Basic Calculus - 1

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Lecture 5

Functions - Part 1

Well, this is lecture 3 of basic calculus one. Today we will be talking about functions defined from a subset of real numbers to real numbers. We will not be talking much about functions. Mainly, we will fix some terminologies, give some examples and make a reportery of what functions we know. If you remember, in the last class we have done something about the absolute value. For any real number x, you define $|x|$ as either x or $-x$, depending on its sign. For example, $|1| = 1$ and also $|-1| = 1$. That really defines a function from real numbers to real numbers, which takes all positives to themselves and all negatives to their corresponding positives, and of course, 0 to 0. So, this is of a function. We are carried out. Well, we will be defining a function soon.

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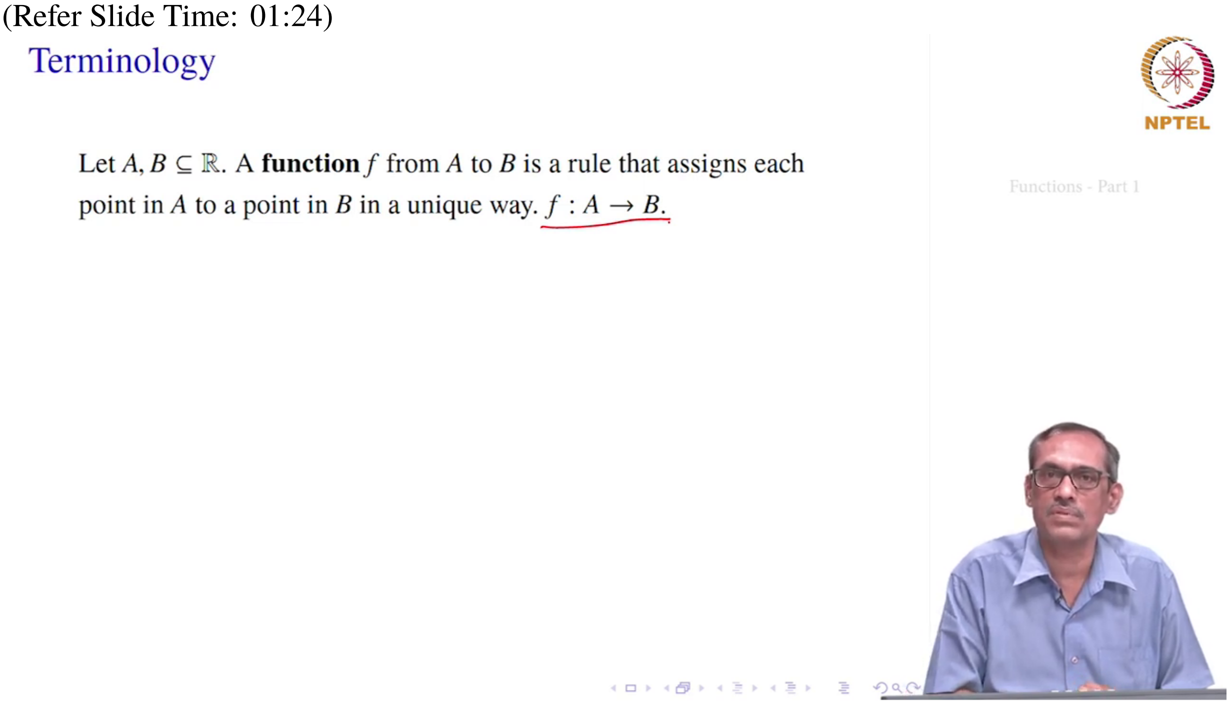

Suppose A and B are subsets of real numbers. We assume them to be non-empty subsets. We will not define any function from empty set to another set. So, suppose A and B are non-empty subsets of real numbers. We denote by f, small f, a function from A to B provided it gives a rule that assigns every number in A to numbers in B. We write it this way. It is a rule that assigns each point in A to a point in B in a unique way. We will certainly say what is the meaning of this 'unique way'. We will write such a function f from A to B. Once we write $f : A \to B$, it will mean that f is a function that is defined on A and it is getting its values in B. For example, your absolute value function will be a function from $\mathbb{R}$ to R; both A and B are R here.

Now, what is the meaning of this unique way? It does not mean that all the elements in A are assigned to one particular element in B. It is not that what it says. It can happen in some of the examples, but that is not what it says in general. It says that if you start with any particular number in A, then it is taken to or it is assigned to or it is associated with another element in B. But that way of assignment is unique to this particular number you have chosen from A. So, we will write this number in B, as $f(x)$. If $x \in A$, then this map is taking this x to the number somewhere in B; that number in B, is written as $f(x)$. Sometimes we say x is taken to $f(x)$ or x is mapped to $f(x)$ by the function f.

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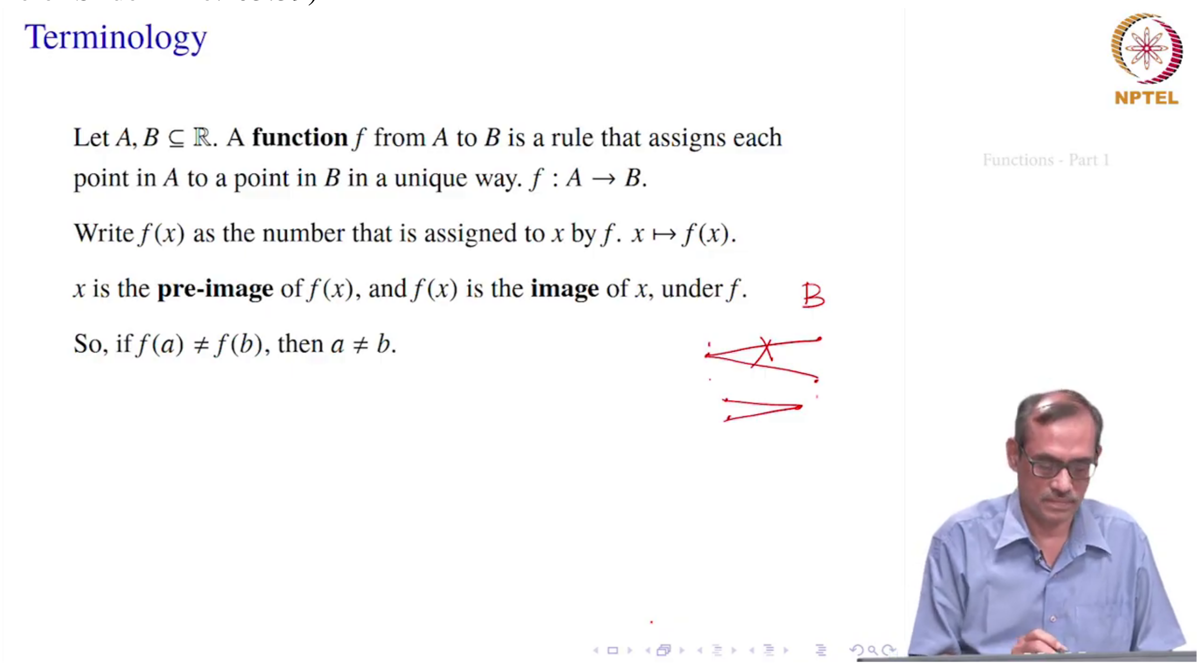

Though f is the function, we will sometimes write $f(x)$ as the function. Suppose we give $f(x) = x^2$. It means 1 is taken to 1, 2 is taken to 4, and so on. We will write that x^2 is a function; which really means f is a function with $f(x) = x^2$.

We will fix some more terminology. We will say that x is the pre-image of the number $f(x)$ under f. It is coming from x, so it is pre-image of $f(x)$. And similarly, $f(x)$ is the image of x under f. This 'unique way' is telling that if you start with two numbers in B and look at from where they have come, then it is never possible that from the same element (some element in A) they have come; this will never happen.

So, we say that if $f(a) \neq f(b)$, then a which is the pre-image of $f(a)$, and b which is the pre-image of $f(b)$ can never be equal. They should be different. That is what the 'unique' means. If a is there in A, then a goes to $f(a)$. But the other side is possible; that if you have a and b, then both of them can go to same thing; that is, $f(a)$ can be equal to $f(b)$. Like in absolute value function, $|1| = |-1|$. That is possible. But the first case is not possible. That is the meaning of this association: f associates elements in unique way.

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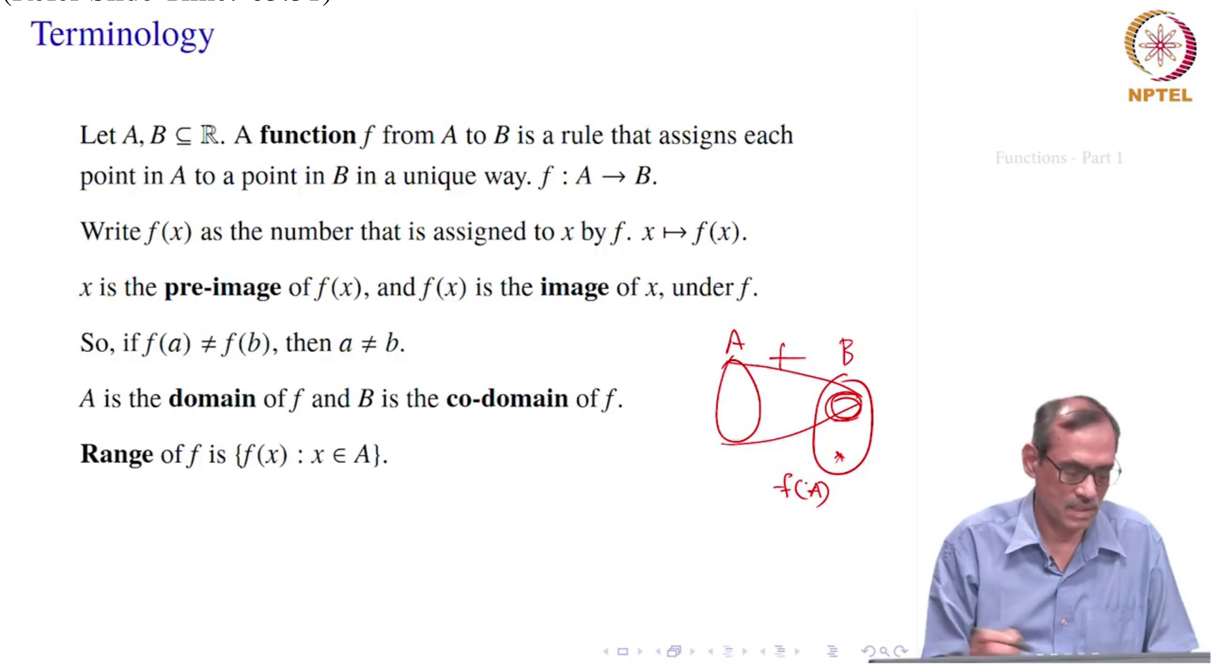

This set A, from where the function is defined is called the domain of f and the set B, or in totality also $\mathbb{R}$, is called the co-domain of f. See, everything of this B may not have been mapped. For example, look at $|x|$. It is taking from $\mathbb{R}$ to $\mathbb{R}$, but the negative numbers in $\mathbb{R}$ are never mapped by $|x|$. There is no number in $\mathbb{R}$ whose mod will be a negative number. But $\mathbb{R}$ is the domain and $\mathbb{R}$ is the co-domain of that. So, co-domain is the right hand side. If $f : A \longrightarrow B$ is a function, the left side is called the domain and the right side is called the co-domain.

But exactly what are really mapped inside B? As in case of mod, the set of non-negative numbers is called the range of f. We will say that range of f is equal to set of all values $f(x)$ such that x belongs to A. It is the subset of B where everything is mapped. You do not find one $x \in A$ such that $f(x)$ will be outside of this range. The numbers outside of the range are members inside the co-domain. So, it is something like your A is here and some set B is here; (it is not appropriate as they are real lines, but let us write it schematically.) then all these things are taking to f maybe somewhere here; that is the range. These are not in the range. B is the co-domain and the range is only this one, which is sometimes also written as $f(A)$. But we can say range of f.

For this $f : A \longrightarrow B$, we sometimes you say that $f(x)$ is a function, and sometimes we say that $y = f(x)$ is a function.

And what is the graph? Sometimes we want to plot it. So, what do we do? Take the set A something like an interval. So, A is something like this, and x is in this interval, and then f takes these values to somewhere, to real numbers. Say, x is here, some point, then it is mapped to $f(x)$, which is somewhere here. Now we take together x and $f(x)$, make one ordered pair; something like $(x, f(x))$. The set of all these order pairs will be called the graph of f. Actually, f and the graph of f are not a different. But since we are telling f is a rule, we use another terminology; and call this the graph of f.

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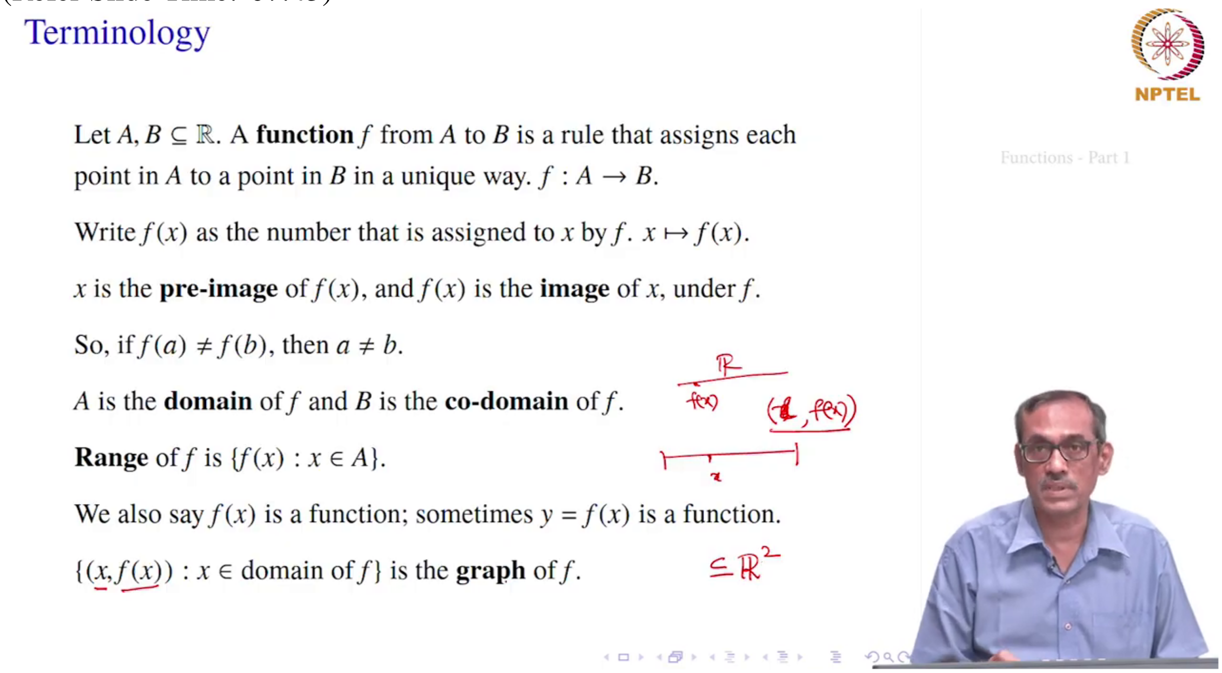

Now we will take some examples. We take the graph of f, it shows how the function looks like; it is the ordered pairs, the set of order pairs. So, the graph of f will be a subset of $\mathbb{R}^2$, because x is from $\mathbb{R}$ and $f(x)$ is from $\mathbb{R}$, so the ordered pair belongs to $\mathbb{R}^2$. The graph of f will be a subset of $\mathbb{R}^2$; usually it will be curve. We will see how it looks.

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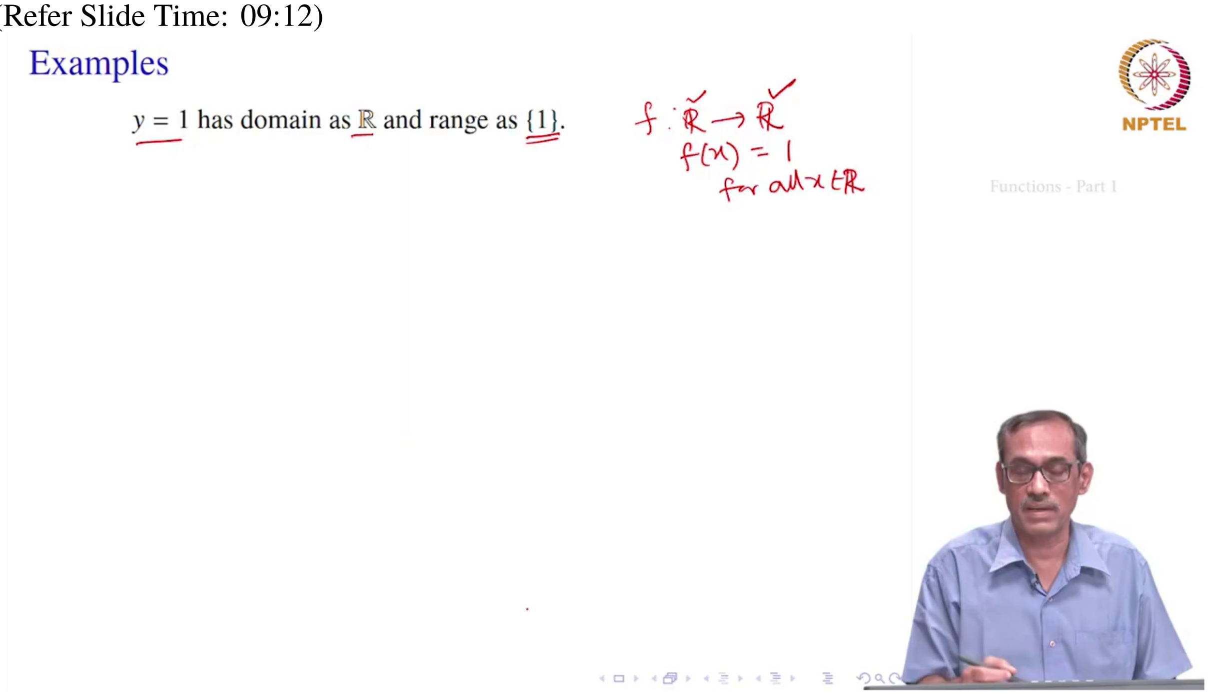

We will consider some examples. Let us take the first and the simplest example: f is given by $f(x) = 1$ for every $x \in \mathbb{R}$. We write it in a short form: $y = 1$ is a function, using the terminology that $y = f(x)$ is a function. Actually, if you write it, the function will be $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 1$ for all $x \in \mathbb{R}$. This is how the function will look like. But in short form, we will write

$y = 1$ is the function. Now this is defined for every element in $\mathbb{R}$. So, we say that its domain is $\mathbb{R}$ and its co-domain is also $\mathbb{R}$.

But then everything in the co-domain is not mapped. The only thing mapped is 1, so its range is the set containing 1, the singleton $\{1\}$. There, the domain is the whole of $\mathbb{R}$. Sometimes the domain will not be given. We will just say: consider the function $y = 1$, or consider the function $f(x) = 1$. There, we may have to find out the largest possible subset of $\mathbb{R}$ where it is defined, if it is so possible. In this case, it will be whole of $\mathbb{R}$ because it is defined for the whole of $\mathbb{R}$. And this is called a constant function whose value is 1. Instead of 1, suppose you write some other number say some π, then that will be the constant function $f(x) = \pi$. But that is also the constant function, and this is also called a constant function. It is a generic one; by changing this constant, we will get different types of constant functions.

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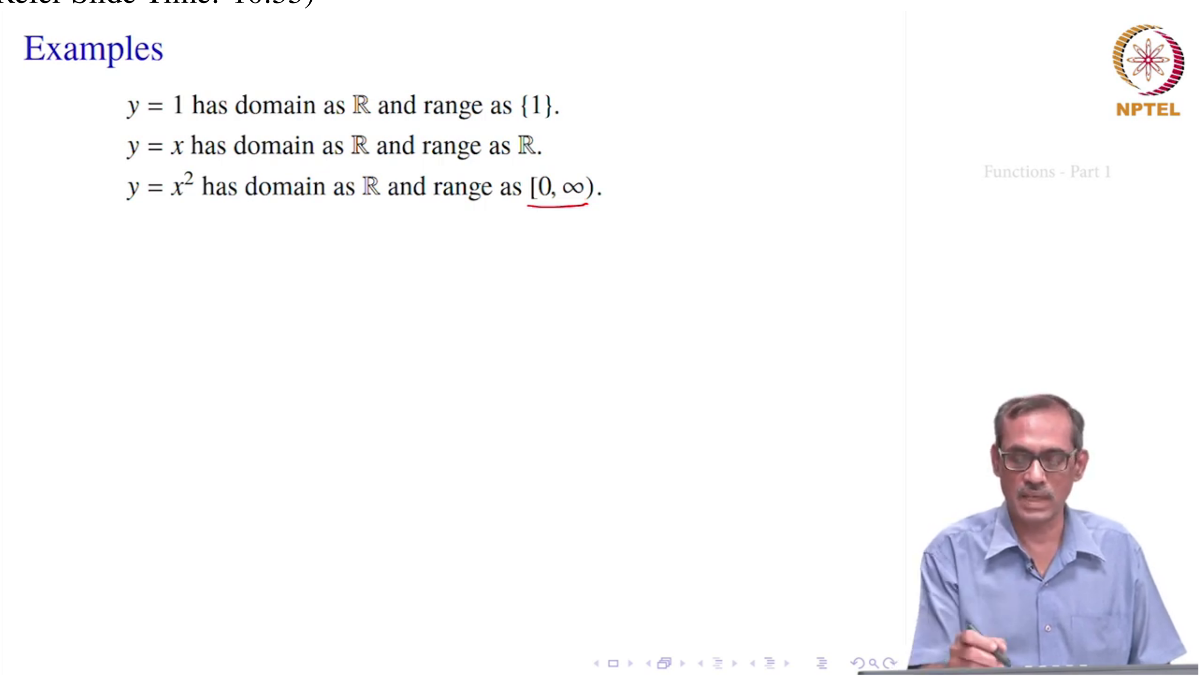

We will take one more example, say, $y = x$. So that means, it is $f(x) = x$ for every $x \in \mathbb{R}$. This is called the identity function; its domain is $\mathbb{R}$, co-domain is $\mathbb{R}$ and range is also $\mathbb{R}$.

Let us take $y = x^2$. This function is defined by mapping x to x^2. If you take any real number, its square is defined, therefore its domain is $\mathbb{R}$, and co-domain is of course, $\mathbb{R}$. Then what is its range? We see that x^2 can never be negative. So, at best it can be $\mathbb{R}_+ \cup \{0\}$ or the positives and 0 included. It is so because for any point in 0 to ∞, including 0, you will get one point there whose square is this. That is why its range is equal to the set of all non-negative real numbers, which is $[0, \infty)$.

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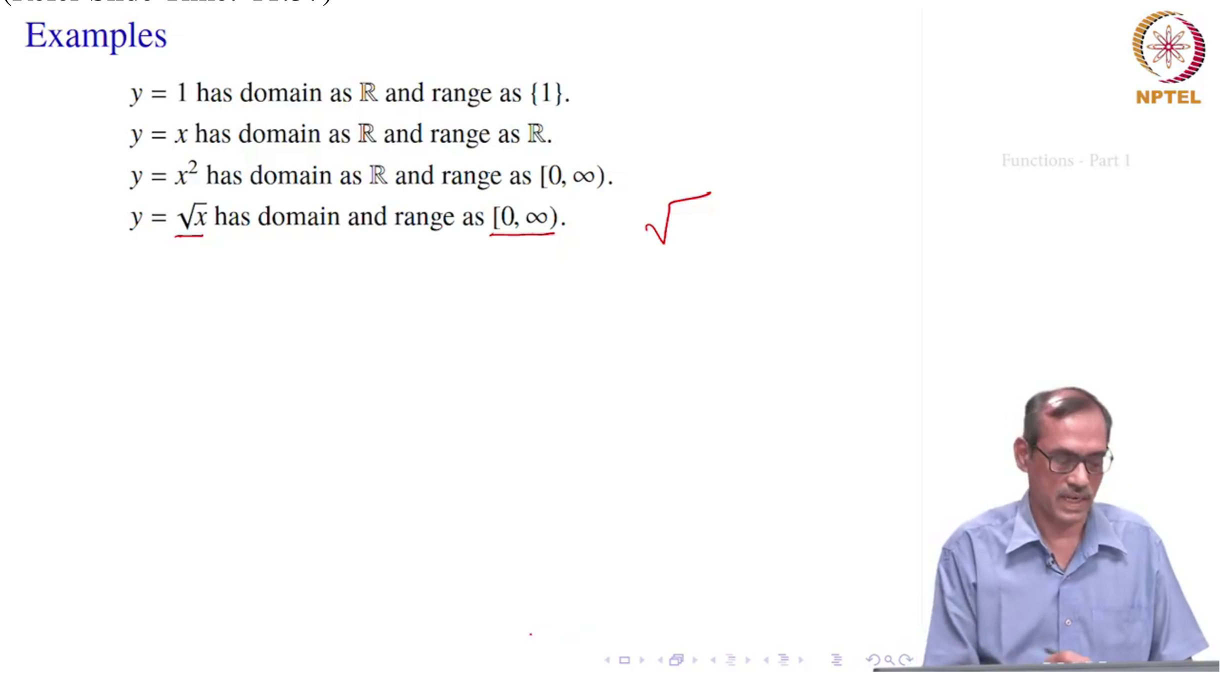

Take $y = \sqrt{x}$ as another function. Now, this is not defined for every real number. Because square roots of negative numbers are not defined; only square roots of positive and square root of 0 are defined. So, its domain is the set of all non-negative real numbers. And, when you take square root, the usual convention is that we will be taking the non-negative square root; that is the meaning of writing the square root symbol. Like, $\sqrt{9}$ will never be considered as -3, it will be $+3$; that is the convention with this symbol. Therefore its range is also the set of all non-negative real numbers.

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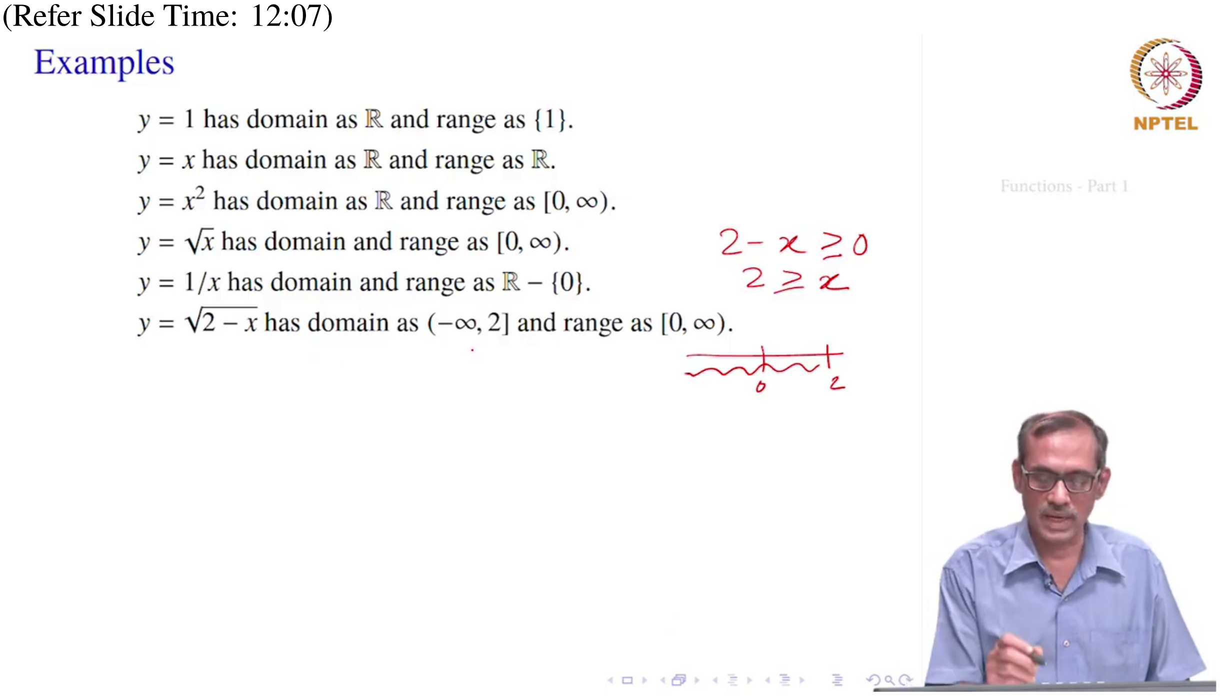

Consider the function $y = 1/x$. This function is not defined for $x = 0$, because we do not have anything called $1/0$. Our axioms of real numbers say that 1 by something is defined only for non-zero quantities. So, $y = 1/x$ has domain as $\mathbb{R}$ excluding 0, $\mathbb{R} - \{0\}$, and its range is of course, the same, because 0 is never achieved by 1 by anything. So, 0 is also gone. Therefore, its domain is $\mathbb{R} - \{0\}$ and range is also $\mathbb{R} - \{0\}$.

Let us consider another function $y = \sqrt{2 - x}$. To get its domain, you have to see where it is defined. We know that square root of non-negatives are only defined. So, $2 - x$ must be a non-negative real number. That means x must be such that $2 - x \geq 0$.This says that $2 \geq x$. So, if $2 \geq x$, how will it look like in the real line? x will be here; $x \leq 2$. If 2 is here, then all these numbers are there in the domain for which our function is well defined. That means it is $(-\infty, 2]$; that is the domain. And its range can be any number of course. But it is square root; it has to be non-negative. So, the range is $[0, \infty)$; the point 0 is possibly included. It is included because at $x = 2$, you will get 0; and that is the only point where it is 0. There can be also positive numbers in the range, because you can take anything, say 1. As $(-\infty, 2]$ is the domain, 1 belongs to that. If you put 1 there you get $\sqrt{2 - 1}$ as 1; that is also included. So, there can be positives. Then its domain is $(-, 2]$ and $[0, \infty)$ is the range.

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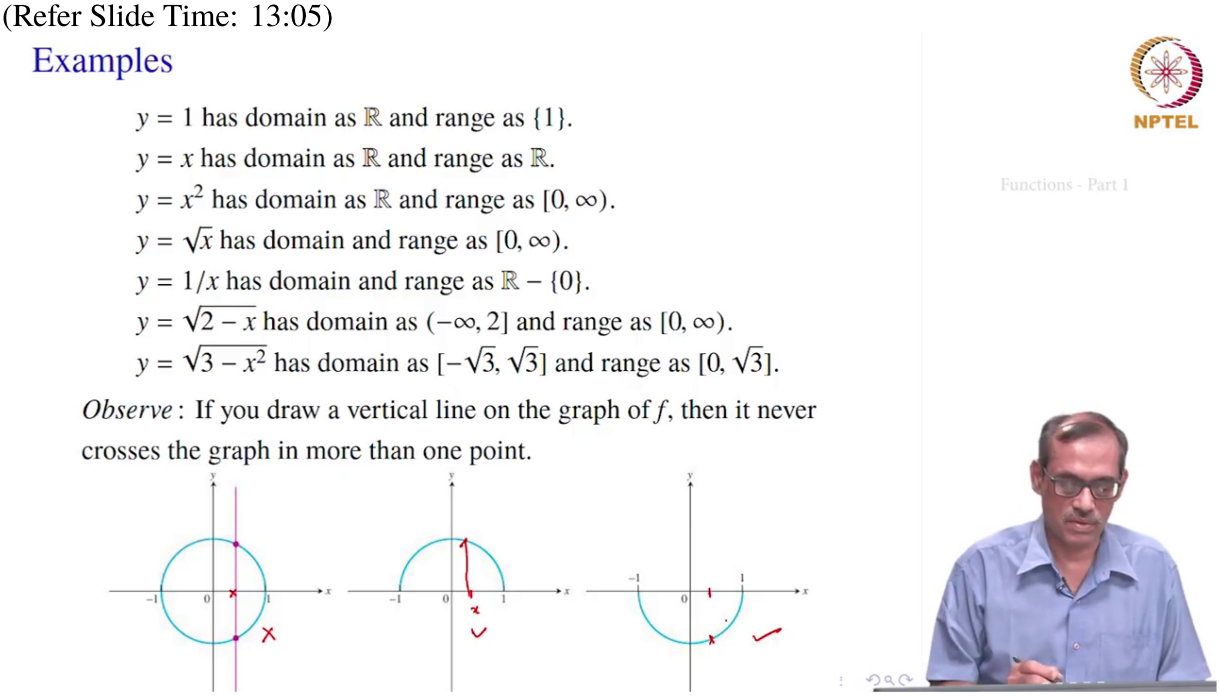

Similarly, if you consider $y = \sqrt{3 - x^2}$, then its domain will be those numbers for which $3 - x^2 \geq 0$, or $x^2 \leq 3$. If you write $x^2 \leq 3$, this is same thing as telling $|x| \leq \sqrt{3}$. And that we know from $|x - a| < \delta$. The interval should be $[-\sqrt{3}, \sqrt{3}]$, the closed interval. That is the domain. And range, of course, can be anything. But now they cannot exceed $\sqrt{3}$, because this is from $-\sqrt{3}$ to $\sqrt{3}$;, so at best x can be 3; at 0 it can be $\sqrt{3}$ also. So, the range can be $[0, \sqrt{3}]$. That is how it would look like.

Once you go for the graph of the function, there is something we should observe. If you take the second picture, which is a circular arc, of course; this is joining -1 to 1 as a circular arc of radius 1. This is a function because if you take any x here corresponding to it, there is one $f(x)$ on this. But if you go to the other side, the first picture, and take any x here, then corresponding to it, there are two points now; and that is excluded from the definition of the function. From a point it should take to a unique point. This is not unique now; there are two points corresponding to that x. Therefore, this is not a function.

This is not the graph of a function. This can be graph of a function and this can be graph of a function. All that you need to know is if you draw a vertical line on the graph of f, then it never crosses the graph in more than one point. That is what happens here. That is why it is not giving a function. So, the circle does not depict a function in this sense. But the semicircles, one above the x-axis or one below the x-axis depict functions. If you take x here you get a unique point corresponding to that x. So, that is a function; we can see that. Find out which graphs really are possibly functions and which are not.

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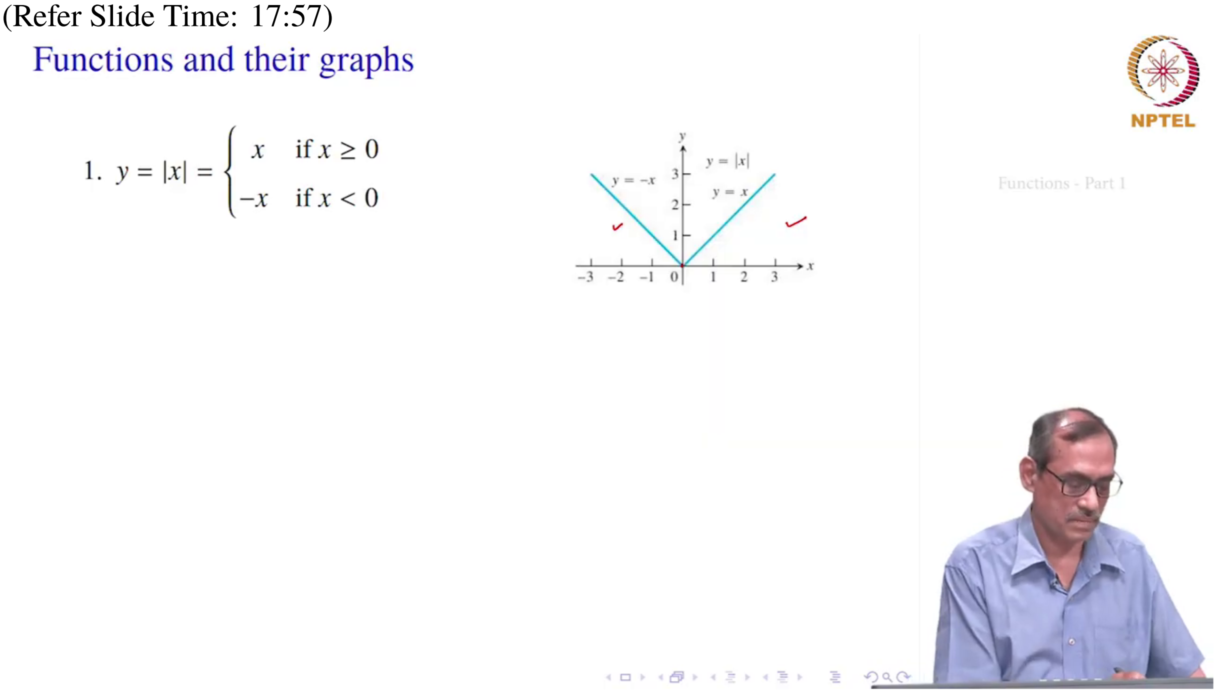

Let us consider some more examples. As we discussed in our last lecture, absolute value defines a function. The function is this: $y = |x|$, where, it is x if $x \geq 0$, and it is $-x$, if $x < 0$. In this case, we have seen its graph also. Again, we say a few lines. When x belongs to the right of 0 or even at 0, $y = f(x) = x$. So, it will be the line $y = x$, which is depicted here on the right side. And, if you take on the left side, that is, if x is negative, then $y = |x| = -x$. So, $y = -x$ is the line, this line. Therefore we got the graph of the function $|x|$ in blue lines: it is the union of those two blue lines. That is how the graph would look like.

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Let us consider another. Suppose we take y as a function, which is defined this way conditionally in three cases. It is $-x$, if x is negative and it is x^2 if x lies between 0 and 1, including 0 and 1 also, and it is 1 if $x > 1$. It is a constant function if $x > 1$. Let us take the case that is simpler to see: y is constant. It is equal to 1 for all those x bigger than 1. Here it is 1; and if $x > 1$, then y is also 1. It is really the constant function which we depicted as $y = 1$, where on the y-axis this distance is 1. For $x > 1$, this is the function. And when $x = 1$, the second case is applicable, which gives you x^2, or it is 1^2, which is also 1. So, at that point, this is also 1, $y(1) = 1$. And, within 0 to 1, $y = x^2$, which is a parabola. So, this is really the portion of the parabola from 0 to 1. And when it is negative, y is $-x$, so $y = -x$. It is that straight line. Therefore you get the blue curve as the graph of the function.
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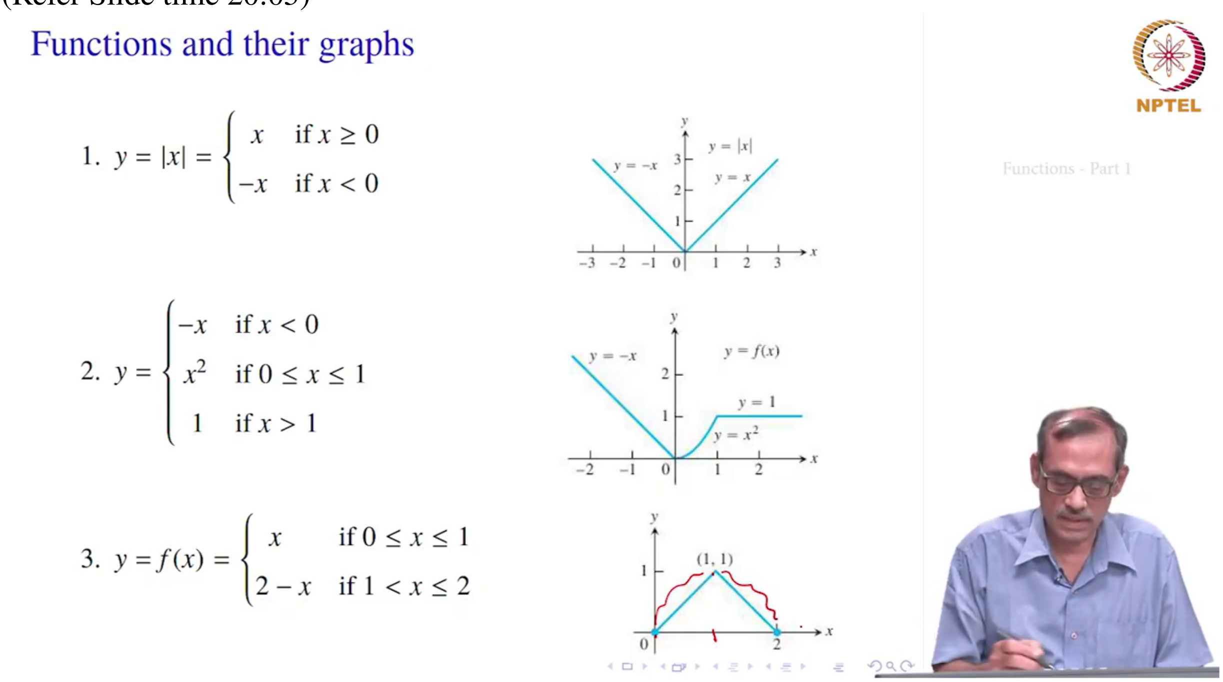

Let us take another. Say, $y = f(x)$, which is given as x, when x lies between 0 and 1; both 0 and 1 includes; and it is $2 - x$ if $1 < x \leq 2$. So, at 1, the first one is applicable; its value is 1. At 0 it is 0, and it is $y = x$ between 0 to 1; so it is this line from 0 to 1. It is this line. And, it is $y = 2 - x$ again, it is a straight line. So, it will be from this place to this place, x from 1 to 2. So, that is how the graph looks like. It is a hat function. It is x from 0 to 1, $2 - x$ from 1 to 2.
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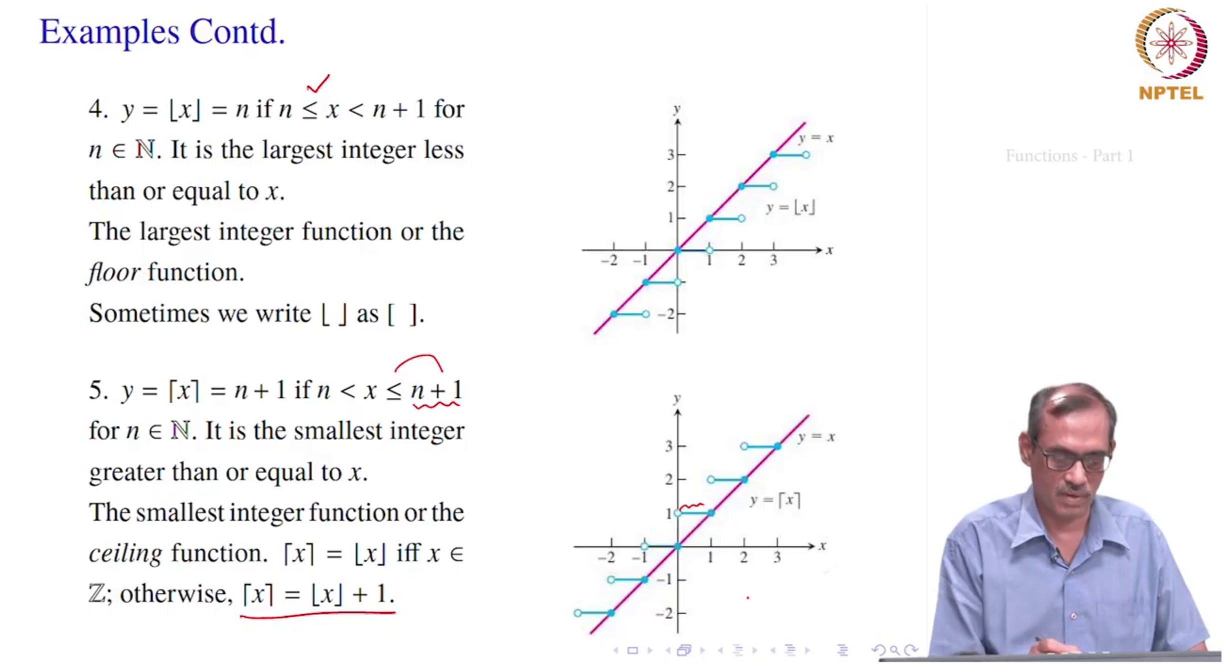

I will give some more examples slowly. Let us take $y = \lfloor x \rfloor$. This symbol is read as floor of x. So, $y = \lfloor x \rfloor$. By definition, it is taken as n, an integer, if $n \leq x < n + 1$. For example, take $x = 1.5$. When $x = 1.5$, it is here. Now, $\lfloor x \rfloor = n$, if $n \leq x < n + 1$; this x, which is 1.5 lies between 1 and 2. So, its value will be 1; $y = 1$ at that point. At $x = 2$, its value is also 1. Is it so? If $x = 2$, then this is not applicable; but when n becomes 2, yes, its value goes to the other side. Because y becomes equal to 2; at 2 it is 2.

Look at this solid dot. This solid dot means that, it is included. If it is blank dot, there is a gap here; it is a not a solid one; it is a hollow circle. That means that point is not included. So, when you say x is lying between 1 to 2, but not equal to 1 or equal to 2 its value is 1; when x is equal to 1, its value is 1; and at x equal to 2 it jumps to the next interval. In the definition, the case $2 \leq x < 3$ will be applicable. So, you can see that $f(1) = 1$, $f(2) = 2$, $f(3) = 3$ and so on. On the integers, it is like the identity function. On the other ones, when it is between 1 and 2, it is a constant function. Again, 2 to 3 it is a constant function. So, its graph looks like this. This is the floor function. Sometimes we write it as the integral value also, but there is a confusion in notation. We will write floor now.

There is a similar related function which is called the ceiling. What does it do? It takes to the next value; that is, $y = \lceil x \rceil$ is $n + 1$ in the same interval. But look at the equality sign here: $n = x$ in the floor, but here in the ceiling, $x = n + 1$. The equality sign is moved to the next interval. That means, $y = \lceil x \rceil$ means, it is equal to $n + 1$ on the interval n to $n + 1$. For example, if x is 1, then this one is applicable, so n must be equal to 0; then, from 0 to 1, its value will be 1. So, 0 to 1, its value is 1, and at 1 its value is 1. But not at 0? At 0 its value will become 0, because the last one will be applicable. So, its graph will look like this. It is coming to the right, the hollow circle on the left side.

That is how the ceiling function looks like. So, what is the difference between them, if you see the ceiling, it will look something like ceiling function equal to floor function plus 1, if x is not an integer; and if x is an integer, then ceiling and floor will be the same. At 1 it is 1, at 2 it is 2, at 3 it is 3 and so on. The same thing for the floor: at 1 it is 1, at 2 it is 2 and so on. So, that is the difference between those two. One is floor, the other one is ceiling.