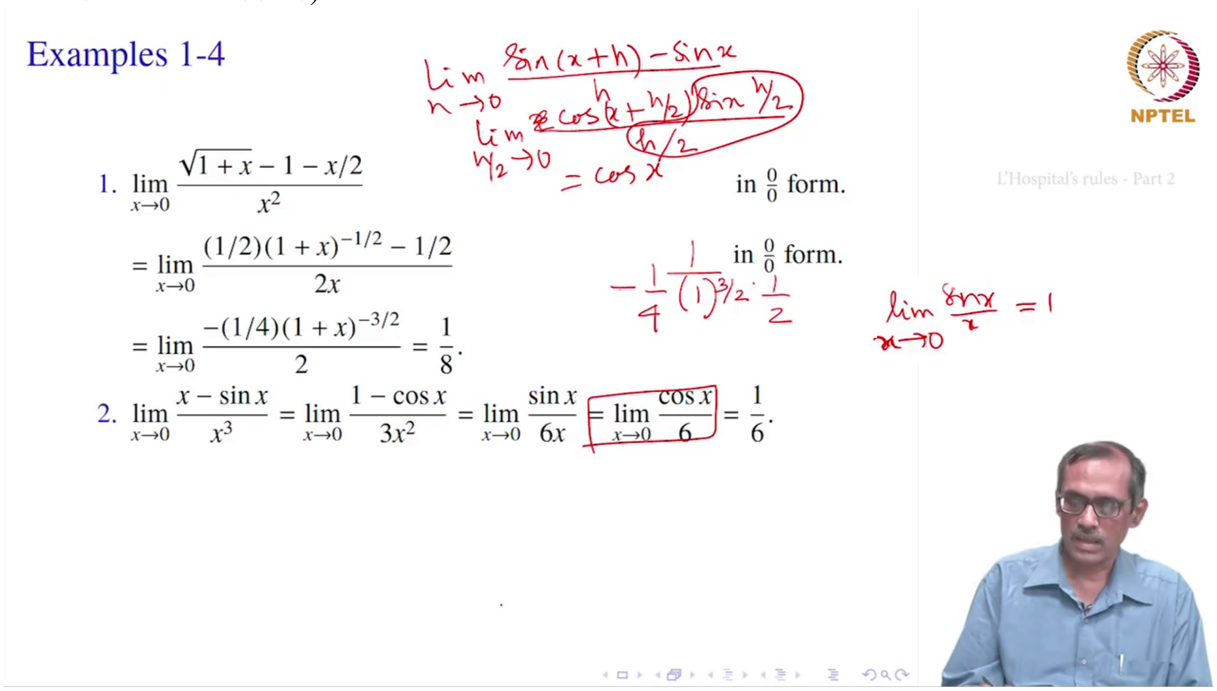
Basic Calculus - 1
Professor. Arindama Singh
Department of Mathematics
Indian Institute of Technology Madras
Lecture 24 - Part 2
L' Hospital's Rules - Part 2

(Refer Slide Time: 00:16)



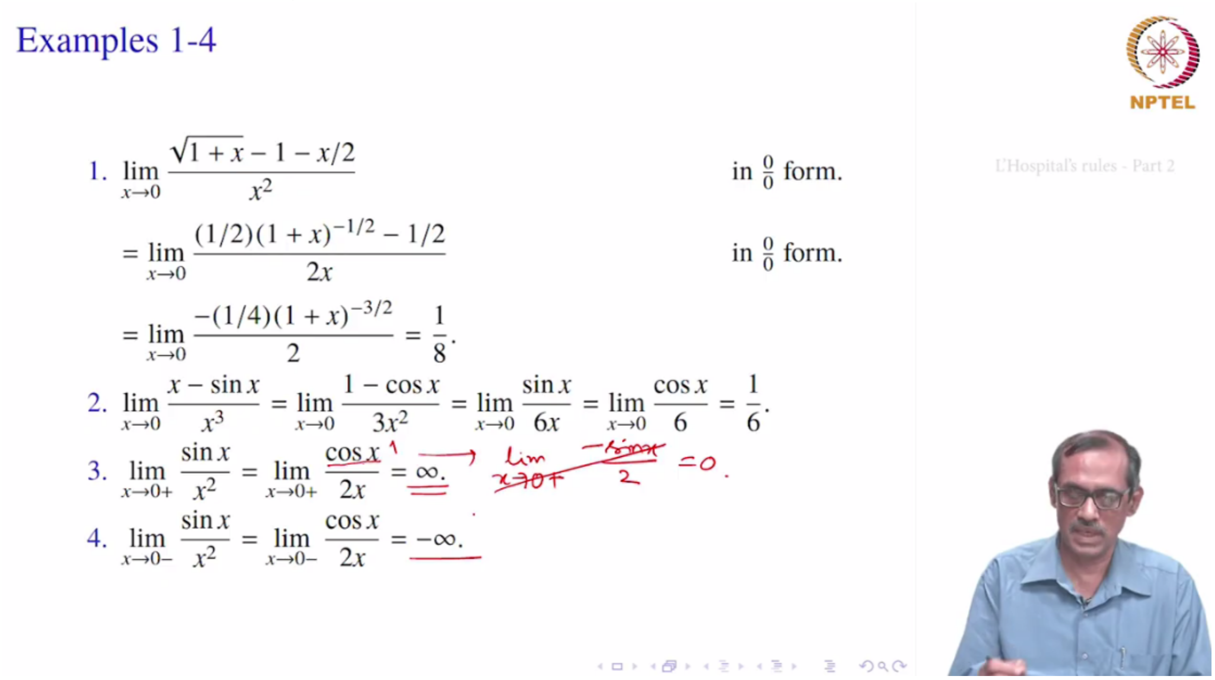
Let us go to second problem or second example. Here, we have the numerator as $x - \sin x$ and the denominator is x^3. When x is 0, $x - \sin x$ is 0 and x^3 is also 0. It is in $0/0$ form. We try to apply L'Hospital's Rule. If this limit exits then it will be equal to the limit as x goes to 0 of $1 - \cos x$ by $3x^2$. It is the other way around. If the the right limit exits then the left also will exist.

As $x \to 0$, the numerator of the right one, which is $1 - \cos x$ also goes to 0, The denominator $3x^2$ also goes to 0. Again, it is in $0/0$ form. We can apply L'Hospital's Rule once more. That gives the derivative of $-\cos x$ is $\sin x$ and of $3x^2$ is $6x$; so, we get $\sin x$ by $6x$. Now, what do we do? We may think of applying L'Hospital's Rule again. That gives $\cos x$ on the top, and 6 on the down simplifying to $1/6$.

It is correct; there is nothing wrong; however there is a comment. See, when evaluating $\sin x$ by x what we have done above is that we differentiated the numerator, that gives $\cos x$ and also differentiated the numerator, that gives 1. But while doing that we are committing some sort of circularity. Well, how do you compute the derivative of $\sin x$? The derivative of $\sin x$ will be the limit of $[\sin(x + h) - \sin x]/h$ as $h \to 0$. Then you use your $\sin A + \sin B$ formula or even the half-angle formulas; that will give you $2\cos(x + h/2)\sin(h/2)/h$. We can bring this 2 here, and make it $h/2$. This limit is same as limit as $h/2$ goes to 0. You see, $\cos(x + h/2)$ as $h/2 \to 0$ gives $\cos x$. What about $\sin(h/2)/(h/2)$? We have proved independently that it is equal to 1. Therefore, this will give us $\cos x$. That means in computing the derivative of $\sin x$, we are using the limit of $\sin x$ by x as $x \to 0$. Therefore, it is really circular to use L' Hospital's Rule to evaluate this limit.

You would better say the limit above is equal to $1/6$ directly because we know that the limit of $\sin x$ by x as $x \to 0$ is equal to 1. But this is not wrong; it will give the same answer. Anyway, this limit is equal to $1/6$.

(Refer Slide Time: 3:23)



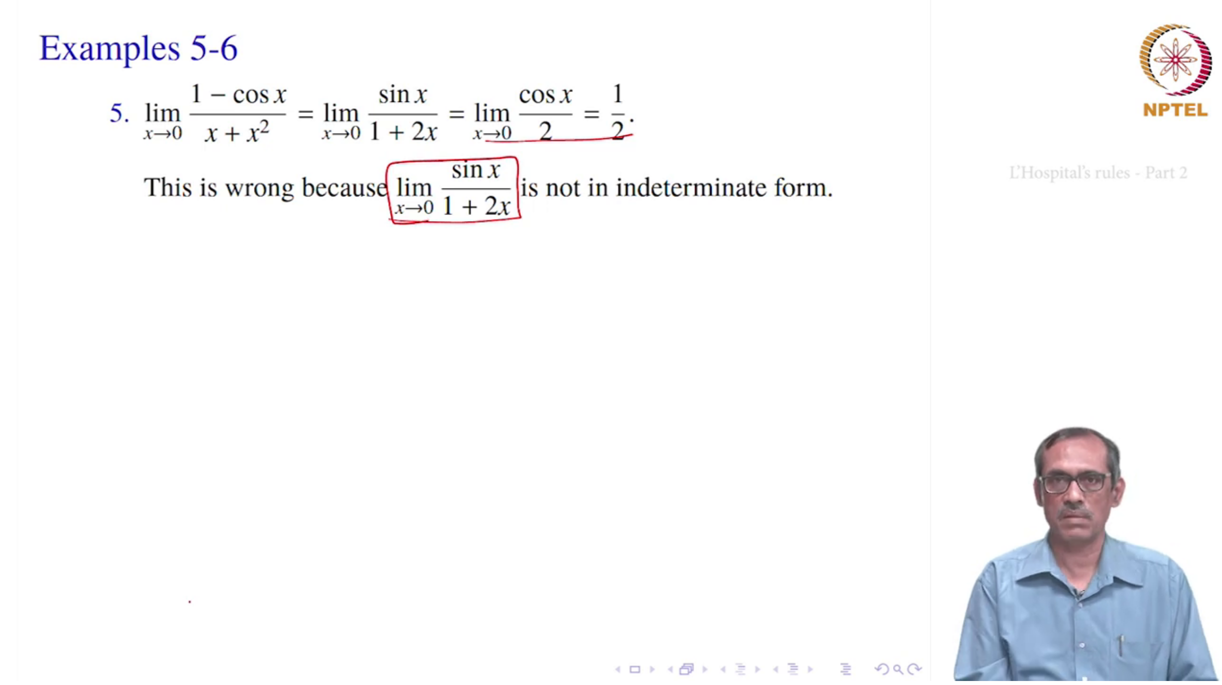
Let us go to the third problem. It asks to evaluate $\sin x$ by x^2 as x goes to 0+. We are interested in one sided limit only. Again it is in $0/0$ form. So, we differentiate the numerator and differentiate the denominator. The numerator gives $\cos x$ and the denominator gives $2x$. Now this is again in $0/0$ form; you can use L'Hospital's Rule. That would give the limit as x goes to 0+ of $-\sin x$ by 2. And that would give 0.

So, what is wrong? Why does it give 0 whereas it should give ∞? There is a simple reason. Reason is, when you apply L'Hospital's Rule you must see that it is in $0/0$ form. Now, when you have $\cos x$, as $x \to 0+$, this $\cos x$ really goes to 1. So, it is not in $0/0$ form. That is exactly the reason. It is in 1 by 0 form as x goes to 0+; and that gives the answer as ∞, not 0. We have to be really observant instead of applying the rule mechanically. We have to see whether it is satisfying all the conditions in L'Hospital's Rule or not. It is the limit as x goes to 0+ of $\sin x$ by x^2; and it is ∞.

Let us see the fourth one. It is similar to the third. We have $\sin x$ by x^2, as $x \to 0-$; this is in $0/0$ form. It gives $\cos x$ by $2x$ as $x \to 0-$. When x is negative $\cos x$ is positive, but $2x$ becomes negative. So it becomes $-\infty$. That is, the limit as x goes to 0 of $\sin x$ by x^2 does not exist, it is neither ∞ nor $-\infty$ even.

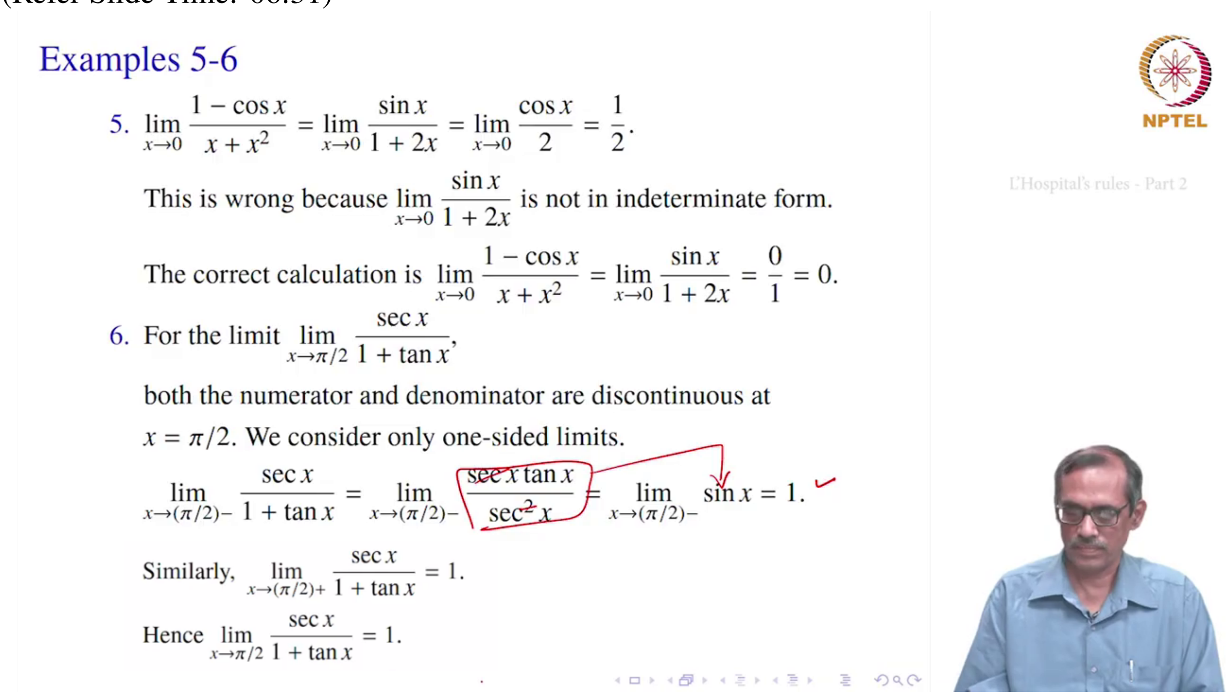
Let us see the fifth one. Here, we have the numerator as $1-\cos x$ and the denominator is $x+x^2$. Both got to 0 as $x \to 0$. So we can apply L'Hospital's Rule.

(Refer Slide Time: 05:26)



Differentiation make 1 as 0, $-\cos x$ as $\sin x$, then $x + x^2$ s $1 + 2x$. Here, $\sin x$ goes to 0, but the denominator does not go to 0. So, we cannot apply L'Hospital's Rule again. If x goes to 0, then the top one goes to 0, down one becomes 1; so the limit should be 0. But if you go in the wrong way that would give, by differentiation, $\cos x$ by 2, whose limit would be $1/2$, which is wrong. The reason is: this is not in the indeterminate form $0/0$. Its limit should be equal to 0 as you see.

(Refer Slide Time: 06:31)



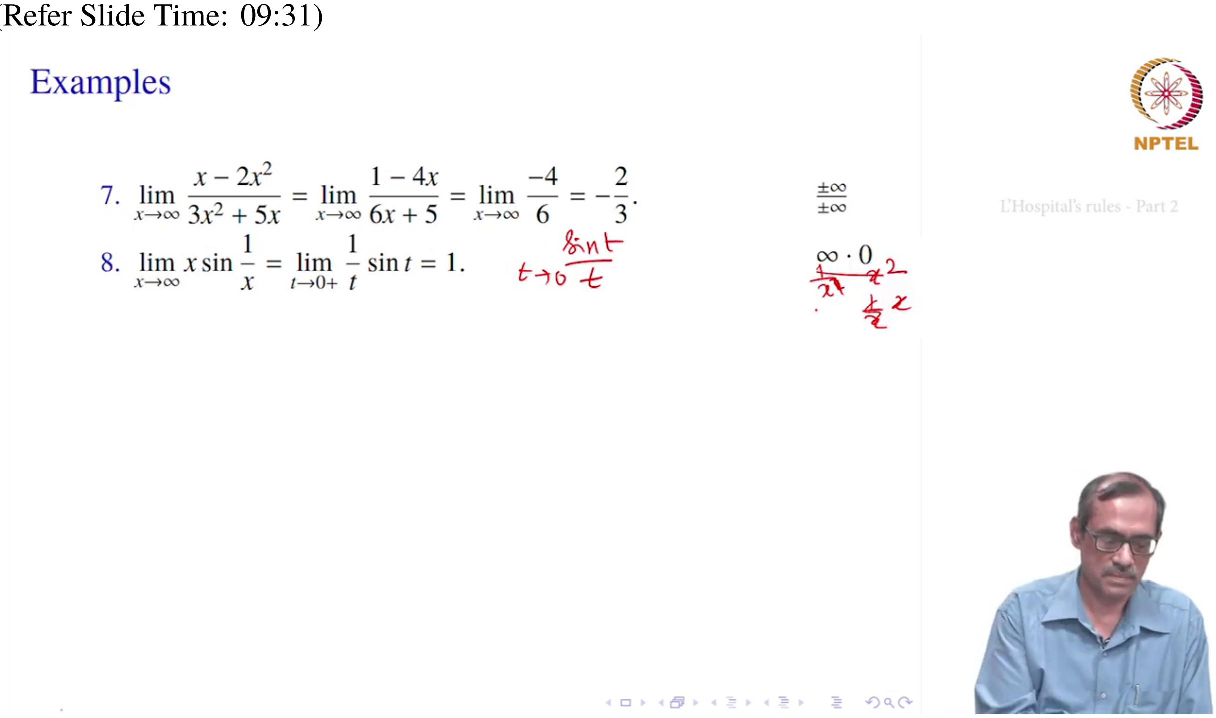
Let us go to the sixth one. Here, we try to find the limit as x goes to $\pi/2$ of $\sec x$ x by $1 + \tan x$. The term $\tan x$ has problem as $x \to \pi/2$; it goes to ∞; also $\sec x$ goes to ∞. So, it is in ∞/∞

form, not $0/0$ form. Then of course, we can use L'Hospital's Rule. We have commented about that earlier. But there is another problem. The function $\tan x$ is not continuous at $x = \pi/2$. So, we cannot apply the rule as $x \to \pi/2$. We thus consider the one-sided limits. We may take the limit as $x \to \pi/2+$, when we should consider the right side neighborhood $(\pi/2, \pi/2 + \delta)$. We can also take the limit as $x \to \pi/2-$, taking the left side neighborhood as $(\pi/2 - \delta, \pi/2)$. Both these things we have to consider separately. We cannot consider those at a time because $\tan x$ is not continuous there. So, let us take the one sided limits.

When $x < \pi/2$, but x is near $\pi/2$, we have $\sec x$ by $1 + \tan x$ in ∞/∞ form. We apply L'Hospital's Rule. The derivative of $\sec x$ gives $\sec x \tan x$ and the derivative of $1 + \tan x$ $\sec^2 x$. So, it is this limit. Again, when x goes to $\pi/2$, it is in ∞/∞ form. But here, one $\sec x$ can cancel. Canceling one $\sec x$, we have $\tan x$ by $\sec x$. Writing $\tan x = \sin x$ by $\cos x$, we have $\sin x$ by $\cos x \sec x$. This whole thing is equal to $\sin x$. So, we do not have to apply L'Hospital's Rule blindly again, because it simplifies to $\sin x$. When $x \to \pi/2-$, the limit of $\sin x$ will be $\sin(\pi/2)$, which is 1.

Now, let us take the right side limit. When you take the right side limit, $x \to \pi/2+$. The top one becomes ∞ and so is the down one. So, we continue along the same line and reach at the limit of $\sin x$ as $x \to \pi/2+$, which again gives 1. Therefore, we say that the limit of the given function is equal to 1. Sometimes we have to take one sided limits this way.

(Refer Slide Time: 09:31)



Let us go to next problem. Here, we want to find the limit as $x \to \infty$. It is not one point like $\pi/2$, it is ∞. Naturally it will be one sided limit. From the left side we are approaching, making x larger and larger. When $x \to \infty$, the function $x - 2x^2$ divided by $3x^2 + 5x$ is in $-\infty/\infty$ form. So, applying L'Hospital's Rule, we have in the numerator: x gives 1, $-2x^2$ gives $-4x$; and in the denominator: $3x^2$ gives $6x$, $5x$ gives 5. It is okay. We obtain the limit of $(1 - 4x)/(6x + 5)$. As $x \to \infty$, the numerator goes to $-\infty$ and the denominator goes to ∞. Again it is in $-\infty/\infty$ form.

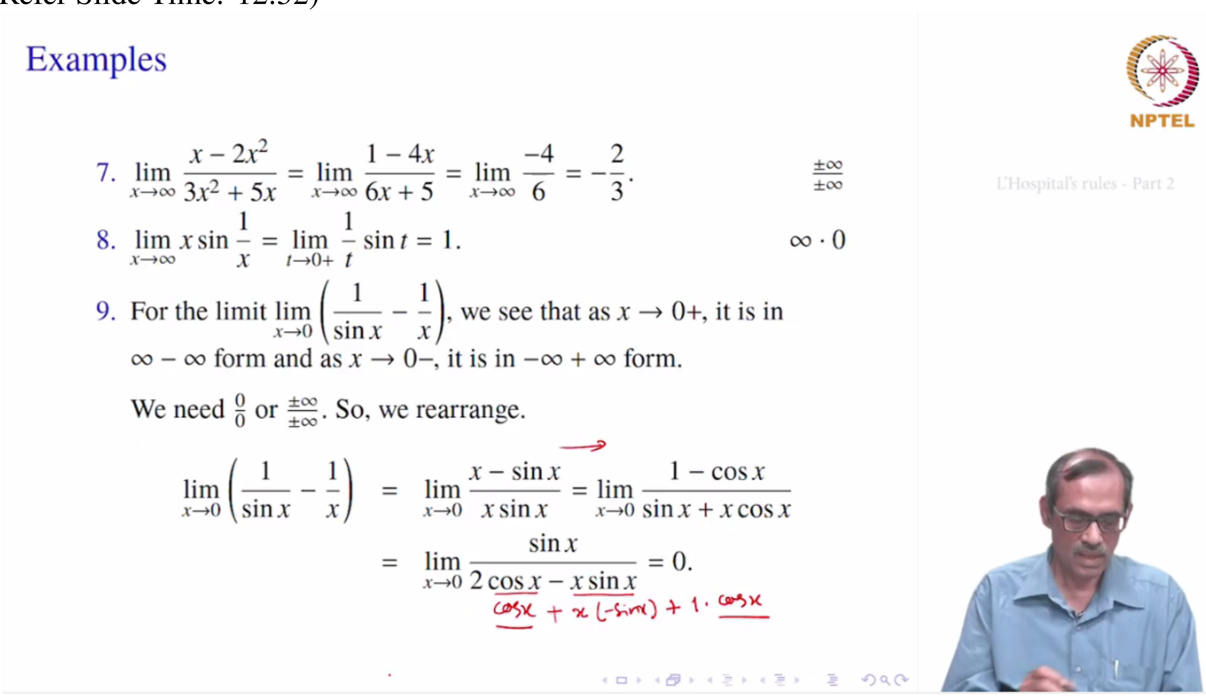
Applying L'Hospital's Rule again, we get the limit of $-4/6$, which is $-4/6$ or $-2/3$.

The indeterminate forms we have met here are of the form ∞/∞ or $-\infty/\infty$ and so on. These are quite straight forward. But the limit is now computed as $x \to \infty$. Similarly, let us find the limit of $x\sin(1/x)$ as $x \to \infty$. Here, $\sin(1/x)$ can give trouble; that is why it is put this way. We have done it in a different way earlier, but let us see.

As $x \to \infty$, $\sin(1/x)$ goes to 0. So, it is in the form $\infty \cdot 0$; which is also an indeterminate form. What is the reason that $\infty \cdot 0$ is indeterminate? You may think of ∞ as the limit of x^2 as $x \to \infty$ and the 0 as the limit of $1/x$ as $x \to \infty$. When you multiply them, it gives x, which will be ∞. But if you take x for ∞ $1/x^2$ for 0, then then multiplying them you get $1/x$, which will give 0 as $x \to \infty$. So, different ways of approaching ∞ and 0 gives different answers. That is why these are called indeterminate forms.

With this indeterminate form, let us see what happens when you apply L'Hospital's Rule. But L'Hospital's Rule requires it in the form $0/0$ or $\pm\infty/\pm\infty$, not $\infty \cdot 0$ directly. So, we look at it in a different way. We take $t = 1/x$. As $x \to \infty$, $t \to 0+$. Then, the given expression becomes $1/t$ into $\sin t$. Now it is directly in the form $\sin t$ by t as $t \to 0+$. We know that this limit is 1; we do not need to apply L'Hospital's Rule here. But if you apply, of course, you will get the same thing.

(Refer Slide Time: 12:52)



In the next problem, we want to compute the limit of $1/\sin x - 1/x$ as x goes to 0 provided it exists. When x goes to $0+$, it is positive so that $1/\sin x$ becomes ∞ and $1/x$ also becomes ∞; so, it is in $\infty - \infty$ form. This is also an indeterminate form because this can become finite, a number or it can become ∞ or it can become $-\infty$ depending on what way you approach these infinities. That is why it is called an indeterminate form. So, what do we do?

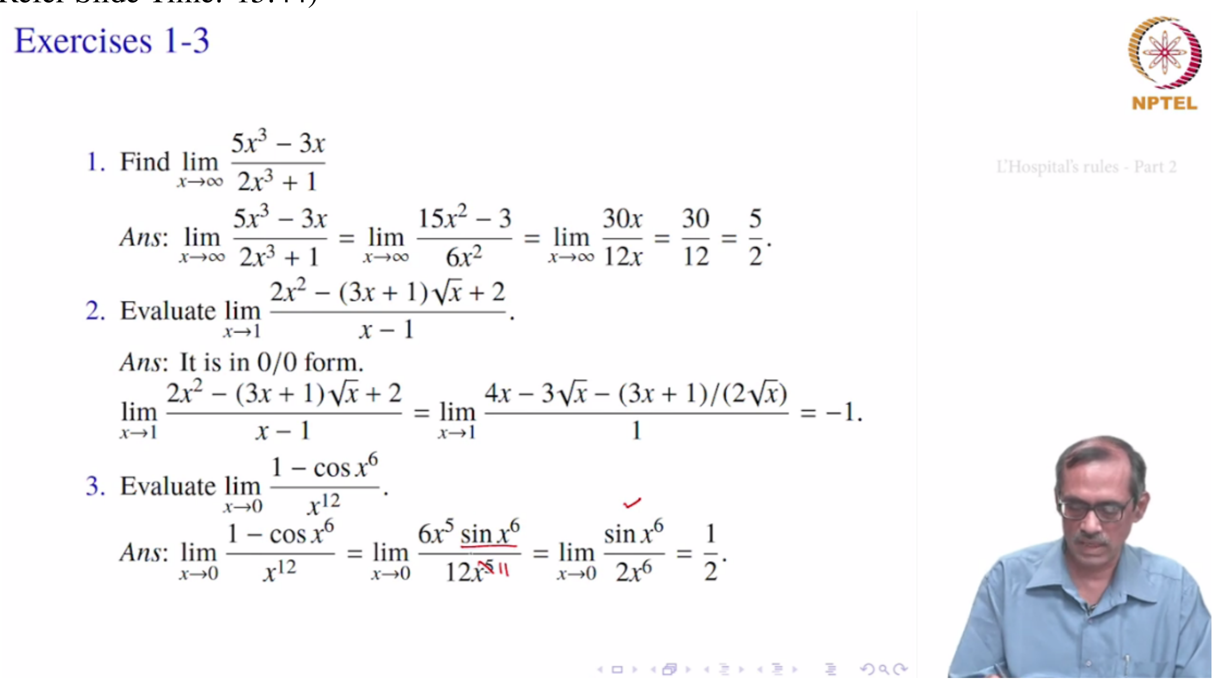
It is neither in the form $0/0$ nor in ∞/∞ form. We have to really play with it and bring it to that form, and then we can apply L'Hospital's Rule. We rearrange and write $1/\sin x - 1/x$ as

$(x - \sin x)/x \sin x$. Now, $x - \sin x$ goes to 0 as $x \to 0$ and $x \sin x$ also goes to 0 as $x \to 0$. So, it is in $0/0$ form. You can apply L'Hospital's Rule here.

Applying L'Hospital's Rule, the derivative of the numerator gives $1 - \cos x$ and that of $x \sin x$ is $\sin x$ into the derivative of x, which is 1 plus x into the derivative of $\sin x$, which is $\cos x$. It gives $\sin x + x \cos x$. When $x \to 0$, the numerator goes to 0 because $\cos x$ goes to 1. And the denominator also goes to 0. Therefore you can apply L'Hospital's Rule once more. The derivative of $1 - \cos x$ gives $\sin x$ and for the derivative of $\sin x + x \cos x$, you again apply the product rule and obtain $\cos x$ plus x into the derivative of $\cos x$ plus the derivative of x into $\cos x$. These two $\cos x$ are written here as $2 \cos x$. So, you get $\sin x/(2 \cos x - x \sin x)$.

Here, the numerator becomes 0 as $x \to 0$ and the denominator does not go to 0; it goes to 2 because $x \sin x$ goes to 0 and $\cos x$ goes to 1; so it is 2. Therefore, the limit is 0. So, we should not apply again L'Hospital's Rule thinking that it is in that form; we have to verify whether it is in that form or not.

(Refer Slide Time: 15:44)



Let us take some more problems. Here, we are required to find the limit of $(5x^3 - 3x)/(2x^3 + 1)$ as x goes to ∞. As $x \to \infty$, this $2x^3 + 1$ goes to ∞ and the numerator is in $\infty - \infty$ form. But it can be thought of as ∞ because x^3 dominates x; it will be much larger than x. Therefore, when you subtract it you would remain still large. Though it is in $\infty - \infty$ form, it is really ∞. Hence, the original limit is in ∞/∞ form. Then, we can use L'Hospital's Rule.

The derivative of $5x^3 = 3x$ gives $15x^2 - 3$, and that of $2x^3 + 1$ gives $6x^2$. Once more, it is in ∞/∞ form. We differentiate it the numerator and denominator separately. The numerator gives $30x$ and the denominator gives $12x$. Now we do not need differentiation. Of course, you can use differentiation, but you can cancel this x and then take the limit. That gives $30/12$ or $5/2$.

Here, this argument is necessary. We need to say that it is in infinity by infinity form because

$5x^3 - 3x$ will be larger than x for x bigger than something, say, for $x > 10$. If $x > 10$, we can see that $5x^3 - 3x > x$. Therefore, the limit of the numerator $5x^3 - 3x$ as $x \to \infty$ is ∞. This argument is necessary.

Let us look at the second problem. Here, we want to evaluate the limit of $2x^2$ minus $3x + 1$ into $\sqrt{x+2}$ divided $x - 1$ as x goes to 1. As $x \to 1$, the denominator goes to 0 and the numerator goes to 2 minus 4 times 1 plus 2, which is 0. That is, the limit is in $0/0$ form. The derivatives will be a bit difficult, but let us try.

We take the derivative of the numerator. Here, $2x^2$ gives $4x$, plus 2 of course gives 0, and the other one is in f/g form. we take the derivative of a ratio. It will be $\sqrt{x}$ into the derivative of $3x + 1$, which is 3, minus $3x + 1$ into the derivative of $\sqrt{x}$, which is $1/2$ into $1/\sqrt{x}$. That should be plus sign not minus; this is minus; and one minus comes here and divided by this; but no this is not division this is really multiplication. That gives $(3x + 1)\sqrt{x}$. Then, you apply the product rule of differentiation. So, that gives the derivative of $3x + 1$ into $\sqrt{x}$, and same sign is there. That gives $1/2$; is it correct?

So, the numerator is this and the denominator is the derivative of $x - 1$, which is 1. As x goes to 1, the denominator is no more 0 now. The numerator can be evaluated directly by substituting x equal to 1. In that case, you get 4 minus 3, which is 1 minus it is 4 divided by 2 minus 2; so that gives you -1. That is the answer.

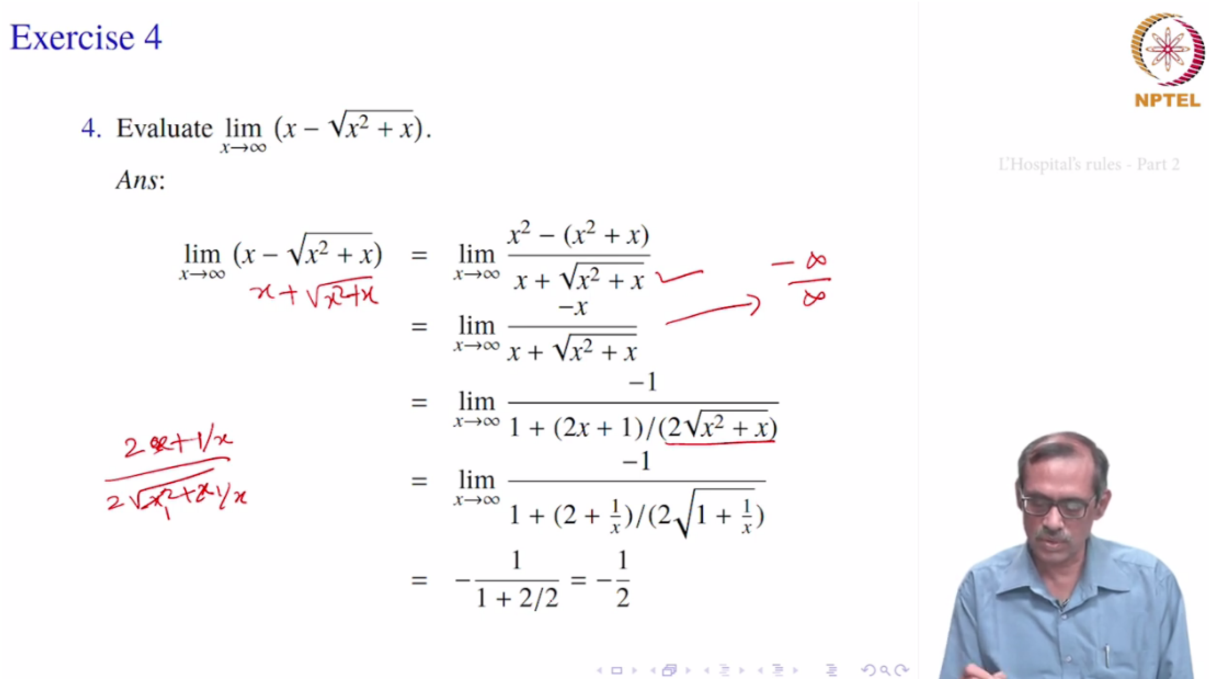
Let us go to next problem. Here, we want to find the limit of $1 - \cos x^6$ divided by x^{12}. When $x \to 0$, x^6 goes to 0, and then $\cos x^6$ goes to 1. In the denominator, x^{12} goes to 0. So, it is in $0/0$ form. We apply L'Hospital's Rule. The derivative of 1 is 0 and of $-\cos x^6$ is the derivative of $-\cos x^6$ with respect to x^6 multiplied by the derivative of x^6 with respect to x. minus cosine x 6th. That gives $\sin x^6$ into $6x^5$. And denominator gives $12x^{11}$. Now, this is $6x^5 \sin x^6$ by $12x^{11}$; here $6x^5$ cancels, and we obtain $\sin x^6$ divided by $2x^6$. As x^6 goes to 0, we can write it as $\sin t$ divided by $2t$ as $t \to 0$. And we know that the limit of $\sin t$ by t is equal to 1 as $t \to 0$. So, the answer is half.

This is pretty straightforward, but this chain rule has to be applied for obtaining the differentiation of $\sin x^6$.

Let us take the next one. Here, we want to evaluate the limit as x goes to ∞ of $x - \sqrt{x^2 + x}$. As $x \to \infty$, this is in $\infty - \infty$ form. We cannot apply L'Hospital's Rule straightforward; it should be brought to a suitable form, such as a numerator by a denominator etc. We multiply $x + \sqrt{x^2 + x}$ and divide it. When you multiply this with $x - \sqrt{x^2 + x}$, you obtain $x^2 - (x^2 + x)$, which simplifies to $-x$. Then, we need to compute the limit of $-x/[x + \sqrt{x^2 + x}]$ as $x \to \infty$. As x goes to infinity, it is in $-\infty/\infty$ form. So, we can use L'Hospital's Rule here.

Using L'Hospital's Rule we differentiate the numerator, which gives us -1. Next, we differentiate the denominator. That gives us 1 plus the derivative of $]sqrtx^2 + x$ with respect to $x^2 + x$ into the derivative of $x^2 + x$ with respect to x. It amounts to $1/2\sqrt{x^2 + x}$ into $2x + 1$. So, the ratio simplifies to -1 divided by $1 + (2x + 1)/(2\sqrt{x^2 + x})$. So, this is correct. Then we take the limit as $x \to \infty$.

(Refer Slide Time: 22:09)



It is no more in any indeterminate form; the numerator is not infinity or 0 or any such, but it is -1. And what happens to the denominator at ∞? We have to check that. It is $1 + (2x+1)/(2\sqrt{x^2+x})$. As x goes to ∞ what happens to this? It is not clear what to do. We divide x so that makes this $(2x+1)/(2\sqrt{x^2+x})$ equal to $(2+1/x)/2\sqrt{1+1/x}$. As x goes to ∞, $1/x \to 0$ in both the places. So you get $1+2/2$, which is 2. Then, the answer is $-1/2$.