Basic Calculus - 1

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Lecture 24 - Part 1

L' Hospital's Rules - Part 1

Well, this is lecture 24 of basic calculus 1. Today we will be discussing an important notion, which helps in evaluating the limits of certain forms using the derivatives. It looks a bit off the place because you define derivatives through the limits. But we are telling that we will be evaluate the limits through the derivatives. Of course, it is not applicable everywhere; it is applicable when the functions for which you want to compute the limit are in certain forms. This is called L' Hospital's rules. We will look at it now.

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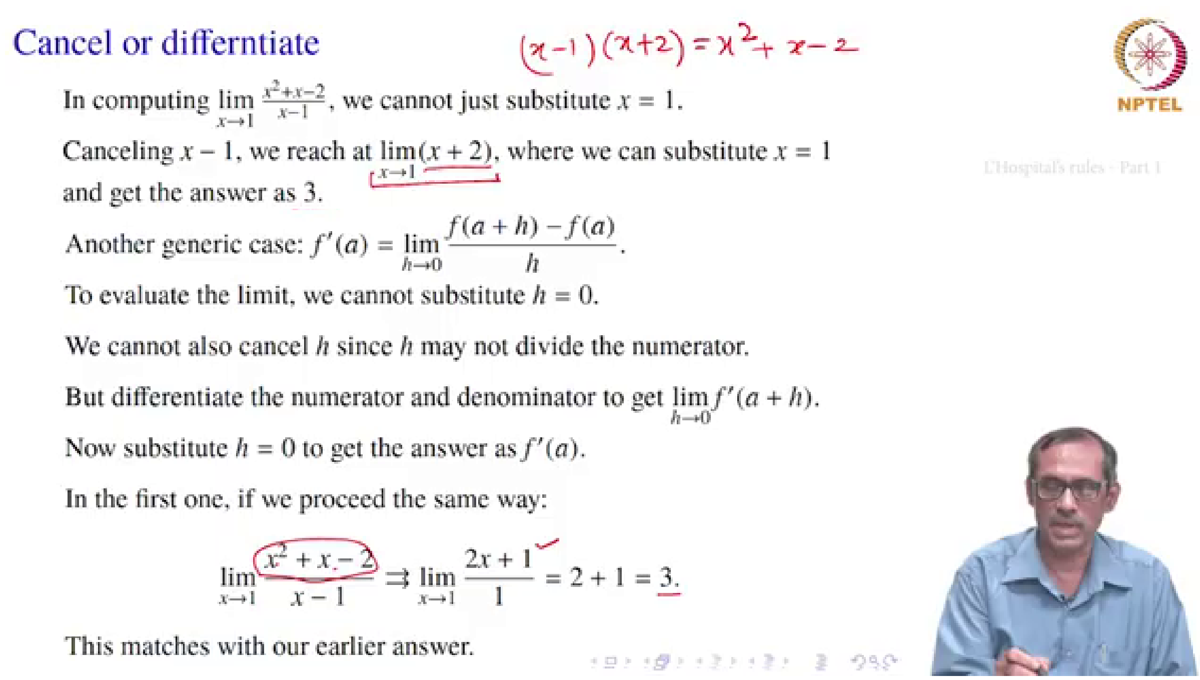

Let us look at evaluating this limit: the limit of $x^+ x - 2$ divided by $x - 1$ as x approaches 1. As you know, if you have a function in the form $f(x)/g(x)$ and the limit of $g(x)$ is not equal to 0, you can take the limits of $f(x)$ and $g(x)$ separately, and then the limit will be equal to the limit of $f(x)$ divided by the limit of $g(x)$. But here that is not applicable, because the denominator $x - 1$ has the limit 0 when x goes to 1. We say intuitively that we cannot just substitute x equal to 1 here.

Note that when the denominator becomes 0 for x equal to 1, the numerator also becomes 0 here. At $x = 1$, it is $1 + 1 - 2$, which is equal to 0. So this is something called $0/0$ form. For this, we note that we cannot just substitute x equal to 1. This means we cannot take the limits separately and divide. But you see that the numerator is $(x-1)(x-2)$. You have $(x-1)(x-2) = x^2 + x - 2$; that is what the numerator is. We can really cancel this $x - 1$ from denominator and numerator so that you get the numerator as $x + 2$ only. The whole ratio is $x + 2$. Now, you can take the limit, and that will be equal to 3, of course. We used to do this everywhere.

But look at the definition of the derivative. The derivative at $x = a$ is equal to the limit as x goes to a of the ratio $f(a + h) - f(a)$ to h. Here you see that if we take h near 0, the denominator goes 0, and also the numerator becomes 0. So, it is again in the same $0/0$ form. The definition of the derivative of any function would give rise to this limit. We cannot just substitute h equal to 0 there. And it is not always obvious whether that h can always be factored away from $f(a + h) - f(a)$ as in the previous case. We do not know how to factor and cancel this h and then take the limit. That is a problem here, right?

But then we can do something else. Suppose, we differentiate the numerator independently and denominator independently. The differentiation will be with respect to h. Then, the derivative of the denominator turns out to be 1 and of the numerator with respect to h gives $f'(a + h)$. Now, taking the limit as $h \to 0$, that is, if you substitute $h = 0$, then you would get back that $f'(a)$.

Similarly, in the previous case also if you differentiate $x^2 + x - 2$ it gives $2x + 1$. The derivative of the denominator $x - 1$ is 1. So, we have $(2x + 1)/1$. Now, taking limit of this as $x \to 1$ gives us 3. That matches with the other answer.

In both the cases we see that if it is in the $0/0$ form, then you differentiate the numerator, differentiate the denominator, then take the limit, the answer turns out to be the same. That is exactly what do you want to justify.

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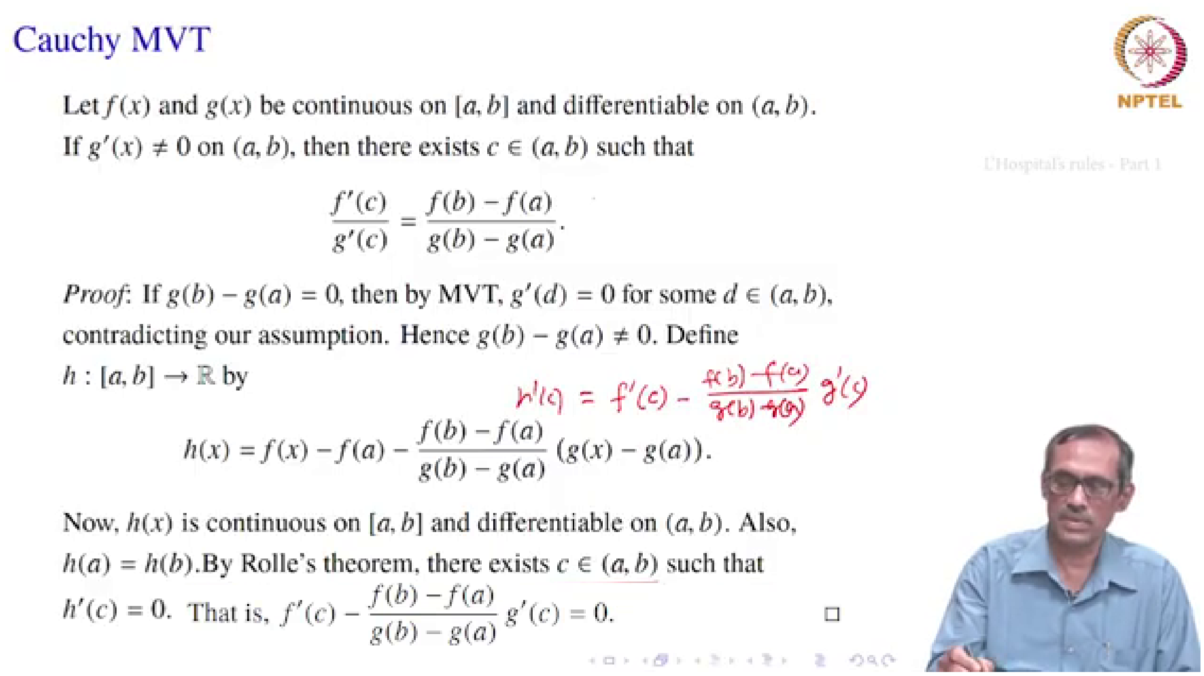

So, to justify this we will prove a slight generalization of our Mean Value Theorem. That will be useful. This generalization is called Cauchy Mean Value Theorem. Here, our assumption is that there are two functions $f(x)$ and $g(x)$, which are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), just like in the Mean Value Theorem. In the Mean Value

Theorem, we have only one function; now we have two functions. Suppose $g'(x)$ is not equal to 0 on the whole of the interval (a, b). That is, it is nonzero at every point in (a, b). Then, the theorem says that there exists some number c between a and b such that if you take this ratio $[f(b) - f(a)]/[g(b) - g(a)]$, then it will be equal to $f'(c)/g'(c)$.

Now, why is it a generalization of the Mean Value Theorem? Well, in the Mean Value Theorem, you have $f'(c) = [f(b) - f(a)]/(b - a)$. It suggests that we take $g(x) = x$ itself. The function $g(x) = x$ is the identity function. It satisfies this condition that $g(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). So, there exists a point c such that $f'(c)/g'(c)$ is equal to that expression. Now, $g'(c)$ is the derivative of $g(x)$ evaluated at c, which is (x') evaluated at c is 1. So, on the left side you get $f'(c)$, and on the right side you get $f(b) - f(a)$ divided by $b - a$ as in the Mean Value Theorem.

First, we will prove this generalization or Cauchy Mean Value Theorem. The proof is simple. We observe that $g(b)-g(a)$ should not be 0, since it is in the denominator. But we have not assumed that anywhere. Well, we have not assumed it because it can be derived from the conditions given. How is it so? Suppose $g(b) - g(a) = 0$. On the contrary assume that $g(b) - g(a) = 0$. Now, you can apply Rolle's theorem or Mean Value Theorem to get g' is equal to 0 at some point between a and b, say, $g'(d) = 0$ for some $d \in (a, b)$. But that will contradict our assumption that $g'(x)$ is not equal to 0 in (a, b). Therefore, $g(b) - g(a) \neq 0$.

So, this right side quantity $[f(b) - f(a)]/[g(b) - g(a)]$ is well defined. Then, we apply a little trick just like in the Mean Value Theorem. We define a new function $h; [a, b] \to \mathbb{R}$. What is its definition? At any $x \in [a, b]$, $h(x) = f(x) - f(a) - [f(b) - f(a)]/[g(b) - g(a)] \times [g(x) - g(a)]$. Since we know that $g(b)-g(a)$ is nonzero, we can now divide it. So, this one is a number, provided x is a number in $[a, b]$. That is how h is defined.

Now, what properties does h satisfy? Since f and g are continuous on the closed interval $[a, b]$, and this is just $f(x)$ minus a constant times $g(x)$ plus another constant, it is also continuous on the whole closed interval $[a, b]$. Similarly, both $f(x)$ and $g(x)$ are differentiable on the open interval (a, b). So, $h(x)$ is also differentiable on the open interval (a, b).

Next, what is $h(a)$? To get it, substitute x equal to a. This is $f(a) - f(a)$, which is 0. And here again $g(a) - g(a)$, that is also 0. So, $h(a) = 0$. What about $h(b)$? In $h(b)$, here it is $f(b) - f(a)$ and on this side it is $g(b) - g(a)$ which cancels with the denominator $g(b) - g(a)$. So, you get $f(b) - f(a)$ minus $f(b) - f(a)$; that is also equal to 0. So, you see that $h(a) = h(b) = 0$. We do not need them to be equal to 0 though; we need only $h(b) = h(a)$ to use Rolle's theorem.

Now, Rolle's theorem says that there exists a point c between a and b such that $h'(c) = 0$. What is $h'(c)$? Here, $f(a)$ becomes 0; here is a constant giving $f'(x)$ evaluated at c minus $f(b) - f(a)$ divided by $g(b) - g(a)$, and the derivative of $g(x) - g(a)$ is $g'(x)$, evaluated at c giving $g'(c)$. So, you have one c in (a, b) such that h$h'(c) = 0$. This is $h'(c)$. It gives the conclusion that $f'(c)/g'(c) = [f(b) - f(a)]/[g(b) - g(a)]$.

The main thing here is that the same c works for both the functions just as in the Mean Value Theorem. Suppose you apply the Mean Value Theorem independently on f and g. Then, you

would get $f'(c) = f(b) - f(a)$, and similarly, $g'(d) = g(b) - g(a)$ since for the right side there exists one d such that etc. Then you would have got the ratio $[f(b) - f(a)]/[g(b) - g(a)]$ equal to $f'(c)/g'(d)$. We do not know whether this c and d are equal or not. The potency of this theorem is that you can choose such a point c that works for both. You do not have to take two different points c and d. That is how this theorem says something more than the Mean Value Theorem.
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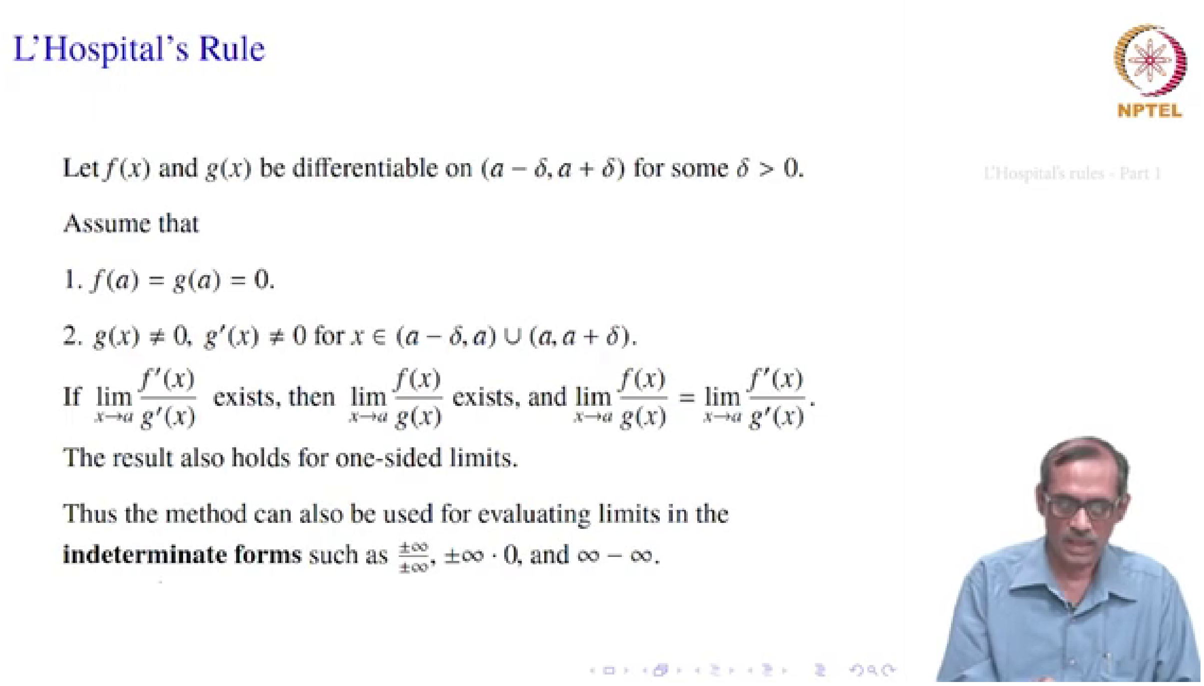

We will see how to apply this for proving L'Hospital's rule. L'Hospital's rule starts with two differentiable functions $f(x)$ and $g(x)$. These are assumed to be differentiable in a neighborhood of a. So, we assume that there exists $\delta > 0$ such that in the open interval $(a - \delta, a + \delta)$, both $f(x)$ and $g(x)$ are differentiable. Suppose $f(a) = g(a) = 0$. That is, we cannot substitute $x = a$ in the ratio $f(x)/g(x)$. We are interested in limits of this kind where both $f(a)$ and $g(a)$ are 0; this is one of our assumptions. And we also assume that $g(x)$ never vanishes in the deleted neighborhood of a; that is, at a it is equal to 0, but except that nowhere else in the open interval $(a - \delta, a + \delta)$, g is 0. Similarly, $g'(x)$ is also not 0 in the same deleted neighborhood. These are the two basic assumptions. Then what is the conclusion? The conclusion says that if the limit of $f'(x)/g'(x)$ as $x \to a$ exists, then the limit of $f(x)/g(x)$ as $x \to a$ also exists, and the value of the value of this limit is equal to the value of the limit of $f'(x)/g'(x)$.

The conclusion means that if you do not know whether the limit of $f(x)/g(x)$ exists or not, you can compute the limit of $f'(x)/g'(x)$. If this latter limit exists, then the former limit exists and the two limits are equal. But these conditions are crucial. We should have the former limit in $0/0$ form; that is, $f(a) = g(a) = 0$; and also $g(x)$ should never vanishes in a deleted neighborhood of a; neither $g(x)$ vanishes nor its derivative $g'(x)$ vanishes. These are the two conditions under which the conclusion holds.

The proof is very easy because all you have to do is go back to our Cauchy Mean Value Theorem.

Now, let us proof L'Hospital's rule.

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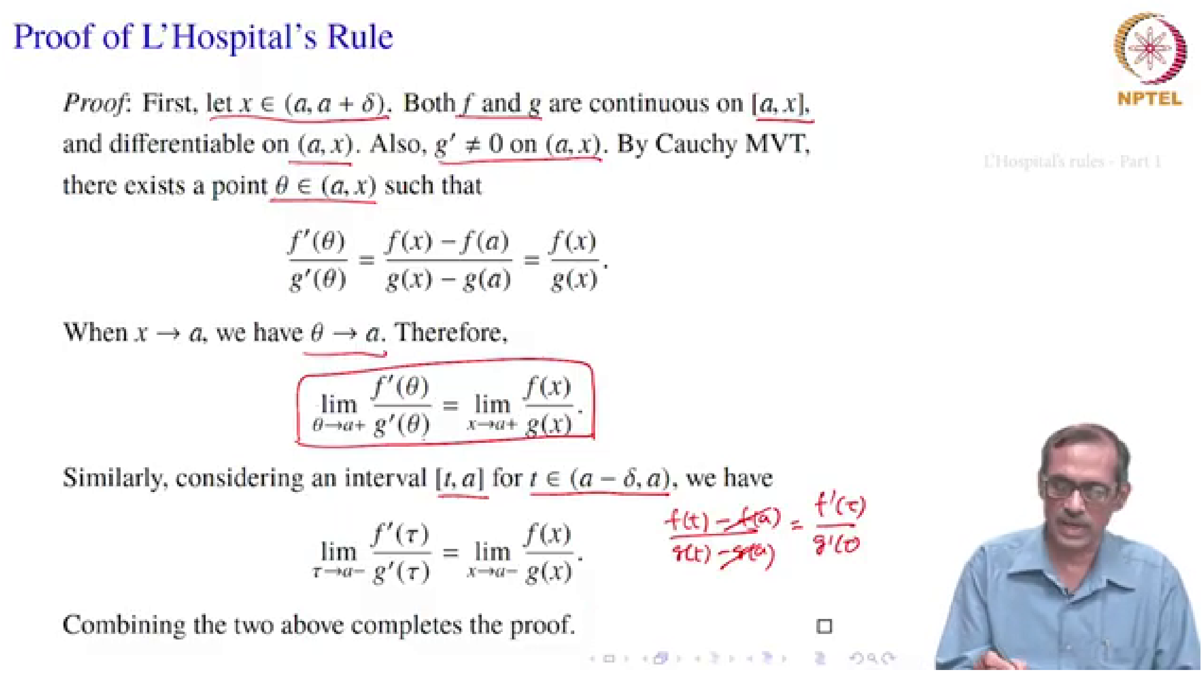

Suppose x is any point inside $(a, a+\delta)$. Let us first consider one sided limits; that will be helpful later. Suppose you take any x between a and $a+\delta$. Then, x is bigger than a but smaller than $a+\delta$. Both f and g are continuous on this closed interval $[a, x]$; that is our assumption. Also, both are differentiable on (a, x). Now, g' is not equal to 0 on (a, x); that is also one of the assumptions. Then, by Cauchy Mean Value Theorem there exists a point θ between a and x such that $f'(\theta)/g'(\theta) = [f(x) - f(a)]/[g(ax) - g(a)]$. We know that $f(a) = g(a) = 0$. So, the right side is $f(a)/g(x)$. We are interested in the limit of $f(x)/g(x)$. As $x \to a$, since θ is in between x and a it also approaches a. Therefore, the limit as θ goes to $a+$ of the left side $f'(\theta)/g'(\theta)$ is equal to the limit as x goes to $a+$ of the right side $f(x)/g(x)$. You see that we have really proved the theorem, but only for the one sided limit: the limit as $x \to a+$. Because, the limit of $f'(\theta)/g'(\theta)$ as $\theta \to a+$ is same thing as the limit of $f'(x)/g'(x)$ as $x \to a+$.

On the other side, for the left sides limit, we take any point $t \in (a-\delta, a)$. In the interval (t, a) we find that similar conditions are satisfied, and a similar conclusion follows. That is, there exists a τ between t to a such that $[f(t) - f(a)]/[g(t) - g(a)]$ is equal to $f'(\tau)/g'(\tau)$. Now $f(a) = g(a) = 0$ so that the left side is $f(\tau)/g(\tau)$. As $t \to a-$, $\tau \to a-$. Therefore, the limit of $f(t)/g(t)$ as $t \to a-$ is equal to the limit of $f'(\tau)/g'(\tau)$ as $\tau \to a-$. We can write both these limits in terms of x rather than t and τ, and obtain our result.

So, both the left side limit and the right side limit behave the same way. Since our assumption is that the limit of $f'(x)/g'(x)$ as $x \to a$ exists, it follows that both the left and right side limits are equal. Therefore, both the left and right side limits of $f(x)/g(x)$ exist and are equal. So, the limit of $f(x)/g(x)$ exists. Again, the same argument shows that the limit of $f(x)/g(x)$ is equal to the limit of $f'(x)/g'(x)$.

You see that the same result holds true for any one-sided limit. That means if we have the interval as (a, ∞) and we want to find the limit of $f(x)/g(x)$ as x goes to ∞, then only the left side limit will be useful.

A similar thing also holds for indeterminate forms $\pm\infty/\pm\infty$ or $\pm\infty \times 0$ or $\infty - \infty$ because in all these cases we are concerned with one sided limits. Fine. We observe that the conclusion also holds when we take the limit as x goes to ∞ or x goes to $-\infty$.

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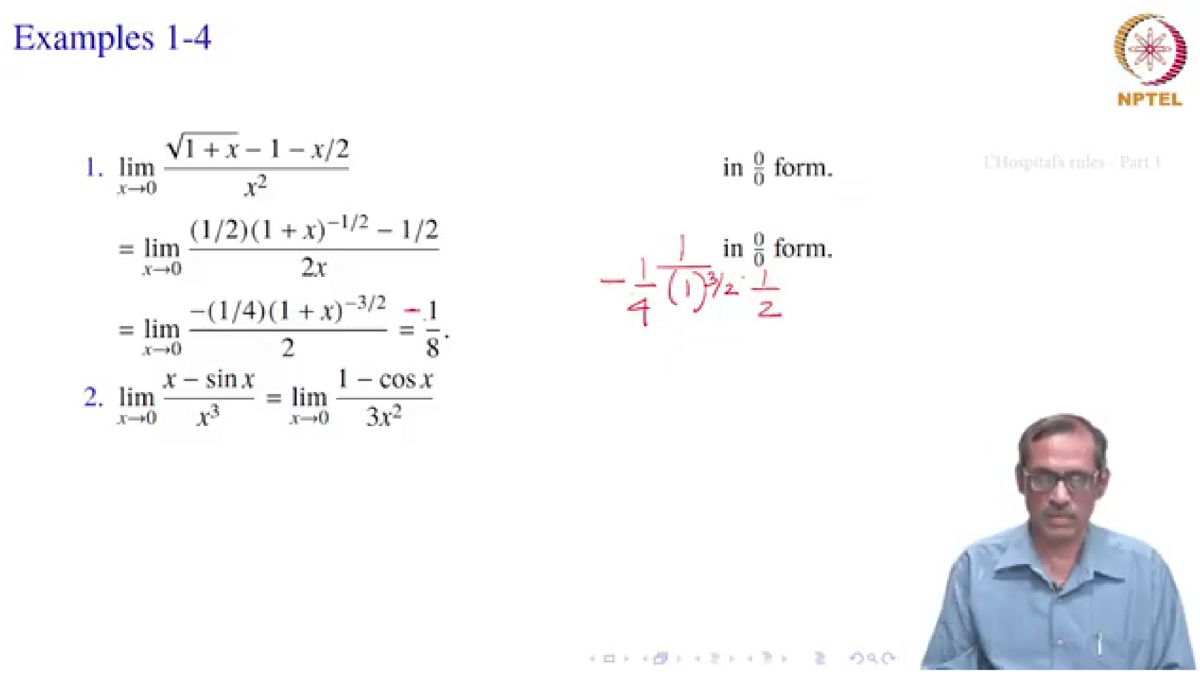

Let us take some examples where we can apply this. Consider finding the limit as $x \to 0$ of $[\sqrt{1+x} - 1 - x/2]/x^2$. As $x \to 0$, its numerator goes to $\sqrt{1} - 1$, which is 0 and the denominator goes to 0. So, it is in $0/0$ form. And you can see the other things: the function $f(x)$ on the top is continuous and also differentiable in a neighborhood of 0; indeed, you can take a δ-neighborhood, where $\sqrt{1+x}$ is well defined, even to the left side. That is, by taking a suitable neighborhood we see that all the assumptions of L'Hospital's rule are satisfied. Therefore, we say that this limit exists and is equal to the limit of the ratio, where we can differentiate the numerator and denominator independently. So, we will just write this limit is equal to this; but remember that it is conditional. Writing this way does not mean that the limit exists and this is equal to this. It says the other way. If the second limit exists, then the first limit also exists and they are equal; that is what this equality symbol here means. Keeping that in mind, we write such equations that this limit is equal to this limit.

Now, differentiate the numerator. The term $(1+x)^{1/2}$ gives $1/2$ times $(1+x)^{-1/2}$; the differentiation of 1 is 0, the differentiation of $x/2$ is $1/2$, keeping that minus sign as it is we get the differentiation of the numerator. And the denominator gives $2x$ for x^2. Now, we can substitute $x = 0$ here to get the limit. If you substitute, you will to get 0 in the numerator and also 0 in the denominator. Again it is in the $0/0$ form.

So, we apply L'Hospital's rule once more. That is, this limit exists provided the next limit exists, which will be the limit as x goes to 0 of the derivative of the numerator by the derivative of the denominator. We write this as equal to the limit as x goes to 0 of the ratio of the derivatives. This gives half into minus half, that is $-1/4$ into $(1+x)^{-3/2}$ divided by the denominator as 2. We can see that the limit of the denominator is not 0. So, when x goes to 0, $1+x$ goes to 1 and its power is minus 3 by 2. See, this is really $-1/4$ into $1^{-3/2}$. This $1^{-3/2}$ simplifies to 1. So, the limit of the ratio is $-1/4$ divided by 2, which gives you $-1/8$.