Basic Calculus - 1
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Lecture 23 - Part 2
Linearization and differential - Part 2

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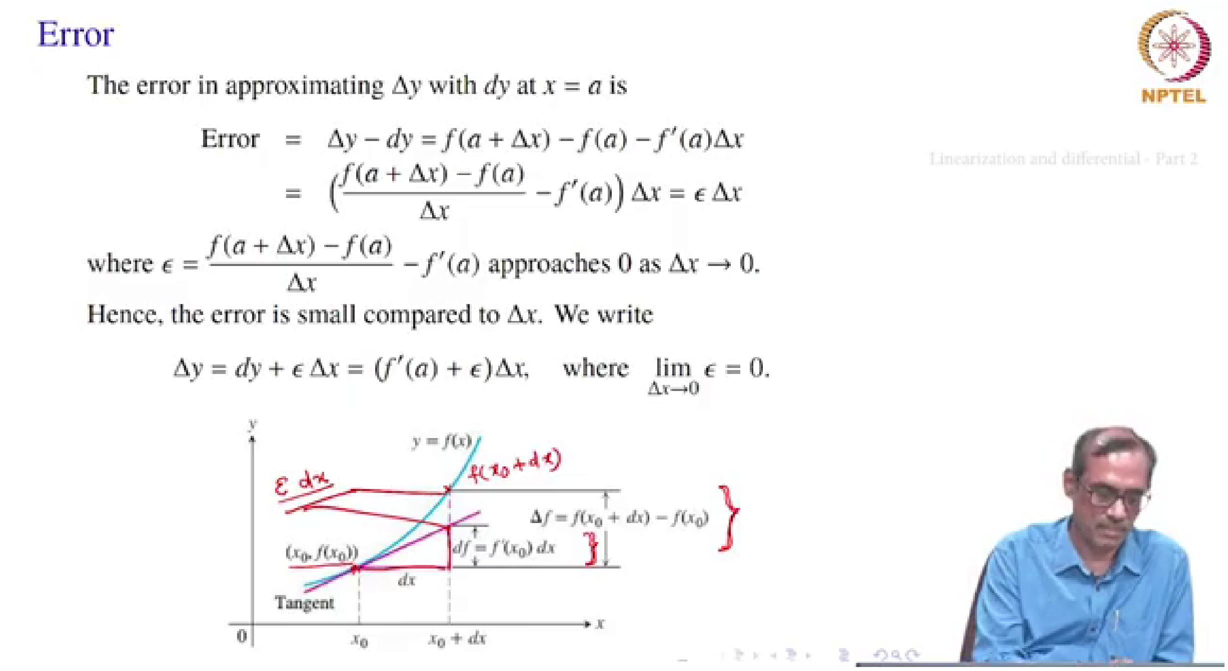

Let us look at the error of linearization. The error was coming from $\Delta y - dy$. As in the earlier example it was 0.1π at the point $x = 10$. This error in approximating Δy with the differential dy at a point $x = a$ is $\Delta y - dy$. Now, Δy is equal to the increment in y, which is $f(a+\Delta x) - f(a)$ and the differential dy is $f'(a)\,\Delta x$. Combining them together we find that the error is equal to $\big[\big(f(a+\Delta x) - f(a)\big)/\Delta x - f'(a)\big]\,\Delta x$. If we write this bracketed thing as ϵ, the error is really $\epsilon\,\Delta x$.

When Δx goes to 0 what happens to this ϵ? We know that the limit of $\big(f(a+\Delta x) - f(a)\big)/\Delta x - f'(a)$ is equal to 0, because the limit of this is equal to $f'(a)$. That means the limit of this expression, which we write as ϵ is equal to 0 when Δx goes to 0. You see that the error is of a bit higher order than Δx; it is equal to something times Δx so that when Δx go to 0, that something also goes to 0.

So, the error is something of higher order than Δx; it is not just linear like Δx. It can be $(\Delta x)^2$ or something like that. The error is $\epsilon\,\Delta x$, where the limit of ϵ is 0 as $\Delta x \longrightarrow 0$. That is how the error looks like.

Look at the figure. Suppose you have the curve $y = f(x)$. At at the point $x = a$, which we write x_0 here, we have the point $(x_0, f(x_0))$ on the curve; it is the blue one. And, we have a tangent at that point $(x_0, f(x_0))$ which is the pink one. We have the point $(x_0 + \Delta x, f(x_0 + \Delta x))$ on the blue one. The height there is $f(x_0 + \Delta x)$. If you look at this point, the corresponding height is $f(x_0)$.

The difference between those heights is the increment in f at the point x_0. And what is df? It is really $f'(x_0)$ times the increment, where $f'(x_0)$ is the slope of the tangent. This slope is this height divided by this dx. When you multiply that with dx, you would get df; this df is really this height, this much. All that we say here is that the error which can be expressed as ϵ times dx or Δx where ϵ goes to 0 as Δx goes to 0. When you plot in the graph, this is how the error looks like.

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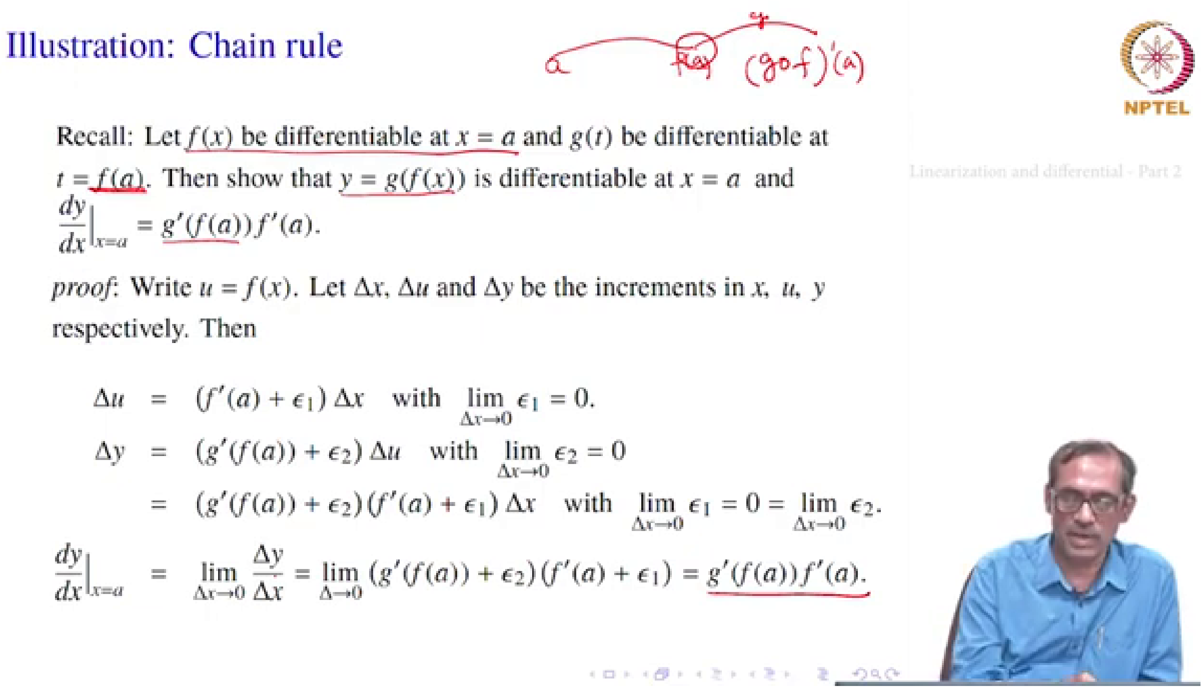

We can apply this notion or this concept of differential to give another proof of the Chain rule. We have already proved that. Now we can give another proof of the same Chain rule using this notion of differential. Recall what the chain rule says. Suppose $f(x)$ is differentiable at the point a, $g(t)$ is another function, which is differentiable at $f(a)$. Here, f goes somewhere, then g goes somewhere. The function f is taking a to $f(a)$; now g works and it takes $f(a)$ to $g(f(a))$. It is assumed that f is differentiable at a and g is differentiable at $f(a)$. We want to find the derivative of $g \circ f$ at a; that is really the chain rule. As we know, $(g \circ f)'(a) = g'(f(a))\, f'(a)$. We want to show that this new function $y = g(f(x))$ is differentiable at $x = a$, and its derivative at $x = a$ is equal to $g'(f(a))\, f'(a)$. That is what the chain rule says.

And here goes the proof. We write $u = f(x)$ just for notational convenience. Then, Δx is the increment in x and Δu is the increment in f at that point a. Write $y = g(f(x))$ and the increment in y as Δy. As we have seen earlier with the error analysis of linearization, you can write the increment in u as $\Delta u = \big(f'(a) + \epsilon_1\big)\Delta x$, where the limit of this ϵ_1 is equal to 0 when Δx approaches 0. Is that clear?

We can write this error equal to this times delta x plus this. So, Δy can be written as $\big(g'(u) + \epsilon_2\big)\Delta u$ at $u = f(a)$. Use that now. We can write $\Delta u = \big(f'(a) + \epsilon_1\big)\Delta x$ where the limit of ϵ_1 is equal to 0. What about Δy? Now, $y = g(f(x))$ and we are taking its increment at $f(a)$. So, $\Delta y = \big(g'(f(a)) + \epsilon_2\big)\Delta u$, where Δu is the increment in u and the limit of ϵ_2 is 0 as Δx goes to 0.

Now, we plug in the earlier expression here to obtain

$$\Delta y = \big(g'(f(a)) + \epsilon_2\big)\Delta u = \big(g'(f(a)) + \epsilon_2\big)\big(f'(a) + \epsilon_1\big)\Delta x.$$

Divide this expression by Δx and see what happens. One expression is $g'(f(a)) + \epsilon_2 g$ and another is $f'(a) + \epsilon_1$. Then, $\Delta y/\Delta x$ is equal to the product of these two underlined factors. And then you take the limit as $\Delta x \to 0$. That gives dy/dx on one side; and we look at the other side. On the other side, taking the limit as Δx goes to 0 gives the product of two limits. As $\Delta x \to 0$, both $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$. So, in the limit the product becomes $g'(f(a))\, f'(a)$. So, the answer is now obtained directly. This is a simpler way of looking at the chain rule using the notion of differential.
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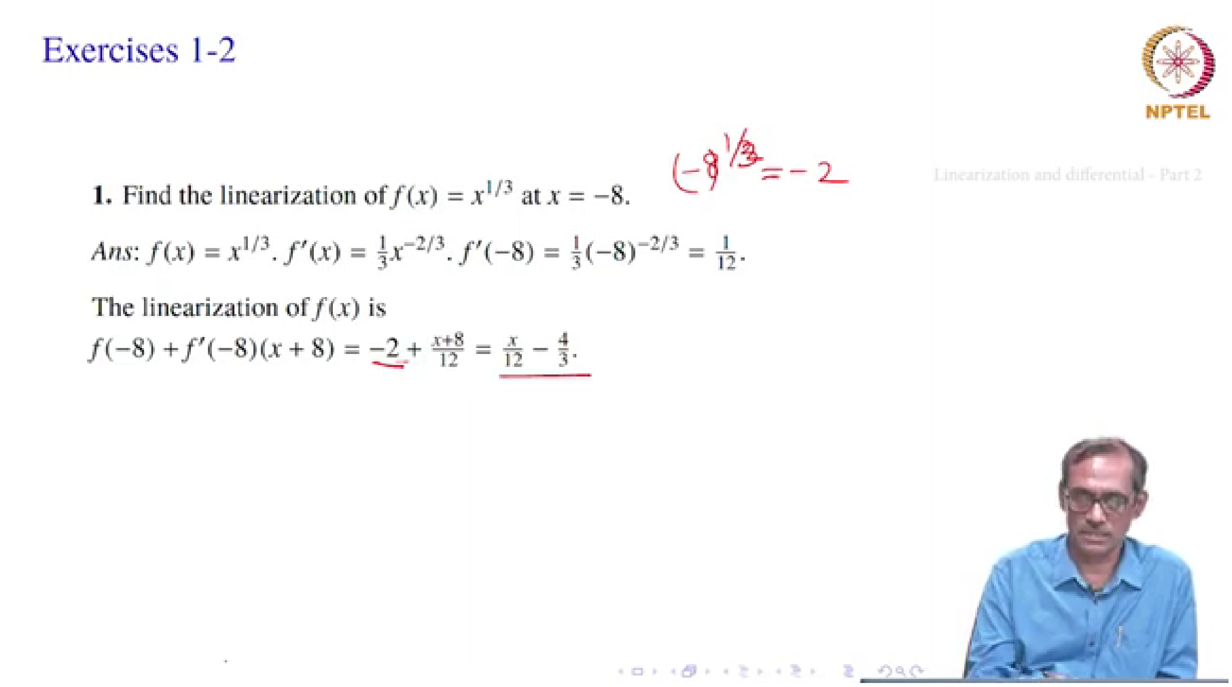

Let us take some problems basing on this idea. We want to find the linearization of the function $x^{1/3}$ at the point $x = -8$. What we do is, we differentiate and then compute the necessary functional values $f(-8)$ and $f'(-8)$, and then use the linearization. As $f(x) = x^{1/3}$, $f'(x) = (1/3)x^{-2/3}$. And when we substitute, we get $f'(-8) = (1/3)(-8)^{-2/3}$. Now, $(-8)^{-2/3}$ is equal to $(-2)^{-2}$, which is $1/4$. So, $f'(-8) = 1/12$. That is $f'(-8)$. And, what is $f(-8)$? It is $f(-8) = (-8)^{1/3} = -2$. So, you get the linearization of $f(x)$ as $f(a) + f'(a)(x - a)$, which is $f(-8) + f'(-8)(x - (-8))$. It is equal to $(-2) + (1/12)(x + 8)$. It simplifies to $x/12 - 4/3$. So, the linearization is $x/12 - 4/3$. That means near $x = -8$, you can approximate the function $x^{1/3}$ with this linear expression $x/12 - 4/3$.

Let us take the second problem. Here, we have the function as $f(x) = \sec x$. We want to find its linear approximation at $x = -\pi/3$. Again, we proceed the same way. As $f(x) = \sec x$, $f(-\pi/3) = 2$. Now we go for its differentiation. We get $f'(x) = \sec x \tan x$. We then evaluate it at $-\pi/3$ to get $f'(-\pi/3) = \sec(-\pi/3)\tan(-\pi/3) = 2 \times (-\sqrt{3}) = -2\sqrt{3}$. Then the linearization is $f(a)$, which is 2 plus $f'(a)$, which is $-2\sqrt{3}$ times $x - (-\pi/3)$. It simplifies to $2 - 2\sqrt{3}(x + \pi/3)$. It says that in a small neighborhood of $-\pi/3$, if you choose any x, then the difference between $\sec x$ and this $2 - 2\sqrt{3}(x + \pi/3)$ will be very small. That is the notion of this linearization.

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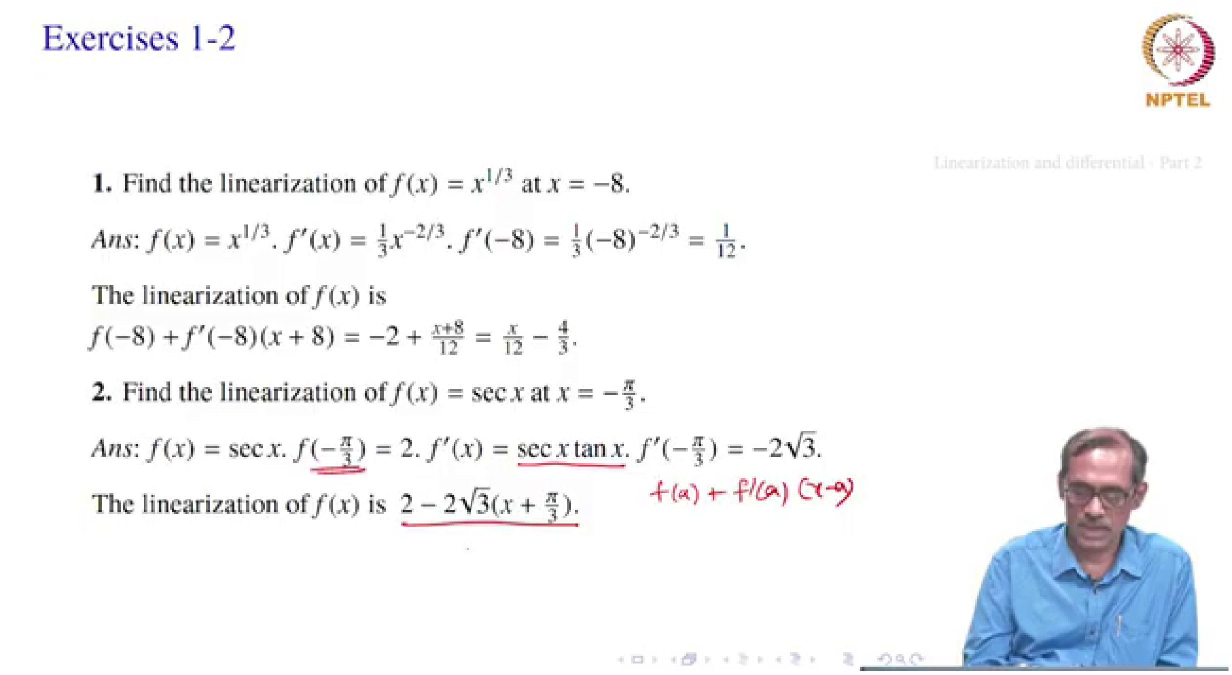

Let us take another problem. Here, we are finding the differential. Given that $2y^{3/2} + xy - x = 0$ find the differential dy. That means the function is implicitly defined. The implicit definition of $y = f(x)$ is given by this equation, and we want to find the differential. As you know, the differential is equal to $f'(a)$ times the increment in x. And what is this a here? Given any a, you can find the differential. Of course when a changes, the value of the differential may change. So, we are suppressing this a in our notation.

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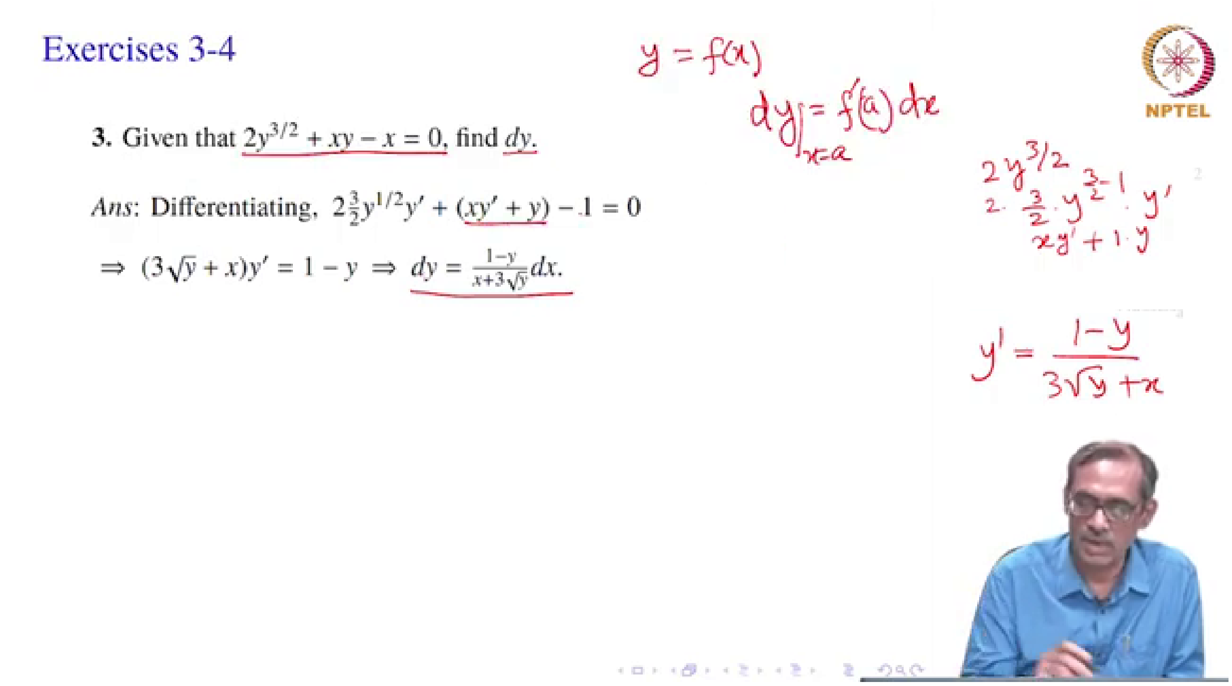

If nothing is asked, that means we need to compute the differential at an arbitrary point a. Since a is arbitrary, you take x itself as an arbitrary point. The question is, what is the differential at the point x? Fine.

Okey, let us proceed. First thing is we have to find the derivative. What is y'? You differentiate the expression itself using implicit differentiation. Differentiating $y^{3/2}$ gives $(3/2)y^{1/2}$ times y'; differentiating xy gives $xy' + (dx/dx)y = xy' + y$. So, you get $2(3/2)y^{1/2}y' + (xy' + y) - 1 = 0$. Solving this, we should get our y'. Fine. That gives $(3\sqrt{y} + x)y' = 1 - y$, or $y' = (1 - y)/(x + 3\sqrt{y})$. And then at any arbitrary point x, the differential of y will be equal to this times dx; that is what it is. So, $dy = [(1 - y)/(x + 3\sqrt{y})]\, dx$.

What does this differential do? It is really an estimate of Δy, the increment in y at any point x by a linear function, called linearization. But it is not exactly Δy; dy is nearby that Δy; that is what this differential means.

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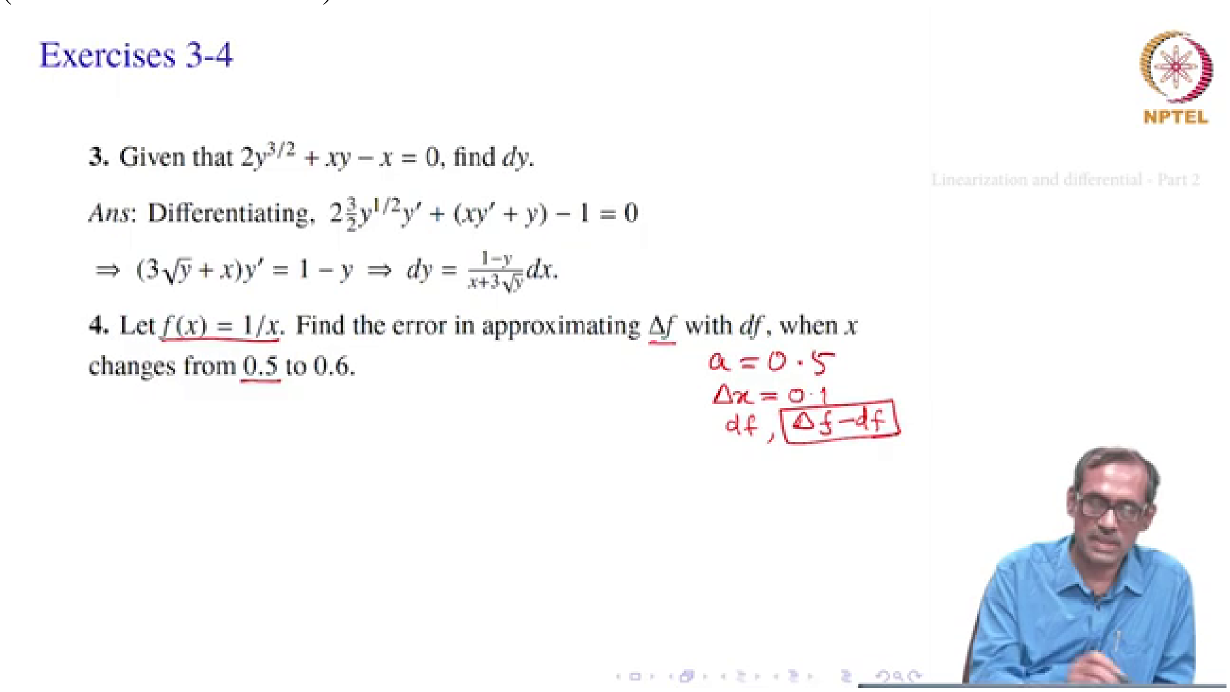

Let us go to the next problem. We have the function $f(x) = 1/x$. We want to approximate the increment Δf at any point x with the differential df. But we are telling that it is not taken abstractly at any point, but at some particular point, say it is at $x = a = 0.5$ That means you have the point $x = a$ for this function, where $a = 0.5$. We say that this point changes to 0.6, which means your Δx is equal to $0.6 - 0.5 = 0.1$. With this Δx, we need to find df, find Δf and then take their difference, which is, the error in approximation. We want to approximate or find out this error. You can really find out exactly the error at $x = 0.5$. Fine.

We have $f(x) = 1/x$. We take its derivative, which is $-1/x^2$. With $dx = 0.1$, we have $\Delta f = f(0.6) - f(0.5)$. When you substitute you get $1/0.6 - 1/0.5 = -1/3$. And what is df? It is equal to $f'(a)$ times the increment in x, which is $f'(0.5) \times 0.1$. You substitute to get $-1/(0.5)^2 \times 0.1 = -2/5$.

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Exercises 3-4

Linearization and differential - Part 2

3. Given that $2y^{3/2} + xy - x = 0$, find dy.

Ans: Differentiating, $2\frac{3}{2}y^{1/2}y' + (xy' + y) - 1 = 0$

$\Rightarrow (3\sqrt{y} + x)y' = 1 - y \Rightarrow dy = \frac{1-y}{x+3\sqrt{y}}dx.$

4. Let $f(x) = 1/x$. Find the error in approximating Δf with df, when x changes from 0.5 to 0.6.

Ans: $f(x) = 1/x$. $f'(x) = -1/x^2$. $dx = 0.6 - 0.5 = \underline{0.1}$. Then

$\Delta f = f(0.6) - f(0.5) = 10/6 - 10/5 = -1/3.$

$df = f'(0.5)(0.1) = -[1/(1/4)](0.1) = -2/5.$

So, Error $= \underline{\Delta f - df} = -1/3 + 2/5 = \underline{1/15}.$

Then what is the error? The error is the difference $\Delta f - df$, which gives you $-1/3 - (-2/5) = 1/15$. Since it is the reciprocal, the reciprocal is linearized and is not a really good approximation. This is what it shows. This difference should go to 0 as our difference between these two points goes to 0, The difference in x is 0.1, but you get this difference as $1/15$. It is almost of the same order as δx; it is not varying much. However, that does not contradict the fact that $\epsilon \to 0$ and $\Delta x \to 0$. That will still hold. Also 0.1 is not really very close to 0; so, even though we get this error, we see that it does not go beyond this 0.1. That is what it shows.

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Exercises 5-6

Linearization and differential - Part 2

5. Show that at $x = a$, the linearization of $f + g$ is the sum of linearizations of f and of g.

Ans: $L(f) = f(a) + f'(a)(x - a)$. $L(g) = g(a) + g'(a)(x - a)$.

$L(f + g) = \underline{(f + g)(a)} + (f + g)'(a)(x - a) =$

$\underline{f(a) + g(a)} + [f'(a) + g'(a)](x - a) = \underline{L(f)} + \underline{L(g)}.$

Let us take the next problem. We want to show something here. We have two functions f and g. They are of course defined in a neighborhood of a point a, and we consider their sum $f+g$. Now $f+g$ is a new function. We have the linearization of f at a; we have the linearization of g at a; and also we have the linearization of $f+g$. What is the relation between those three linearizations? That is being asked. Sctually, we want to show that this sum of the linearizations is equal to the linearization of the sum. That is to be shown, and that should be quite easy.

What do we do? Let us first find out the linearization of f. That is equal to $L(f) = f(a) + f'(a)(x-a)$, where x is in a neighborhood of a. And what is the linearization of g? Instead of x in $L(x)$, we are writing here $L(f)$ and $L(g)$ for ease notation. So, $L(g) == g(a) + g(a)(x-a)$. And what is $L(f+g)$? $L(f+g) = (f+g)(a) + (f+g)'(a)(x-a)$. But $(f+g)(a) = f(a) + g(a)$ and similarly, $(f+g)'(a) = f'(a)+g'(a)$. When you come to expand $L(f+g)$, you get $f(a)+f'(a)(x-a)$ plus $g(a) + g'(a)(x-a)$. And, that is exactly $L(f) + L(g)$. That is what we have shown. That is, the linearization of the sum is equal to the sum of the linearizations.

Let us go to next problem. It asks to get a differential formula that estimates the change in the lateral surface area of a right circular cone when the radius changes from a to $a + dr$, while the height remains constant. So, you have a circular cone; there is some radius of the base, say it is a; the height remains same, and you are changing this right circular cone to another with the difference in the base radius being dr; so this distance is dr.

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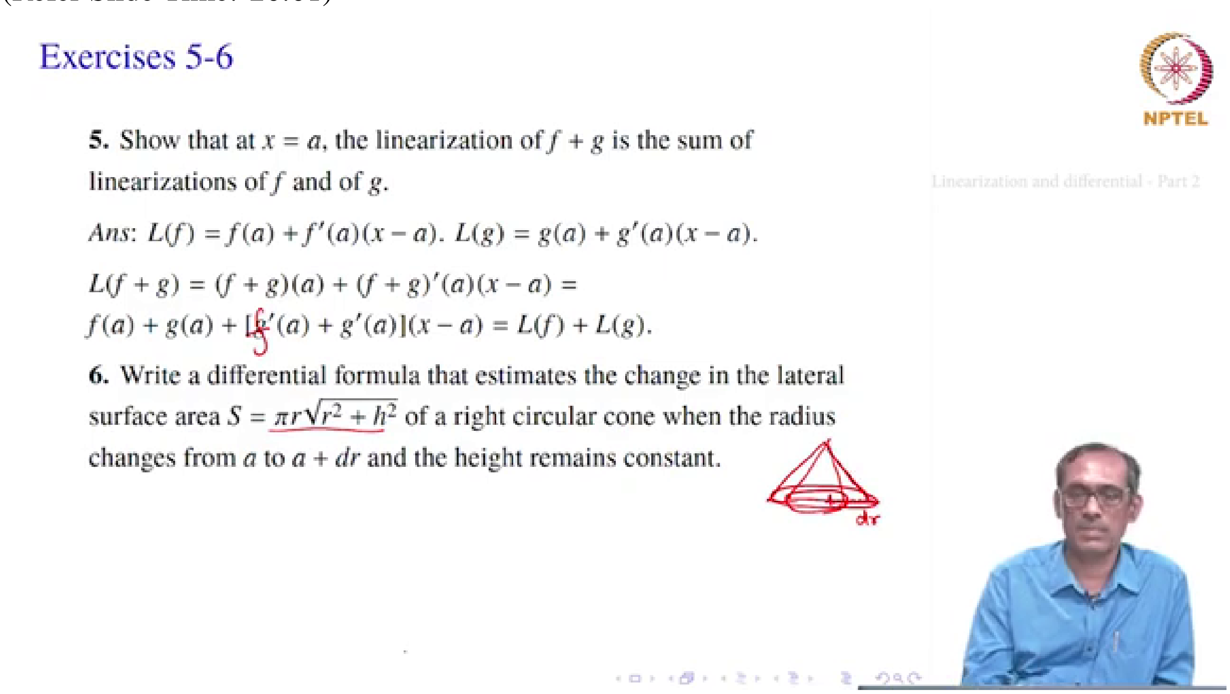

It asks to find the change in the lateral surface area. Of course you know the formula for the surface area and the lateral surface area. The lateral surface area is π times the base radius times the lateral height: $\pi r\sqrt{r^2+h^2}$. We want to find the change in this function.

Here, we take $S(r)$ as a function, it is $S(r) = \pi r\sqrt{r^2+h^2}$. The height is remaining constant, so h is constant. We take dS/dr, where $S = \pi r\sqrt{r^2+h^2}$. It is a product; and its derivative is the derivative of r, which is 1, times $\sqrt{r^2+h^2}$ plus r times the derivative of $\sqrt{r^2+h^2}$. That is giving you this expression: $[\pi(r^2+h^2)+\pi r^2]/\sqrt{r^2+h^2}$.

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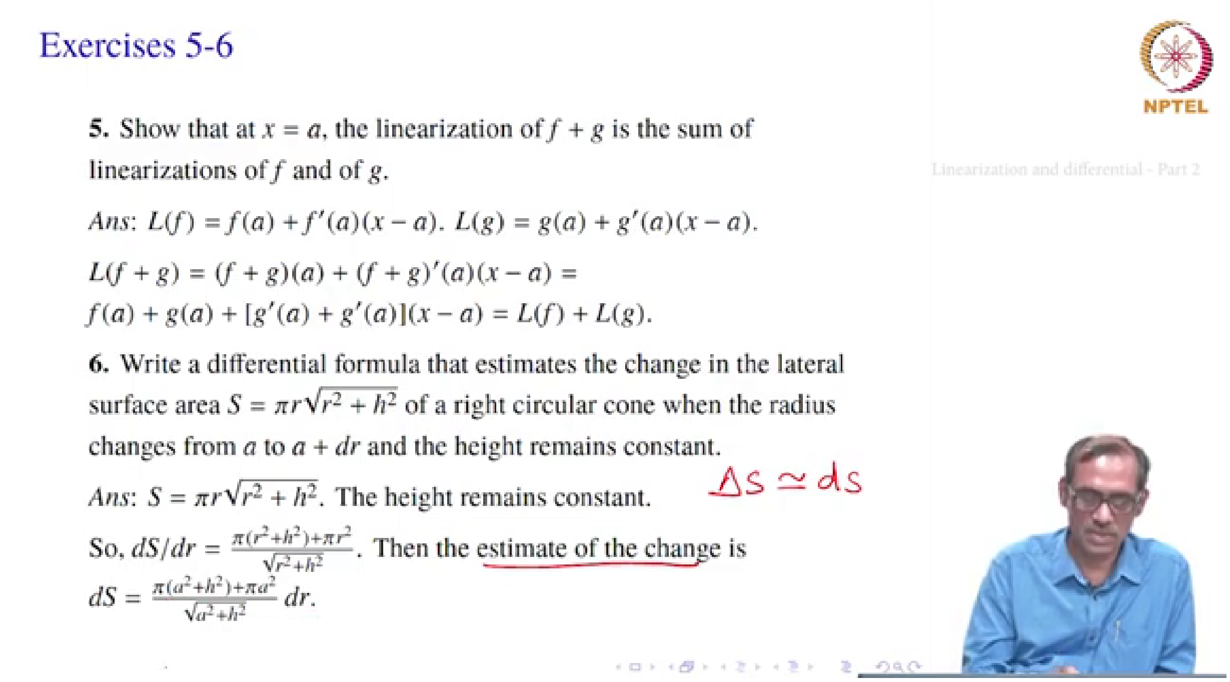

We want to estimate the change, not find the exact change, right? Of course, you can find the exact change. Our estimation is through the differential, that is, ΔS is approximately equal to dS. You find dS equal to f' that is, $S'(r)$ times dr. That is really an estimation of the change. Is that fine? So, let us stop here.