antiderivative of a continuous function $f(x)$ can be given by $F(x) = \int_a^x f(t)\,dt$. Because, once you differentiate this, you would get back $f(x)$. So, this $F(x)$ is an antiderivative of $f(x)$.

However, we know that $f'(x) = g'(x)$ implies $f(x) = g(x) + C$ for some constant C. So, in general, the antiderivative can be written or should be written this way. It is because two functions having the same derivative might differ by a constant. If we take another function g, that g will be equal this antiderivative of $f(x)$ plus C. It means that an antiderivative can be written as $\int_a^x f(t)\,dt + C$. We do not know what is this constant C. In general, we will be writing or giving another symbol for the general antiderivative. We will write $\int_a^x f(t)\,dt + C$ as an integral without any limits, that is, as $\int f(x)\,dx$. It is also called as the indefinite integral of the function $f(x)$.

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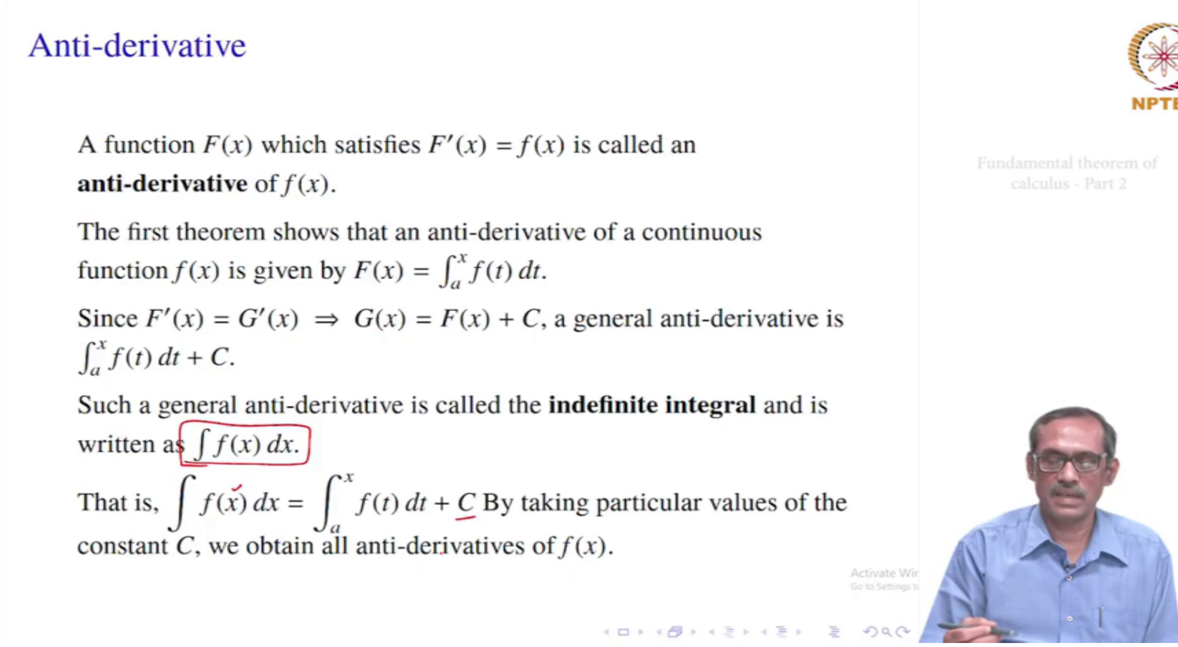

That means the indefinite integral $\int f(x)\,dx$ is equal to $\int_a^x f(t)\,dt + C$ for some constant C. In particular, we may take $C = f(a)$, the functional value of f at $x = a$. In fact you can take any other point instead of a. The constants are accommodated in an indefinite integral. So, in general, we will be talking of that indefinite integral as this, where we do not write the limits bof integration such as a to b or a to x. Fine. We see that the indefinite integral $\int f(x)\,dx$ is the definite integral $\int_a^x f(t)\,dt$ plus a constant C. When you take a particular value of C, we would get back an antiderivative. So, all the antiderivative are given by this where C varies over all real numbers. These are all possible antiderivatives.

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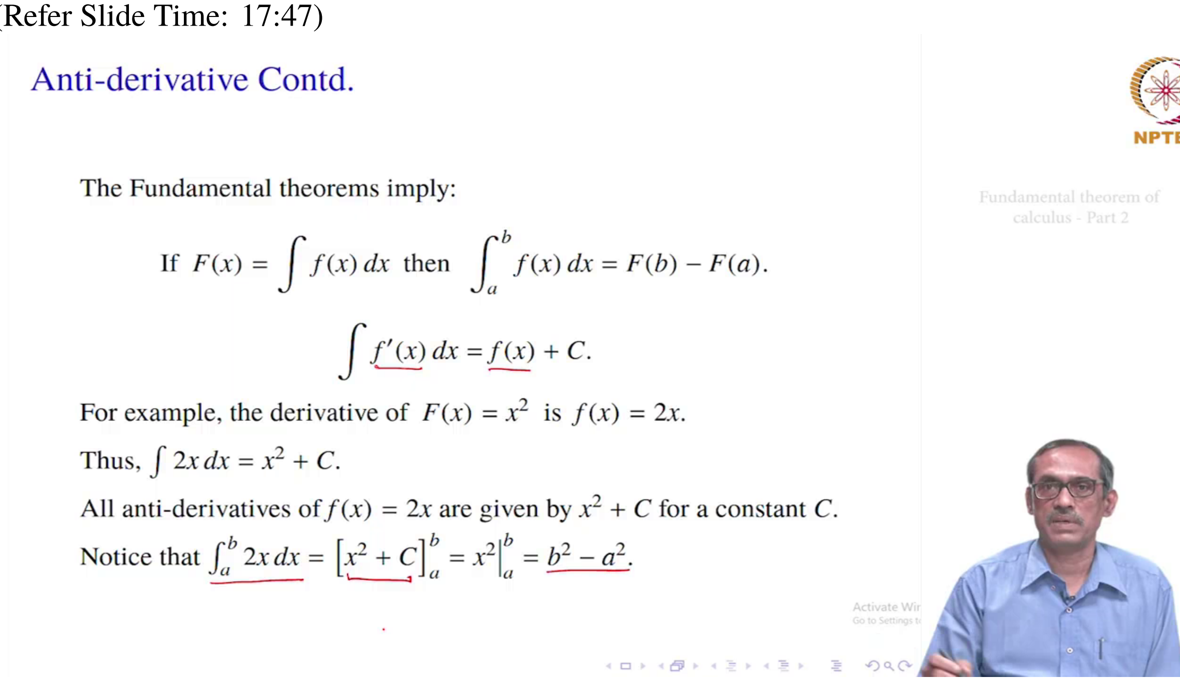

Look at the fundamental theorems. We can write them in terms of the antiderivatives. If you write $F(x)$ equal to this indefinite integral $\int f(x)\,dx$, then once we put the limits of integration, this indefinite integral turn into a definite integral, and then it will be equal to $F(b) - F(a)$. For instance, take $F(x) = x^2 + c$. It is an antiderivative of $2x$ as $x^2 + c = \int 2x\,dx$. Then you take $F(b) - F(a)$. That will give $b^2 + c - (a^2 + c) = b^2 - a^2$. You see that constant c is gone. That should be equal to $F(b) - F(a)$. This is same thing as $\int_a^b 2x\,dx$, that is, x^2 evaluated at b and at a, and then subtracted. We say that if $F(x)$ is the indefinite integral, then the definite integral $\int_a^b f(x)\,dx = F(b) - F(a)$.

Also, we have seen the second fundamental theorem. It says that the indefinite integral of $f'(x)$ is equal to $f(x) + C$. In terms of the definite integral, it looks like $\int_a^x f'(t)\,dt = f(x) - f(a)$. The constant C here is $-f(a)$. In the indefinite integral that $-f(a)$ is accommodated by talking C to be arbitrary.

These are the main two results, which are called the fundamental theorems. If $F(x)$ is the indefinite integral, then that when evaluated from a to b, we get the definite integral. That is, $\int_a^b f(x)\,dx = F(x)\big|_a^b = F(b) - F(a)$. And, the indefinite integral of $f'(x)$ is $f(x) + c$. That is, $\int f'(x)\,dx = f(x) + C$.

After this, we will not always read it as an indefinite integral, we just say "integral $f'(x)\,dx$", and for the definite integral, we will say "integral from a to b $f(x)\,dx$.

And we will evaluate the definite integrals by using the above notation. For example, in evaluating $\int_a^b 2x\,dx$ we use the fact that the derivative of x^2 is $2x$. So, the indefinite integral of $2x$ is $x^2 + C$. Of course, this C will get canceled when we take $F(b) - F(a)$. So, we write it as $\int_a^b 2x\,dx = x^2\big|_a^b = b^2 - a^2$. This will help us in evaluating the integrals provided you know relevant results about the derivatives. If you know that the derivative of f is F, if you know how to connect them by $\int F = f$. For instance, we know that the derivative of $\tan x$ is $\sec^2 x$. So, you can say that $\int \sec^2 x\,dx = \tan x + C$. That is the way we will be using the derivatives to come back to integrals.

These fundamental theorems will help us in finding out the definite and indefinite integrals, provided we have the knowledge of the derivatives of some elementary functions. That is how the derivatives and the integrals are related. If we go back to our earlier repository that such a function has such derivative, then we can now rewrite those in terms of the indefinite integral.