Basic Calculus - 1
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Lecture 26 - Part 1
Properties of Integral - Part 1

So, this is lecture 26 of Basic calculus 1. In the last lecture, we had introduced the notion of definite integral. Given a function from a closed interval $[a, b]$ $\mathbb{R}$ we compute its integral or definite integral by dividing $[a, b]$ into n-many sub-intervals $[x_{i-1}, x_i]$; we call the set of all endpoints of the sub-intervals a partition P; we define the norm of the partition as the maximum of the lengths of these sub-intervals; we choose points c_i in the ith sub-interval and write the set of all choice points as C; we form the Riemann sum $S(P, f, C) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$. If the limit of the Riemann sum exists as the norm of P goes to 0, that is, as the lengths of sub-intervals become smaller and smaller approaching 0, then we say that $f(x)$ is integrable (over $[a, b]$); and then we define this limit as the definite integral $\int_a^b f(x)\,dx$.

Today, we will look at some properties of this definite integral, which will help us in evaluating integrals of complicated functions, provided we know the integrals of the simpler functions. So, let us try these properties.

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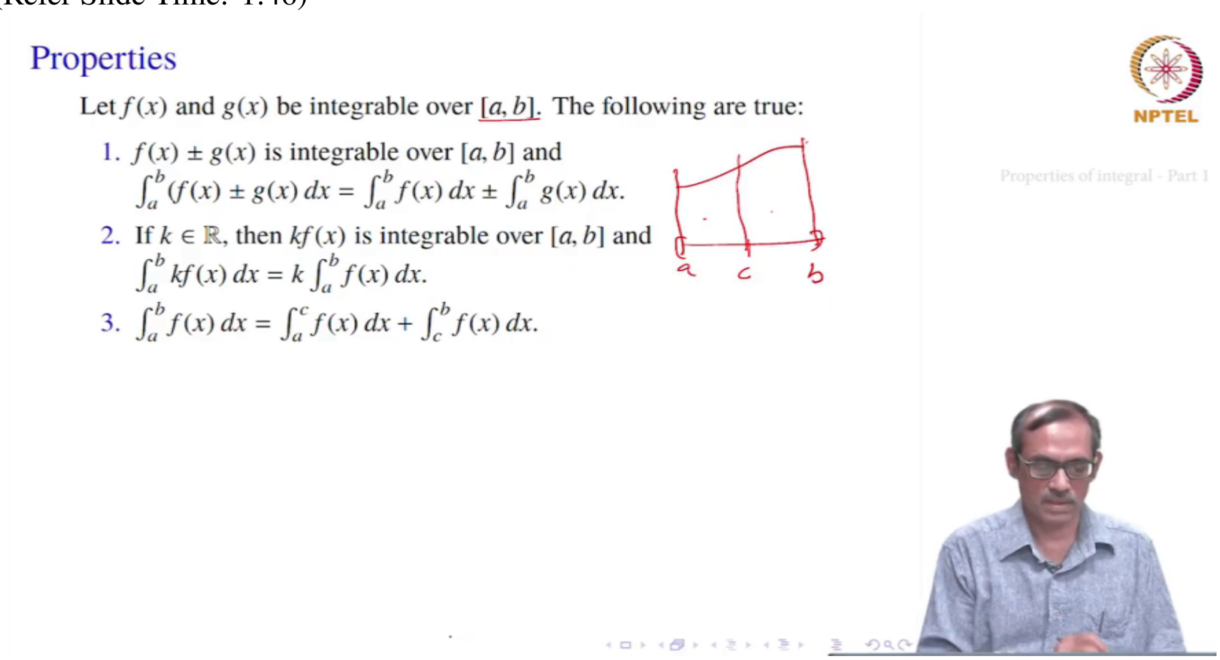

Assume that $f(x)$ and $g(x)$ are integrable over $[a, b]$. That means the integral exists. The first property says that if $f(x)$ and $g(x)$ are integrable, then their sum $f(x) + g(x)$ is also integrable, and also the integrals add up. That is easy to see; because the limit of the Riemann sum exists for whichever c_is you choose. So, you choose c_is for $f(x) + g(x)$ and then split that Riemann sum to Riemann sum for f plus the Riemann sum for g. In the limit, they will become integral $f(x)$

plus integral $g(x)$. Therefore, it is integrable and the integral of the sum is equal to some of the integrals. Similar thing happens when the plus sign is replaced by a minus sign.

Suppose k is any real number, that is a constant, then k times $f(x)$, which is another function, is also integrable. And again, similar things hold. That is, the integral of $kf(x)$ is equal to $k$times the integral of $f(x)$. This follows directly from the Riemann sum definition.

We go to the next one. In all these things we assume that $a < b$. If c is a point between a and b, then you can break the interval $[a, b]$ into $[a, c]$ and $[c, b]$.

What it says is that the integral over $[a, b]$ will be equal to the integral over $[a, c]$ plus the integral over $[a, b]$. This is easy to see because the integral is simply an area. The area bounded by this, if you draw a line here, the whole area is equal to sum of these two areas; that is what this property means.

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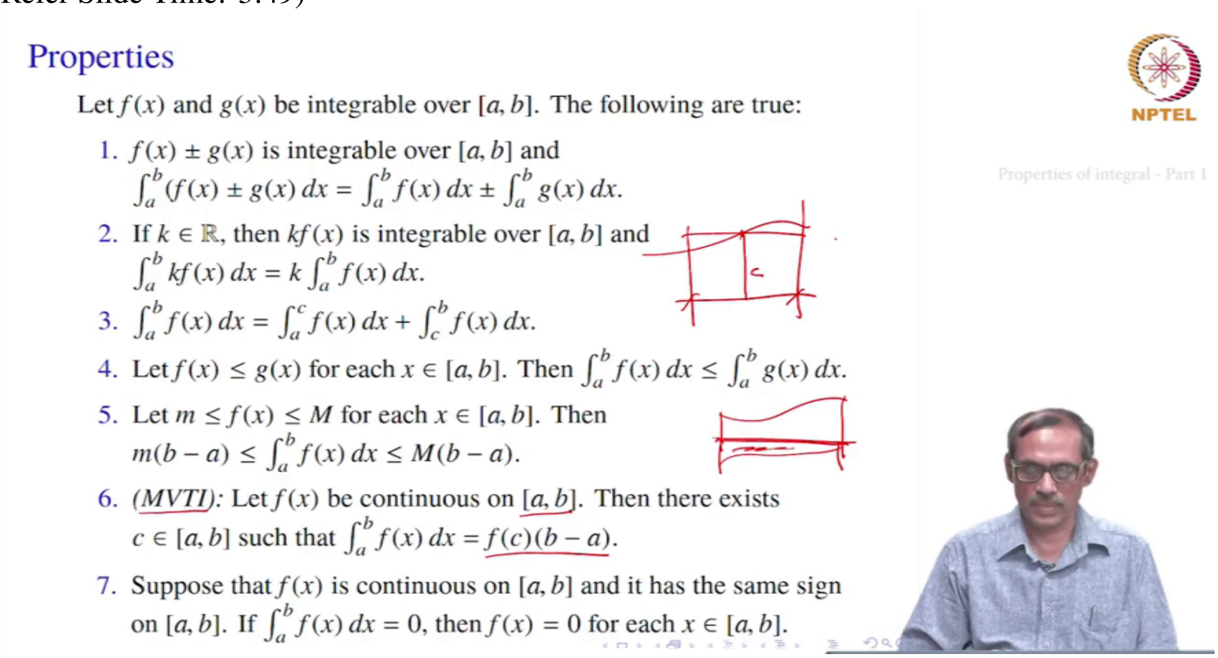

Suppose $f(x) \le g(x)$ at every point $x \in [a, b]$. That is, g is dominating f. Then the same inequality also will hold for the integrals. That is also clear from the Riemann sum definition. If you see geometrically, it means that f is some curve, and g is lying above it; maybe at some points, yheir values are equal like this. Then, the area under g will be bigger than ar equal to the area under f; that is what it says.

Suppose $f(x)$ is bounded over this $[a, b]$, where m is a lower bound for $f(x)$ and M is an upper bound for $f(x)$. This means $m \le f(x) \le M$ for every $x \in [a, b]$. Then, something similar holds for the integral, but now we multiply $b - a$ to both the sides. The reason is very clear geometrically. You have some function $f(x)$ and you have this area that is the integral of $f(x)$ over $[a, b]$. If you take m as the minimum or even which is smaller than every $f(x)$, say, this one is your m, then $m(b - a)$ is the area of this rectangle. If you take capital M as anything which is larger than every $f(x)$, then $M(b - a)$ is this area. Clearly, the area under $f(x)$ which is the integral lies between these two rectangles. That is what it says.

The next property is not so obvious looking. It is called the Mean Value Theorem for Integrals, just like our Mean Value Theorem for differentials, which we just write as the Mean Value Theorem. Here, we have the Mean Value Theorem for Integrals. It says the following. Assume that $f(x)$ is a continuous function on $[a, b]$. Then, it is integrable; of course, we know that it is integrable; but its integral will satisfy something else. Its integral can be expressed as $f(c)(b - a)$ for some c between a and b.

What does it mean geometrically? If you have some function $f(x)$, then this area is equal to the area of some rectangle with height as c. That is what it expresses. This area can be seen as a rectangle of height c, where c lies between a and b. Of course, we will give proofs of some of these, others are obvious like Parts 1 and 2.

Now, seventh property assumes that $f(x)$ is continuous on $[a, b]$ and it has the same sign on $[a, b]$. This means either $f(x) \geq 0$ for all $x \in [a, b]$ or $f(x) \leq 0$ for all $x \in [a, b]$. Suppose $f(x)$ is continuous on $[a, b]$ and it has the same sign on $[a, b]$, say, $f(x) \geq 0$ on for all $x \in [a, b]$. Then, the integral of $f(x)$ over $[a, b]$ is also greater than or equal to 0. Similarly, if $f(x) \leq 0$ for all $x \in [a, b]$, then its integral is also less than or equal to 0.

This is looking a very non-trivial kind of thing, but it is not so. Suppose $f(x)$ has the same sign, say, positive or even equal to 0 on $[a, b]$. If the function is not equal to 0, that is, it is not coinciding with the x-axis, then the area can never be 0, the area will become positive; that is what it says in this case. Similarly, if $f(x)$ has the negative sign on $[a, b]$, then its integral is either 0 (when f is identically 0), or this area will remain negative. That is what this property says.

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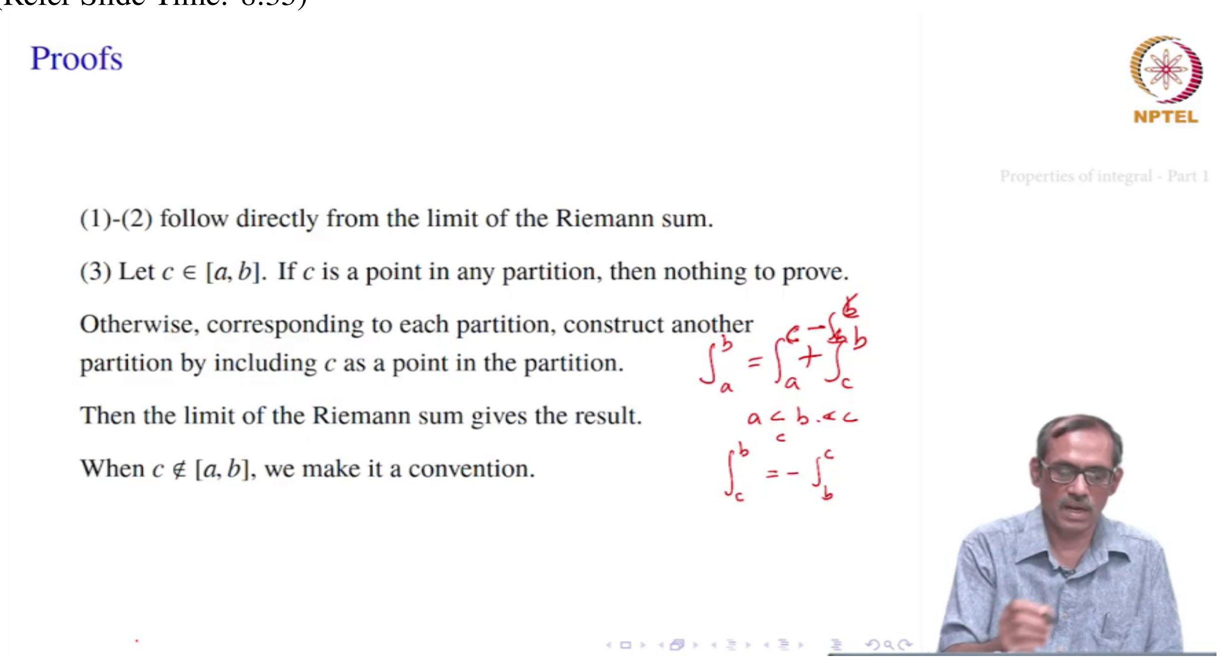

These properties are fairly easy, and they are intuitively alright. But we have to see how to give proofs for at least some of them.

So, Properties 1, 2 and 3 follow directly. We have discussed those already. Along with 3, we have another point. It says that the integral from a to b is equal to the integral from a to c plus the integral from c to b, where our assumption is that a is always less than b, and c is a point which is in between a and b. Now, we will make it a convention for the case that c is outside this interval a to b. Let us say that $c > b$. It is somewhere here. Then, a to c plus c to b will also be equal to this. That means along with this, we assume that if $c > b$, then integral the integral c to b is equal to minus of the integral b to c.

We put this up as a convention, because that is not covered in the definition. In this convention, look at the integrals here: the integral from a to c minus the integral from c to b. If you transfer it to other side, it becomes a to b plus b to c; and this is equal to a to c. This is what we have shown in Property 3. And that is why we put this convention so that we need not have to worry whether this upper limit c lies between a and b or not. The same symbol will carry over the meaning. So, this is an extension of the definition, given as a way of convention. Even if c does not belong to $[a, b]$, we will assume that this property holds.

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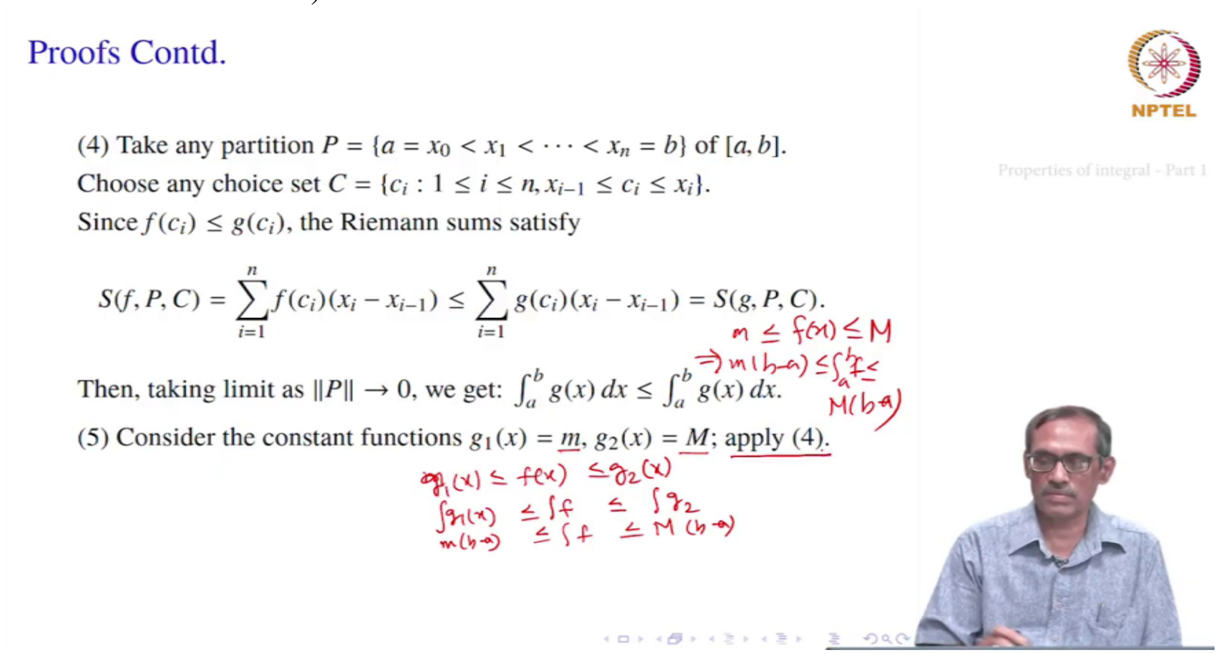

We now come to the fourth property. It says that if $f(x) \le g(x)$ for each $x \in [a, b]$, then the integrals also will satisfy the same inequality. How to do it? We have $f(x)leqg(x)$, then it should imply that $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$. We will see from the definition directly. You take any partition of $[a, b]$ for for computing the integrals of f and g. Then you take any choice set, that is, c_is lying between the consecutive points x_{i-1} to x_i. Since $f(c_i) \le g(c_i)$, we get $f(c_i)(x_i - x_{i-1}) \le g(c_i)(x_i - x_{i-1})$. Then, the corresponding sums also satisfy the sme inequality. That is, $F(f, P, C) \le S(g, P, C)$, the Riemann sums for f and g are also related the same way. When you take the limit of that as $\|P\| \to 0$, we get the integrals; so that is quite straightforward.

Now we come to the fifth one, where we have $m \le f(x) \le M$; then that would give $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$. That is our fifth property. That is easy to see now. What do we do, we take $g_1(x) = m$, the constant function, and $g_2(x) = M$, another constant function. Then, $g_1(x) \le f(x)$ and $f(x) \le g_2(x)$ for all $x \in [a, b]$. Now, what is the integral of $g_1(x)$? By Property 2, we see that it is equal to $m(b-a)$. By using Property 4, you get the integral of $g_1(x)$ less than or equal to that of $f(x)$; and this gives $m(b-a) \leq \int_a^b f(x)\,dx$. Similarly, you get the other inequality: $\int_a^b f(x)\,dx \leq M(b-a)$. That is how the proof will go; it is just an application of the fourth property.

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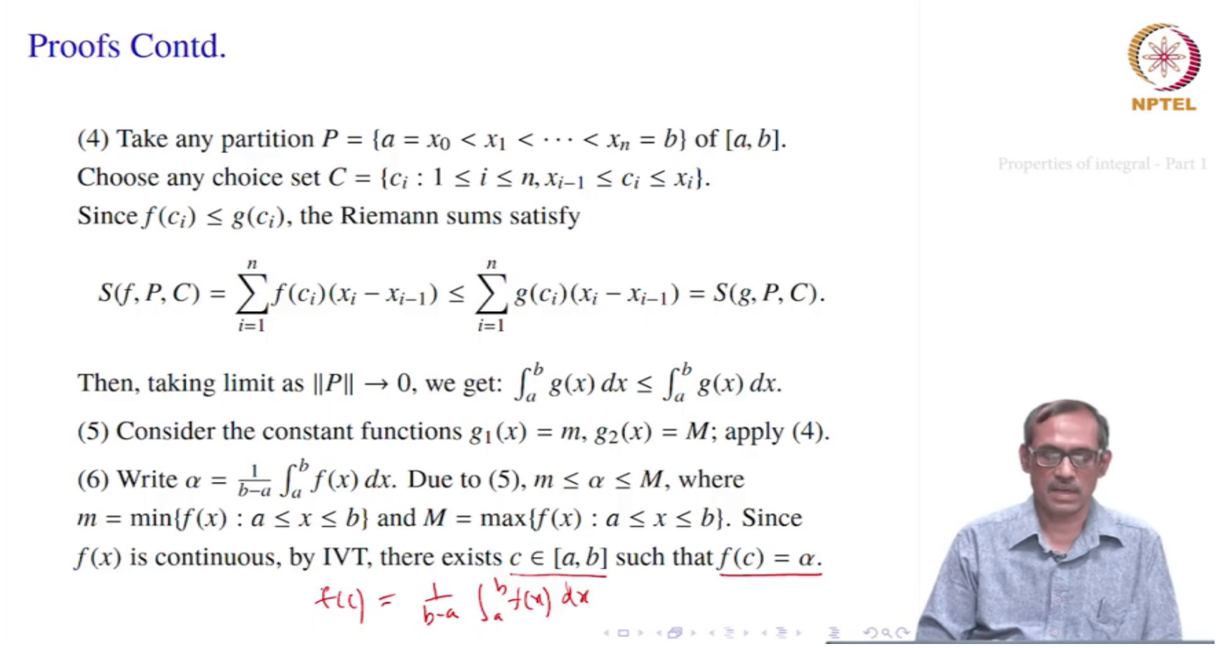

Let us come to sixth one. This is the Mean Value Theorem for Integrals, It says that you will have some point c between a and b such that the rectangle at with height c will have area equal to that area of region under the graph of the function f. Well, let us write $\alpha = (b-a)^{-1}\int_a^b f(x)\,dx$; it is a real number. Due to Property 5, we see that $m \leq \alpha$. Why? Property 5 says that if f lies between m and M, then then the integral of f will lie between $m(b-a)$ and $M(b-a)$. When you divide by $b-a$, your integral is divided by $b-a$ and this quantity will lie between m and M. Notice that since f is continuous on $[a, b]$, such numbers m and M exist. Also, since f is continuous, by the Intermediate Value Theorem, we can find c between a and b such that $f(c) = \alpha$ because alpha lies between the minimum and maximum values of the continuous function $f(x)$. So, α is achieved at some point. So, there exists a point c inside $a, b]$ such that $f(c) = \alpha$. I think that finishes the proof. Why? Because $f(c) = \alpha$ means that $f(c) = (b-a)^{-1}\int_a^b f(x)\,dx$. The mean value is achieved there; that is why it is also called mean value theorem for integrals.

Let us go to the seventh one. It says that if $f(x)$ is continuous on $[a, b]$ and it has the same sign on $[a, b]$ (this is a crucial assumption), then the integral equal to 0 will imply that the function is equal to 0. It has the same sign; that is an assumption. It can happen that $f(x) = 0$ already, so there is nothing to prove. Assume that it is not equal to 0, that is, suppose $f(x) \geq 0$ for each $x \in [a, b]$. A similar proof will go for $f(x) \leq 0$. Since we say that $f(x)$ is not equal to 0 for every x, there exists at least one point t inside the interval $[a, b]$ where $f(t) > 0$. Because this is not equal to the zero function, but it is greater than or equal to 0, there is at least one point where its value is positive.

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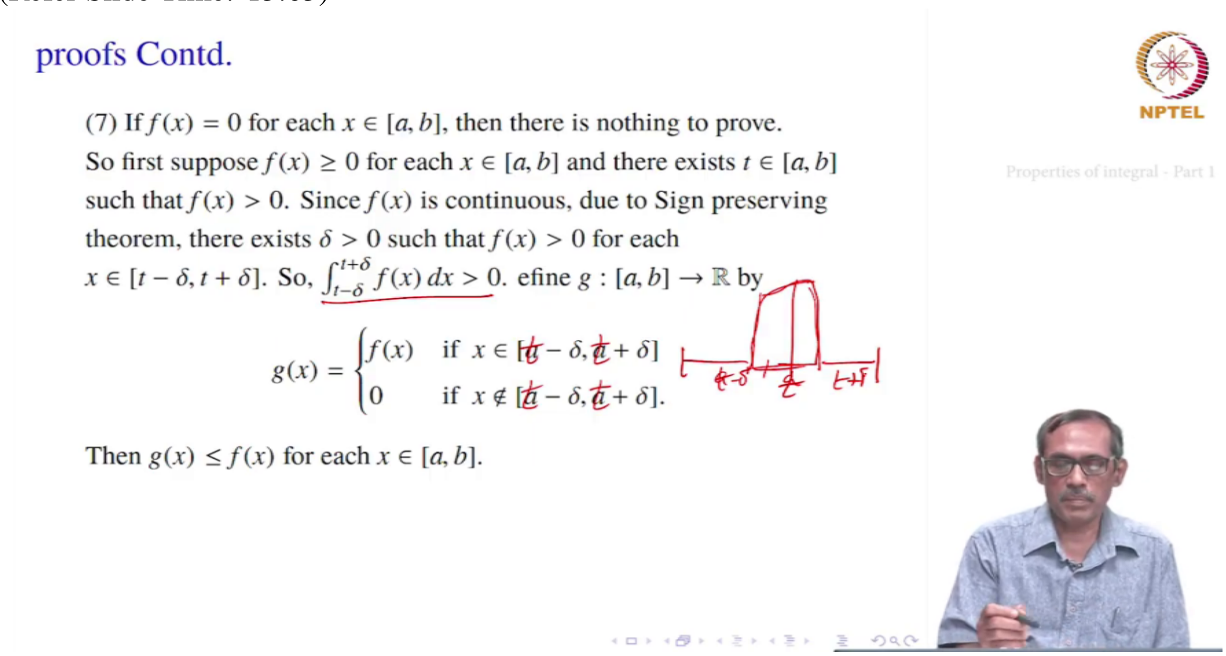

Now we use continuity. The Sign Preserving Theorem for continuous function says that there is a neighborhood of this t such that $f(x)$ remains positive through out that neighborhood. Such a neighborhood is of course some open interval. We can chose a small δ so that there the closed interval $[t - \delta, t + \delta]$ lies inside this neighborhood. Now, $f(x) > 0$ for all $x \in [t - \delta, t + \delta]$. Then its integral will be greater than 0; we just apply the earlier result. Now that the integral of f over $[t - \delta, t + \delta]$ is bigger than 0, our integral over $[a, b]$ which is is always greater than or equal to the integral over $[t - \delta, t + \delta]$, is also greater than 0.

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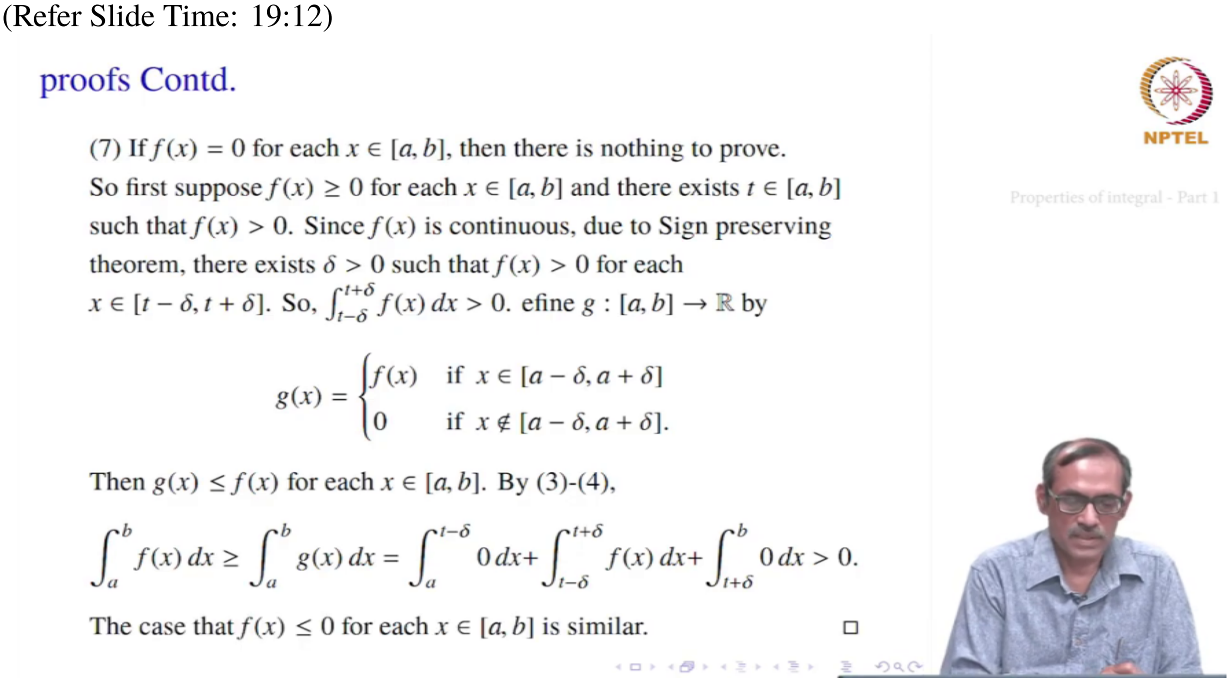

Given our assumption that if the integral is equal to 0, then f is 0; that is what we wanted to show. We have started with the assumption that f is not equal to 0. In the first case, we considered

f to be greater than or equal to 0, and at some point f is greater than 0. Then, we obtain the integral over that smaller neighbourhood is bigger than 0, since $f(x) > 0$ fro each x in that neighborhood. From this we find that the integral over $[a, b]$ is greater than 0. That contradicts the assumption that integral is equal to 0. Notice that it follows since it is given that no where $f(x)$ can be negative; $f(x)$ is greater than or equal to 0. So, there is the contradiction.

To see it another way, you can formally define the function $g(x)$ which is $f(x)$ for $x \in [t-\delta, t+\delta]$ and 0 otherwise. That means you have some point t here, $g(x) = f(x)$ on this interval $[t-\delta, t+\delta]$ so that its graph look s like this. At other points $x \in [a, b]$, $g(x) = 0$. That is how the function looks like. Now, the area under this will be really this area. And, that is always less than or equal to $f(x)$ but it is already bigger than 0 so the integral of $f(x)$ is bigger than 0.

So, this is telling in all its formal detail about how to deduce that the integral over the whole interval $[a, b]$ is greater than or equal to the integral over $[t-\delta, t+\delta]$ and how this latter integral is positive. That proves the result.

Now if $f(x) \leq 0$, then you can use $<$ instead of $>$ everywhere, and the same argument also proves this case. It will give the conclusion that the integral is greater than 0, which is wrong.

These are fairly straightforward results to remember. It is just coming from our intuition by comparing the geometrical area with something else as the case demands.

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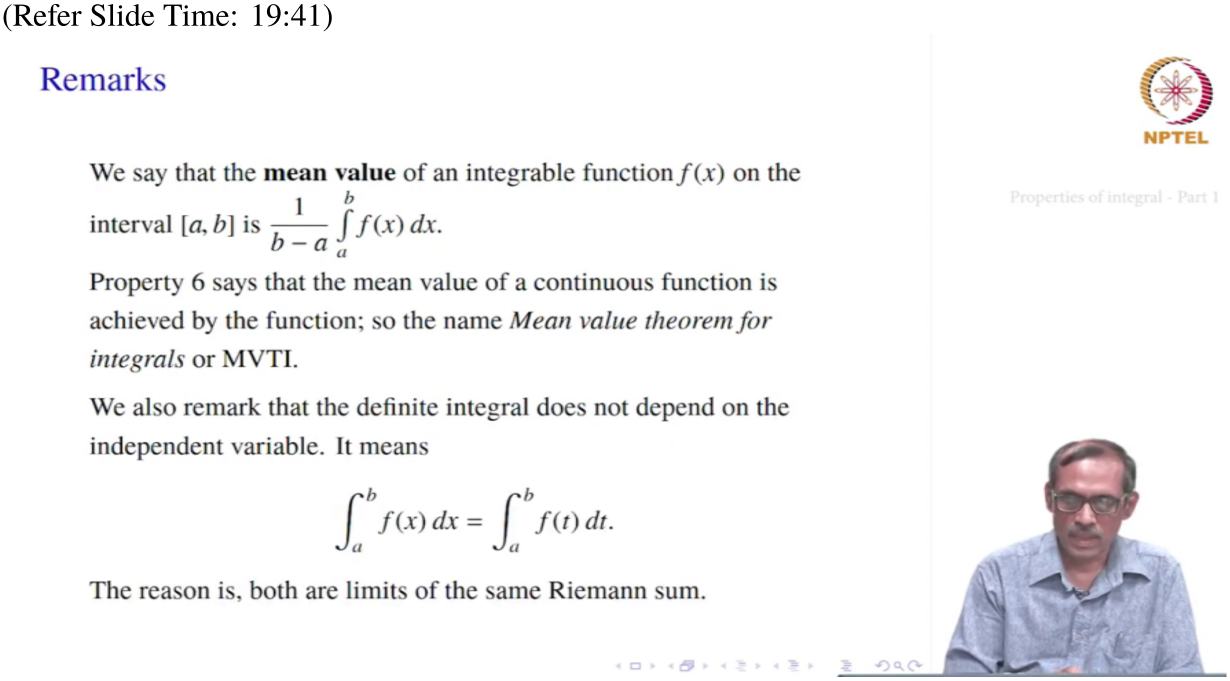

We will be discussing applications of these results. But before that, just for one second, let us recall why it is called the Mean Value Theorem for Integrals. It is called so, because this number which is the integral of $f(x)$ from a to b divided by $b - a$ is also called as the mean value of the function f. And it is so called because it is the sum of this which is the area divided by the length $b - a$. That is why it is called the mean value. So, Property 6, the Mean Value Theorem for Integrals, says that the mean value is achieved by the function in some sense; that is, $f(c)$ is equal to this.

There is another thing to note, which will be useful many times. If you have the integrals: $\int_a^b f(x)\,dx$ and $\int_a^b f(t)\,dt$, then they are really same. It does not matter what this variable is, whether x or t. Right? Because, anyway you will be taking the Riemann sum; and in the Riemann sum, you do not have any x or any t; it is the area below f. Whether $f(x)$ or $f(t)$; it is the same thing, because the curve is same. That is what we say, the definite integral does not depend on the independent variable.

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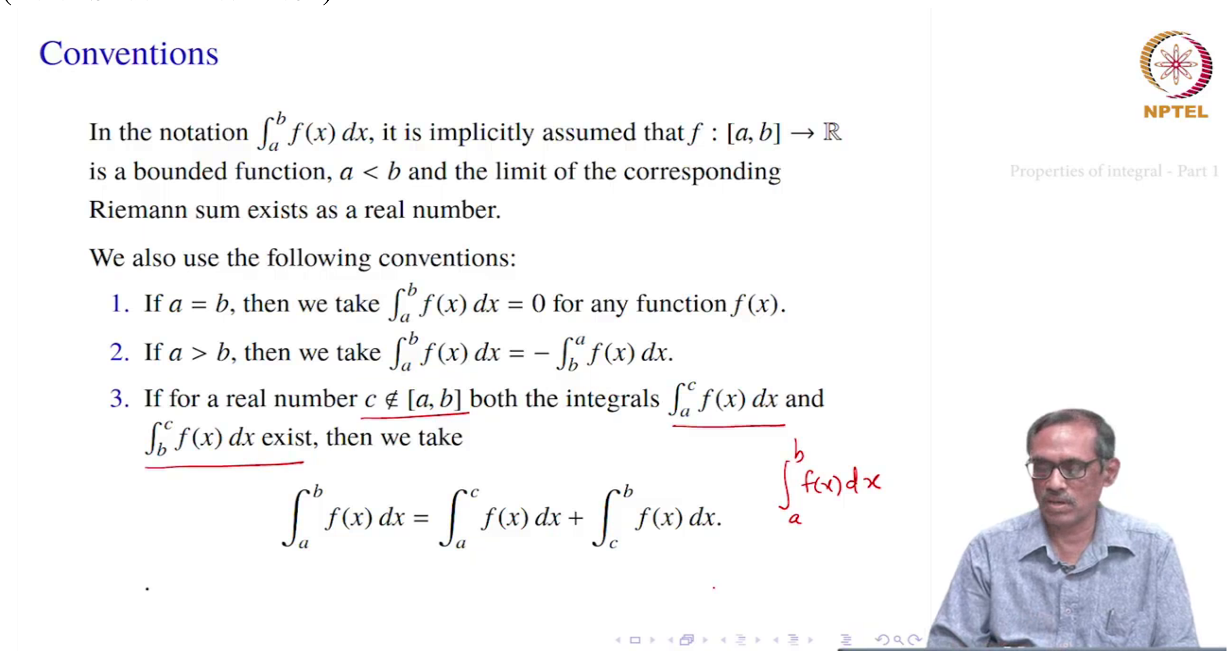

With this, let us go to the convention mentioned earlier. In the notation $\int_a^b f(x)\,dx$, we always assume that $a < b$. But if $a > b$, or $a = b$, then we can give some meaning to the integral, which will be consistent with our properties. If it contradicts some of the properties, we should not put that convention. Otherwise, we can use any such convention. It is really a redefinition of that idea for the case where $a = b$ or $a > b$.

If $a = b$, we take the integral equal to 0 for any function $f(x)$. Intuitively, it says that if you have a function, which is defined at one point only, then the area bounded by that curve, the x-axis, the line $x = a$ and the line $x = b$ is 0. That is what it says; and that is clear from the geometry. So, we put this convention that if $a = b$, then $\int_a^b f(x)\,dx = 0$ whatever be the function $f(x)$.

If $a > b$, then we will put $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. This will go along with our Properties 3 and 4.

If a real number c does not belong to $[a,b]$, but if both these integrals $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ exist, then, in that case, we will say that $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$. As we have seen, if $c > b$, then integral c to b will become now the integral b to c with a negative sign. Then, this convention will comply with our earlier property. Similarly, if $c < a$, then the integral from a to c will be minus the integral c to a. So that the minus sign there will give us the integral a to b as earlier. So, we put this as a convention because it is consistent with our properties.

When an integral appears in the form $\int_a^b f(x), dx$, we need not assume that $a < b$. Of course, that is the crucial case which will decide everything along with all its properties. However, if $a > b$, then it is still all right; it will be negative of the integral $\int_b^a f(x)\,dx$. That is easy enough for us to remember.