Basic Calculus - 1
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Lecture 26 - Part 2
Properties of integral - Part 2

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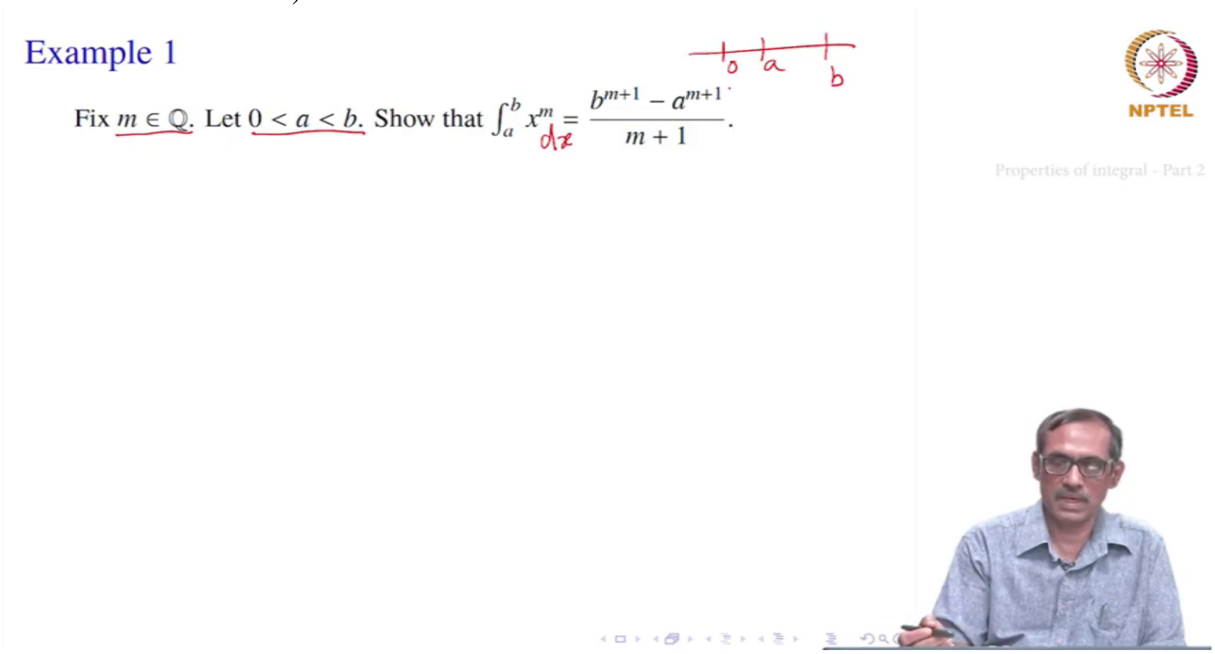

Let us take an example see how these properties help us in evaluating integrals. Suppose we take some m, which is a rational number, because only rational powers we have defined. You can take m to be a real number whenever it is well defined, but it should be continuous function and so on. We make it simpler. Let us take m to be a rational number in the form p/q. Suppose $0 < a < b$. You have the interval $[a, b]$ this way. Then, you take $\int_a^b x^m \, dx$, the integral of x^m on this interval a to b. This is the power function x^m, which is defined on this interval as m is a rational number. (The dx symbol is missing here.) That is equal to $b^{m+1} - a^{m+1}$. It means, we have to really compute the Riemann sum and take the limit, and then see that it happens.

Let us see how do we proceed. We take a partition of the interval $[a, b]$ which is $a = x_0 < x_1 < \cdots < x_n = b$. Suppose this is the partition. Let us fix any i in between this 1 to n. That means we are fixing a sub-interval $[x_{i-1}, x_i]$. In this sub-interval, let us define the function $g(x) = x^{m+1}$. This is a clever way of evaluating the limit of the Riemann sum. We have done it in another problem earlier while finding the integral of $\cos x$.

Let us consider $g(x) = x^{m+1}$ on the interval $[x_{i-1}, x_i]$. Now, $g(x)$ is differentiable in the open interval (x_{i-1}, x_i) with $g'(x) = (m+1)x^m$. That is, for any point x in the open interval we know that its derivative is this. We apply the Mean Value Theorem on g to see that $g(x_i) - g(x_{i-1}) = g'(c_i)(x_i - x_{i-1})$ for some point c_i between x_{i-1} and x_i. That is, there is a point c_i in a every sub-interval (x_{i-1}, x_i) such that this thing happens. Now what do we do, in our Riemann sum we take take these c_is our points of the choice set or choice points.

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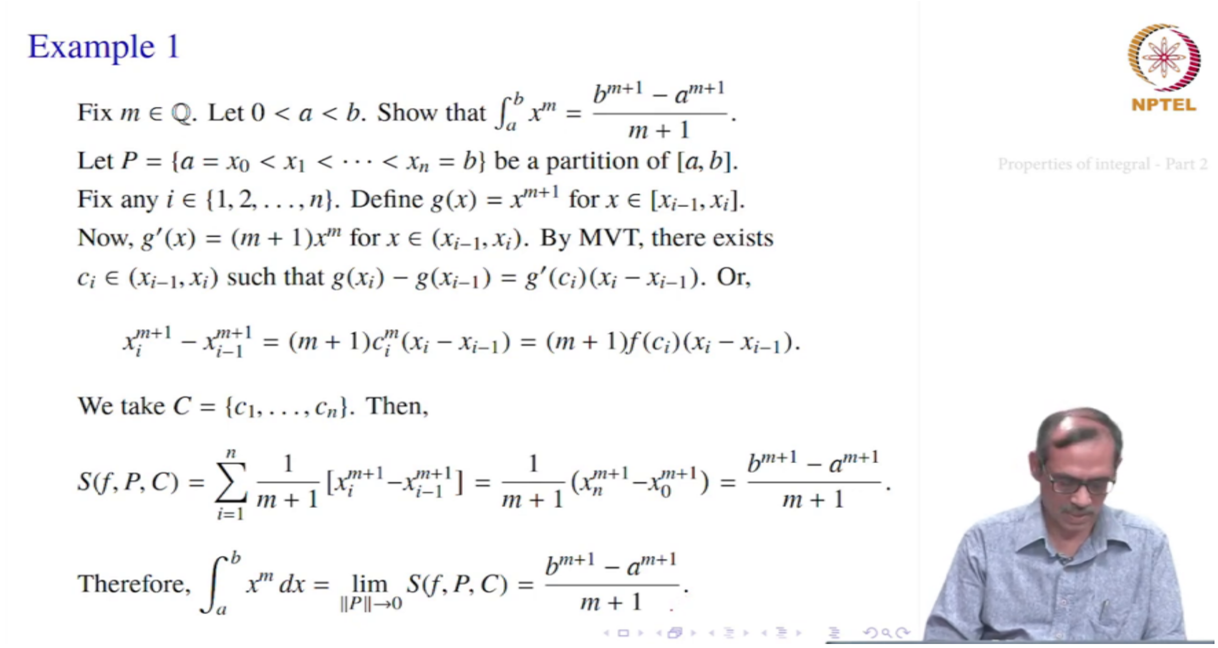

So, take these c_is as the choice points. With these c_is our Riemann sum is equal to $\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$. What is that $f(c_i)$? It is $f(c_i) = c_i^m(x_i - x_{i-1}$. So, $g'(c_i) = (m+1)c_i^m$. The term inside the Riemann sum is $f(c_i)(x_i - x_{i-1})$. This is equal to $g'(c_i)(x_i - x_{i-1})$. We know that it is equal to $g(x_i) - g(x_{i-1})$. The sum of these terms becomes the Riemann sum.

Now, we want to sum it. What will be the sum? This m is not dependent on i; we take it out. Then, the sum is

$$[1/(m+1)]\big([g(x_1) - g(x_0)] + [g(x_2) - g(x_1)] + \cdots + [g(x_n) - g(x_{n-1})]\big).$$

You see that this minus term remains, but this one is cancelled, all of them will be cancelling except the last term, which is $g(x_n) - g(x_0)$. We get $x_n^{m+1} - x_0^{m+1}$. But $x_n = b$ and $x_0 = a$. So, you get $b^{m+1} - a^{m+1}$ divided by $m+1$. This is what we wanted to show.

Due to our choice points c_i, specifically chosen this way we see that the Riemann sum itself becomes a number. Therefore, in the limit the same number stays, and we get the integral $\int_a^b x^m\,dx = [b^{m+1} - a^{m+1}]/(m+1)$.

This trick will be helpful whenever you have a very complicated function, and where you cannot really do anything through the limits. You apply the Mean Value Theorem for the differential. Otherwise, it may not be required. Sometimes we just get it from the function itself. This helps in evaluating the limit.

Let us take another example. Evaluate $\int_{-1}^{1}(2 + 3x^2 + 4x^5)\,dx$. We use the previous example and our properties. First thing is, the integral of 2, a constant function, is simply $2x^0$. That gives, or directly you can get from our property, that $\int_{-1}^{1} 2\,dx = 2\int_{-1}^{1}\,dx = 2(b-a) = 2(2-(-2)) = 4$.

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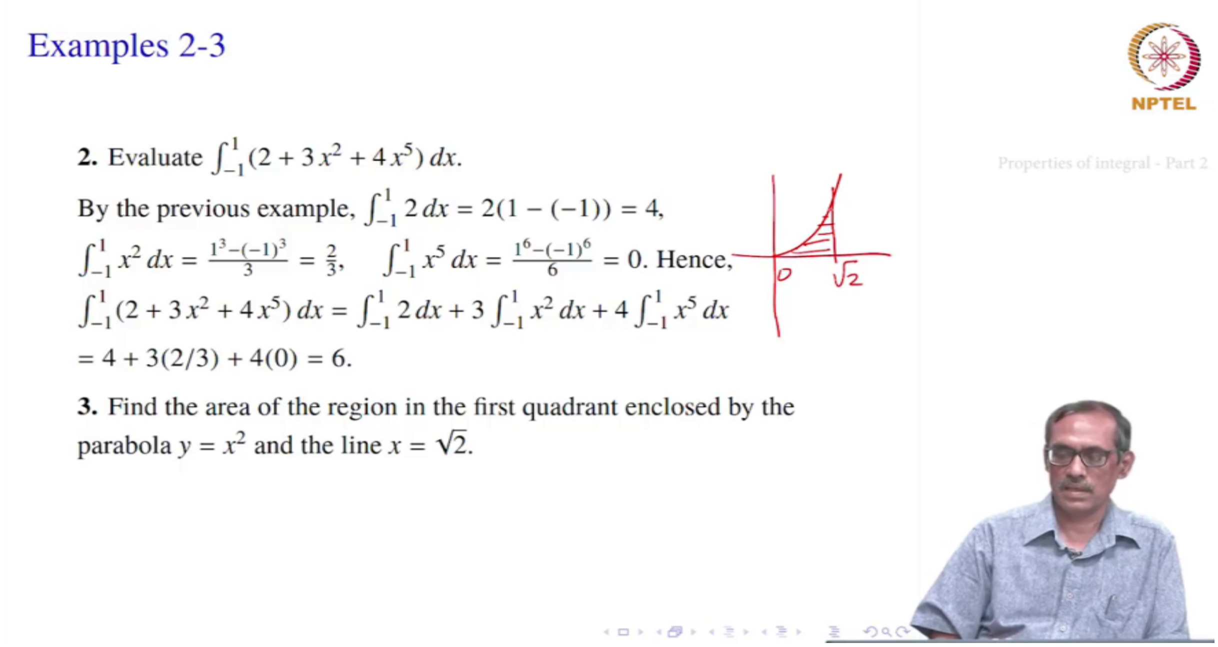

We go slowly for the other one. The integral $\int_{-1}^{1} x^2\,dx$ is evaluated by using the last example again. It is $b^{m+1} - a^{m+1}$ divided by $m+1$, which is $1^3 - (-1)^3$ divided by $2+1$, which is equal to $2/3$.

Similarly, if you take x^5, its integral is $1^6 - (-1)^6$ divided by $6+1$, which is equal to 0.

So, our integral will be this integral plus 3 times, because of property 2, plus four times this integral. Now, it is easy to simplify and see that the answer is 6.

You see how the properties help us in integrating a polynomial. If you know how to integrate the powers, that is, x^m.

Let us see the third example. Here, we are asked to find the area of the region in the first quadrant. We want to compute the area of something in the first quadrant enclosed by the parabola $y = x^2$. It looks something like this $y = x^2$; and the line $x = \sqrt{2}$. This is $\sqrt{2}$, so this is the area we want to compute.

This area will be equal to the integral $\int_0^{\sqrt{2}} x^2\,dx$. The curve is $y = x^2$ and the region bounded by that is this integral from 0 to $\sqrt{2}$ of x^2. We know how to integrate x^2. It is x^3 evaluated at $\sqrt{2}$ and 0, subtracted and then divided by $2+1$. That gives $(\sqrt{2})^3/3$, which is $2\sqrt{2}/3$.

Let us take one more example. You want to show that the integral of $\sqrt{1+\cos x}$ over the interval 0 to 1 is smaller than 1.5. It is an estimation. So, somewhere we have to use some estimation; that is what we will do. You see that for $x \in [0,1]$, $\cos x$ is always less than 1. I think that gives.Let us try. Then, $\sqrt{1+\cos x}$ will be less than $\sqrt{1+1} = \sqrt{2}$. So, $f(x) \le g(x)$ where $f(x) = \sqrt{1+\cos x}$ and $g(x) = \sqrt{2}$, a constant function here, which we can readily integrate.

So, the integral of $\sqrt{1+\cos x}$ from 0 to 1 will be less than the integral of $\sqrt{2}$ from 0 to 1, which is $\sqrt{2}$ times $(1-0)$. It is $k(b-a)$, that is, $\sqrt{2}(1-0)$. And, $\sqrt{2} < 1.5$. Hence the result.

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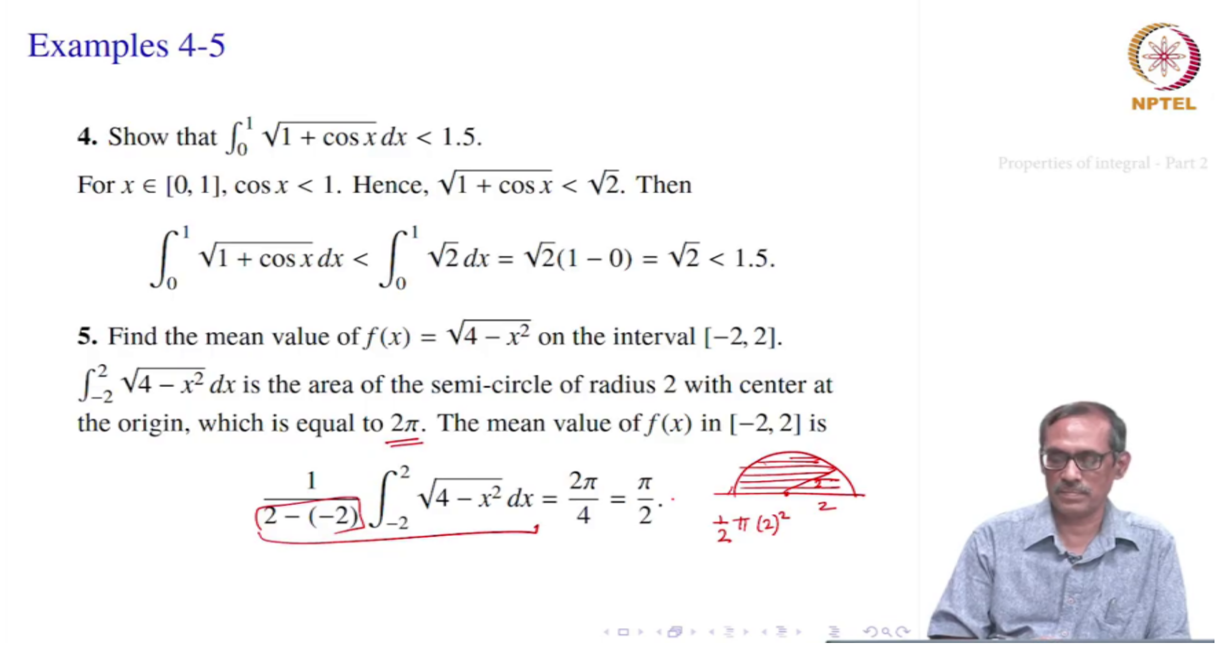

Sometimes, these numbers can be different, and you do not know which property of the integral and which estimation are to be applied. But only trial and experience give you some way. We have chosen $\cos x \leq 1$ here, not any other inequality.

Let us go to the fifth problem, the fifth example. We want to find the mean value of this function on the interval $[-2, 2]$. The mean value is $1/(b-a)$ times the integral $\int_a^b f(x)\, dx$. It is straightforward.

We can find the integral. Our integral is $\int_{-2}^2 \sqrt{4-x^2}\, dx$. The mean value will be this integral divided by $2-(-2)$ or 4. What is this integral really? Here, $y = \sqrt{1-4x^2}$ is a circle; it is a circular arc, a semicircle really. The square root is positive only; so, it lies on the upper half plane. And $\sqrt{4-x^2}$ means that its center is at 0 and radius is 2. Now you see that $x^2 + y^2 = 4$ gives you $y = \sqrt{4-x^2}$; right? That is the curve. We know its area, the area of this semi circular region is equal to 2π, because its radius is 2. The area of the full circle is πr^2; so that of the half circle will be $\pi/r^2/2$ with $r = 2$; it gives 2π. Then, the mean value of this function is equal to 2π divided by this 4, which is $\pi/2$. Since we know the area we proceeded this way instead of computing the integral directly.

Let us take one more. Find the values of a and b so that the integral $\int_a^b (x-x^2)\, dx$ is maximum. What happens here is that if you have the interval $[a, b]$, where a and b are given and a function $f(x)$ is given, then the integral is a number. But here the values of a and b are not given; these are unknowns. So, this number which is the definite integral will depend on both a and b. The question is to choose a and b in such a way that this is the maximum number you will be getting from that. Right? That is what we are being asked.

Let us look at the function first. The function is $x - x^2$, which we can factor as $x(1-x)$. The reason is, we will use a very simple argument. Suppose, the function is something like this, which has some negative values and some positive values. If we want to maximize the area bounded by the function and the x-axis, and these two lines $x = a$ and $x = b$, then you take a to avoid the negative values. For instance, we will take a to be here. Because it needs to be maximum. That is, if at all the function has both positive and negative values, we should choose our a to be here. And we do not know till now where is b. But this is the guiding principle for choosing that also.

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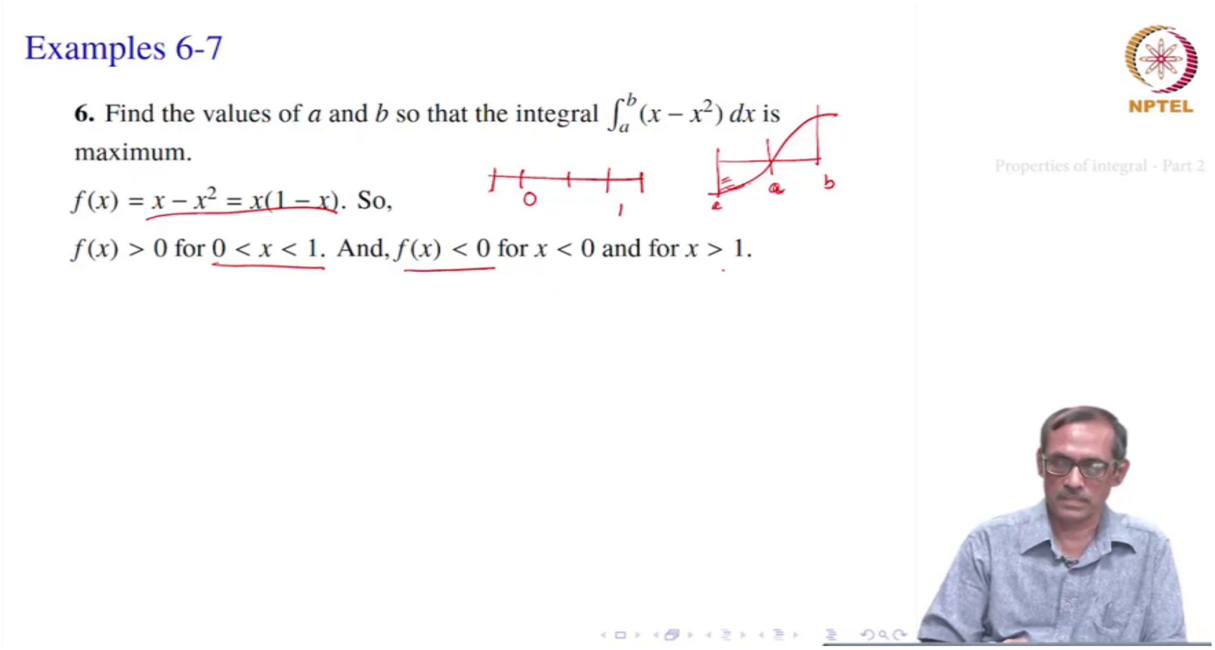

So, suppose $f(x) > 0$ somewhere. Then we can choose our a like this. But for this given function, we should know where it is bigger and where it is smaller than 0. This function is $x(1-x)$. That is, if you take 0 here, 1 here, you take any point x in between 0 and 1, then one of them is $x - 0$ and another is $x - 1$. One of them will be positive and one of them will be negative. This is $1 - x$, so both will be positive and their product will become positive if it is between this 0 and 1.

If it is bigger, say x is here bigger than 1, then $1 - x$ is negative, but x is positive; so, that will be negative. Or if it is less than 0, then x itself is negative, but $1 - x$ is positive; so, it is again negative. What we observe is that for $0 < x < 1$, that is, x between 0 and 1, $f(x)$ is positive. Otherwise, $f(x)$ is negative for $x < 0$ and for $x > 1$.

Once we are going to find this integral to be the maximum, we should find out where the function is positive. The picture would look something like this. Not exactly, but like this. So here it is 0, here it is 1, and between 0 to 1 the function remains positive, and elsewhere it becomes negative. To maximize the value of the integral, we should take only the positive part, and this part should go away. That means we choose our a to be this point, b to be that point where the function remains positive; that is, we choose $a = 0$ and $b = 1$. See, we have not computed the integral, but some geometric argument gives the result.

Let us come to the seventh example. Suppose f is a continuous function defined on the closed interval $[a, b]$, where we generally assume that $a < b$. But suppose along with that you also have $\int_a^b f(x)\,dx = 0$. Then, show that $f(x)$ has a zero in $[a, b]$. It means that there is a point c such that $f(c) = 0$. Well, we can see the geometric argument first to have a feeling of what is going on.

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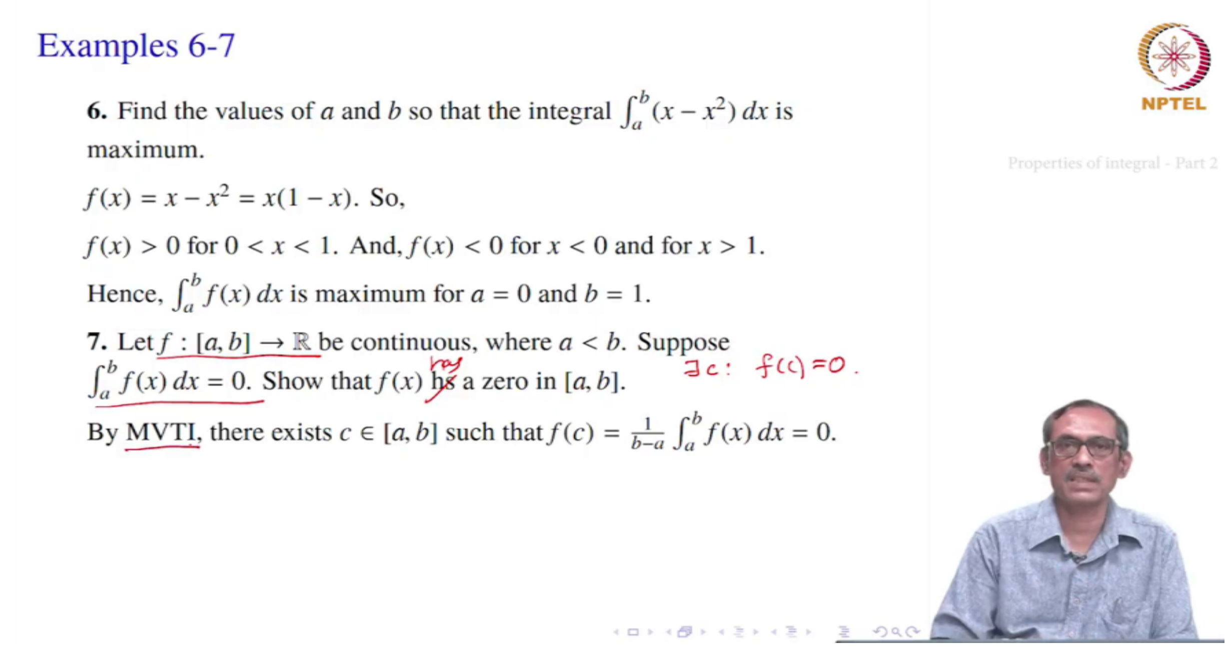

Suppose our function $f : [a, b] \to \mathbb{R}$ is positive throughout $[a, b]$. Then the integral will become positive, which is the area. If f is negative throughout then its integral also will become negative. But the integral is 0. That means at some points f is positive, and at some points f is negative. Since f is continuous, by the Intermediate Value Theorem f must be 0 somewhere. That is the argument we are going to give. But we can also directly do it because we have the Mean Value Theorem for integrals available with us. In fact, that is exactly the proof of Mean Value Theorem for Integrals, whatever argument we thought.

We can now apply the Mean Value Theorem for Integrals directly. By the Mean Value Theorem for Integrals, there exists a point $c \in [a, b]$ such that $f(c) = (1/(b - 1)) \int_a^b f(x)\,dx$. But this integral is 0. So, $f(c)$ must be equal to 0.

The job became easier for us due to this. Alternatively, you can argue whatever way we have just told but using the Mean Value Theorem for Integrals. That is, if $f(x) > 0$ for each x, then its minimum exists and is achieved at some point in $[a, b]$ due to continuity of $f(x)$. Let us say m is equal to the minimum of $f(x)$. Now $m > 0$ and this $m(b - a) > 0$. Then the integral is also bigger than 0.

To repeat, since f is continuous, it achieves its minimum, say m inside the interval $[a, b]$ at some point c. Since $f(x) > 0$ for each x, we have $m = f(c) > 0$. Then, $\int_a^b f(x)\,dx \geq m(b - a) > 0$. Similarly, if $f(x) < 0$ for each x then you would reach at the conclusion that this integral is less than 0. But the integral is given to be equal to 0. Is that okay?

We will do one or two problems and then, may be, conclude today. First problem asks to evaluate the integral $\int_{-1}^{1} (2 - |x|)\,dx$. How do we proceed? If you draw the figure, it is $2 - |x|$. It will be looking something like this. This distance is 2, and you join it something this way. As it is from -1 to 1, at -1, it will be something like this. So, it is this area we have to compute.

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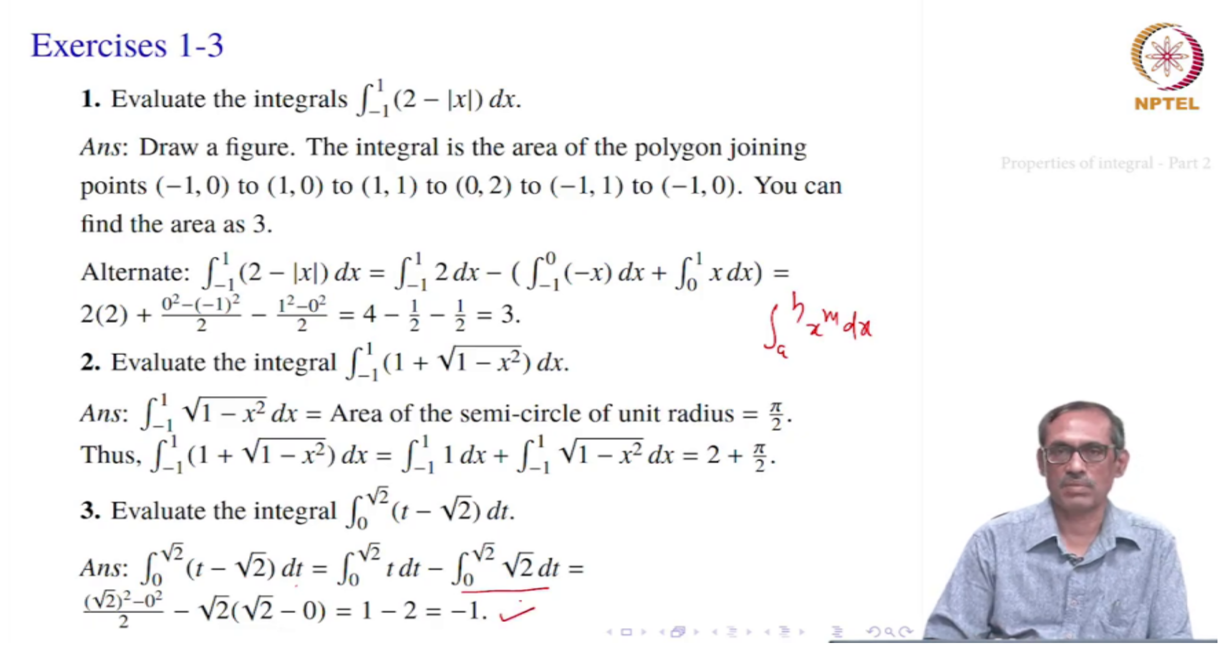

You can see the polygon now. It joins the points $(-1,0)$ to $(1,0)$ to $(1,1)$ to $(0,2)$ to $(-1,0)$. The integral is the area of this polygon, which is easier to compute. This is -1 to 1, so length is 2; this is 1; and this is same thing as this area. So, you have 3 times 1, and the answer should be 3.

You can also do directly through integration. The integral is equal to integral of 2 minus that of $|x|$. Now, $\int_{-1}^{1} -|x|\,dx$ can be broken into two integrals such as $\int_{-1}^{0} -|x|\,dx$ plus $\int_0^1 -|x|\,dx$. In $[-1,0]$, $|x|$ becomes $-x$ and in $[0,1]$, $|x|$ becomes x. So,

$$\int_{-1}^{1}(2-|x|)\,dx = \int_{-1}^{1} 2\,dx - \left(\int_{-1}^{0}(-x)\,dx + \int_0^1 x\,dx\right).$$

Now if you just evaluate them using our earlier formula, you also get back the answer as 3.

Now, we take this integral: $\int_{-1}^{1}\left(1+\sqrt{1-x^2}\right)dx$. We do not know right now how to integrate $\sqrt{1-x^2}$. But you can use the earlier argument. We think of what curve is it, and whether we know its area geometrically or not. In the integral $\int_{-1}^{1}\sqrt{1-x^2}\,dx$ the curve is $y=\sqrt{1-x^2}$. It describes the semicircle of unit radius with center at the origin that lies above the x-axis. So, its area is $\pi/2$. Then, you can use the properties of the integral to see that

$$\int_{-1}^{1}\left(1+\sqrt{1-x^2}\right)dx = \int_{-1}^{1} 1\,dx + \int_{-1}^{1}\sqrt{1-x^2}\,dx = 2 + \frac{\pi}{2}.$$

Let us take the last problem. Evaluate $\int_0^{\sqrt{2}}\left(t-\sqrt{2}\right)dt$. Recall our remark that $\int_a^b f(t)\,dt = \int_a^b f(x)\,dx$. To compute this integral we break it into two integrals: $\int_0^{\sqrt{2}} t\,dt$ minus $\int_0^{\sqrt{2}}\sqrt{2}\,dt$. We can apply our earlier formula that $\int_a^b x^m\,dx = (1/(m+1))(b^{m+1}-a^{m+1})$. That gives the first integral equal to $[(\sqrt{2})^2-0^2]/2$ and the second integral equal to $\sqrt{2}(\sqrt{2}-0)$. Then, the answer simplifies to -1. Let us stop here today.