Basic Calculus - 1
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Lecture 23 - Part 1
Linearization and Differential - Part 1

Well, this is Basic Calculus 1, Lecture 23. Today, we will be talking about approximating a function by a straight line, so to say. The idea is very simple. Suppose you have a function f, you consider its graph. And then at a point say a (on the x-axis), you have a tangent at $(a, f(a))$. For δ small, we have $f(a+\delta)$; then you think of the line $x = a + \delta$. It is the vertical line and it touches the tangent somewhere. The value of $f(a+\delta)$ is approximated by where this vertical line touches the tangent or crosses the tangent.

They are two different things; one number is on the curve, another is on the tangent. Somehow we will be thinking that the point we get there, or the height we get from the tangent is approximately equal to the height we get from the curve. That is the notion of linearization. From the tangent formula you can get it directly; but we have a better way of doing it. We will use the mean value theorem and see how it is being done.

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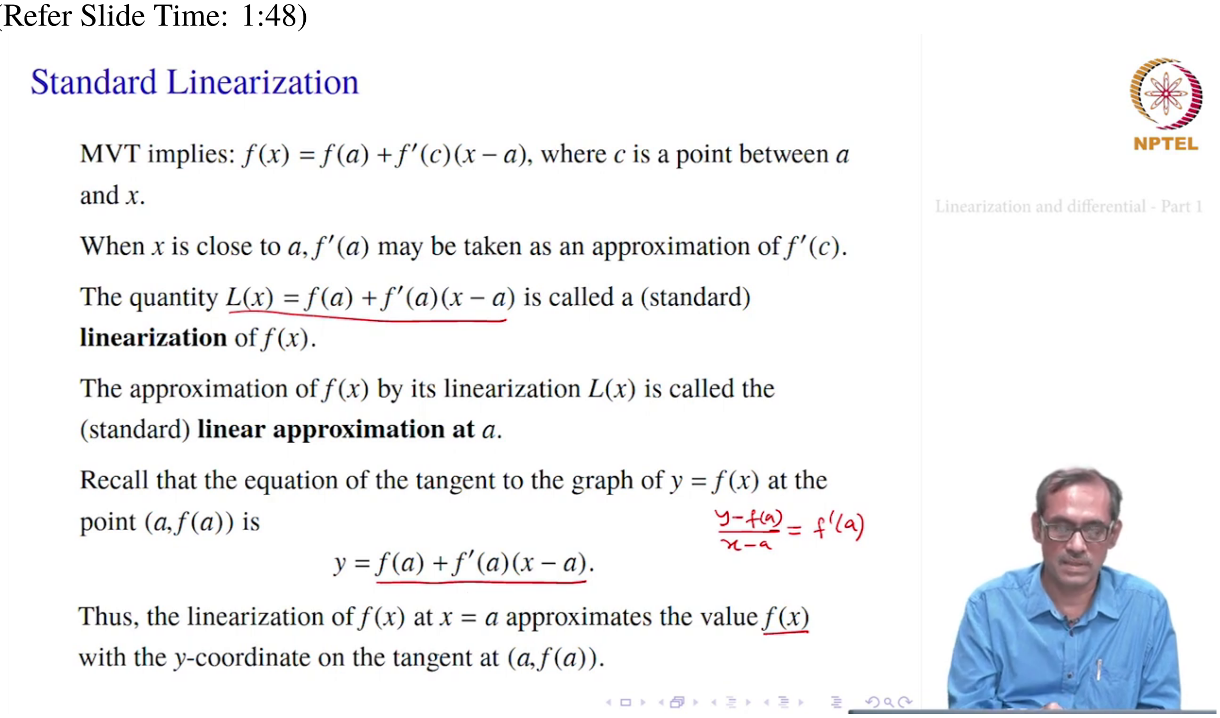

Remember the Mean Value Theorem? Its conclusion says that $f(x) = f(a) + f'(c)(x-a)$, where c is a point between a and x. Of course, this holds under certain conditions, like, $f'(c)$ must exist, at least f is differentiable inside an open interval, and f is continuous over that closed interval, and x is a point within that open interval. So, this c is between a and x; that is how $f(x)$ would look like.

Now, if x is very near a, you may think of approximating the function $f(x)$ by $f(a)+f'(a)(x-a)$, where we take $f'(a)$ instead of $f'(c)$. That is basically the notion of linearization. We say that when x is close to a, we may think of $f'(a)$ is close to $f'(c)$; it is an approximation of $f'(c)$. In that case, we define the new function $L(x) = f(a) + f'(a)(x-a)$ and call this $L(x)$ as the linearization of $f(x)$ at a or near a. Of course, it will depend on a because L itself is depending on a.

The approximation of $f(x)$ by this linearization $L(x)$ is called the standard linearization or standard linear approximation of $f(x)$ at a. Recall the equation of the tangent to $y = f(x)$ at $x = a$. It is $y - f(a) = f'(a)(x-a)$ because it is the straight line passing through $(a, f(a))$ and its slope is $f'(a)$. It gives $y = f(a) + f'(a)(x-a)$. The expression $f(a) + f'(a)(x-a)$ is taken as our linearization, the function $L(x)$.

In a certain sense, since x is variable lying in a neighborhood of a, we will say that f defined over that neighborhood is approximated by this $L(x)$. That is why this is called a linearization. The linearization of $f(x)$ at $x = a$ approximates the value $f(x)$, where the y-coordinate on the tangent at $(a, f(a))$ is that $f(x)$. So, it is really the y-coordinate which comes from the tangent. If you take the equation of the tangent and then compute where it intersects our vertical line, it will give rise to the same expression. That is what it says. But most importantly, $L(x)$ is an approximation to $f(x)$ not only at one point but over a neighborhood of that point. This is the notion of linearization of $f(x)$ at $x = a$.

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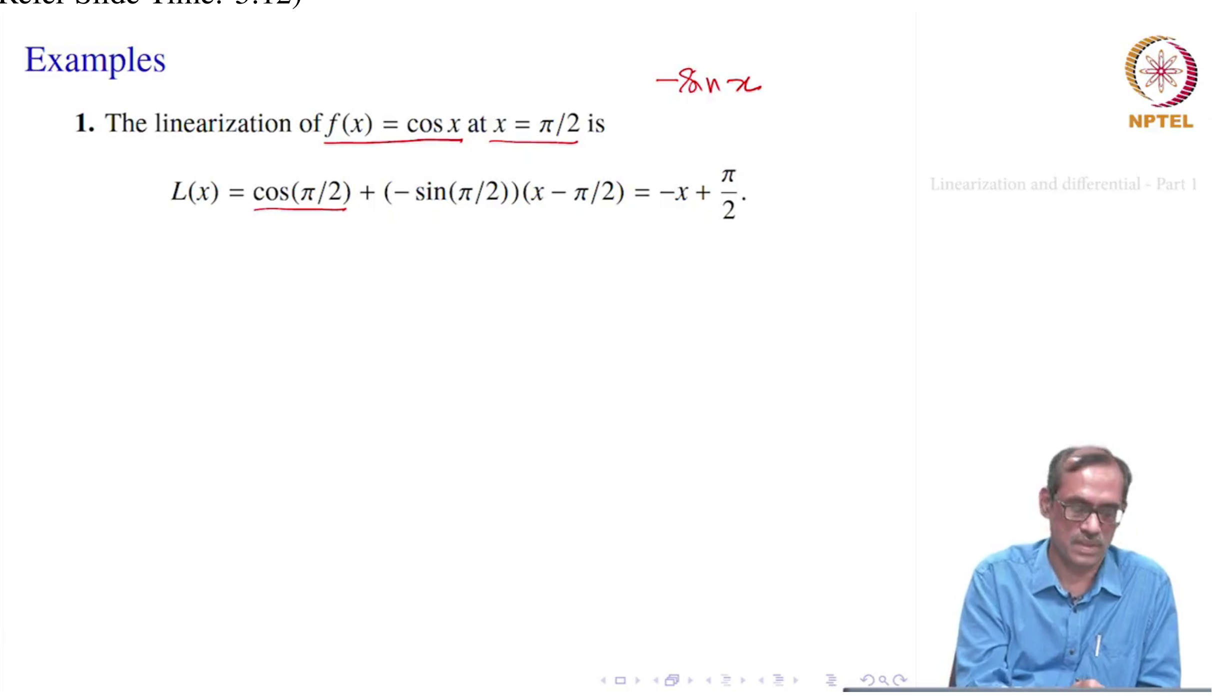

Let us take an example. We take the function $f(x) = \cos x$. Of course, it is defined everywhere. We want to linearize it or take a linearization of this, or a linear approximation $L(x)$ of this function at $x = \pi/2$. Now, $L(x) = f(a)$, which is $\cos(\pi/2) = 0$ plus $f'(\pi/2)$, which is $-\sin(\pi/2) = -1$ times $(x - \pi/2)$. That is, $L(x) = 0 + (-1)(x - \pi/2)$. This simplifies to $-x + \pi/2$. So, this is the linearization of $\cos x$ at $x = \pi/2$. If you take a very small neighborhood around $\pi/2$, then the values

of $\cos x$ will be approximated by the values of $L(x)$ in that neighborhood; that is the notion.
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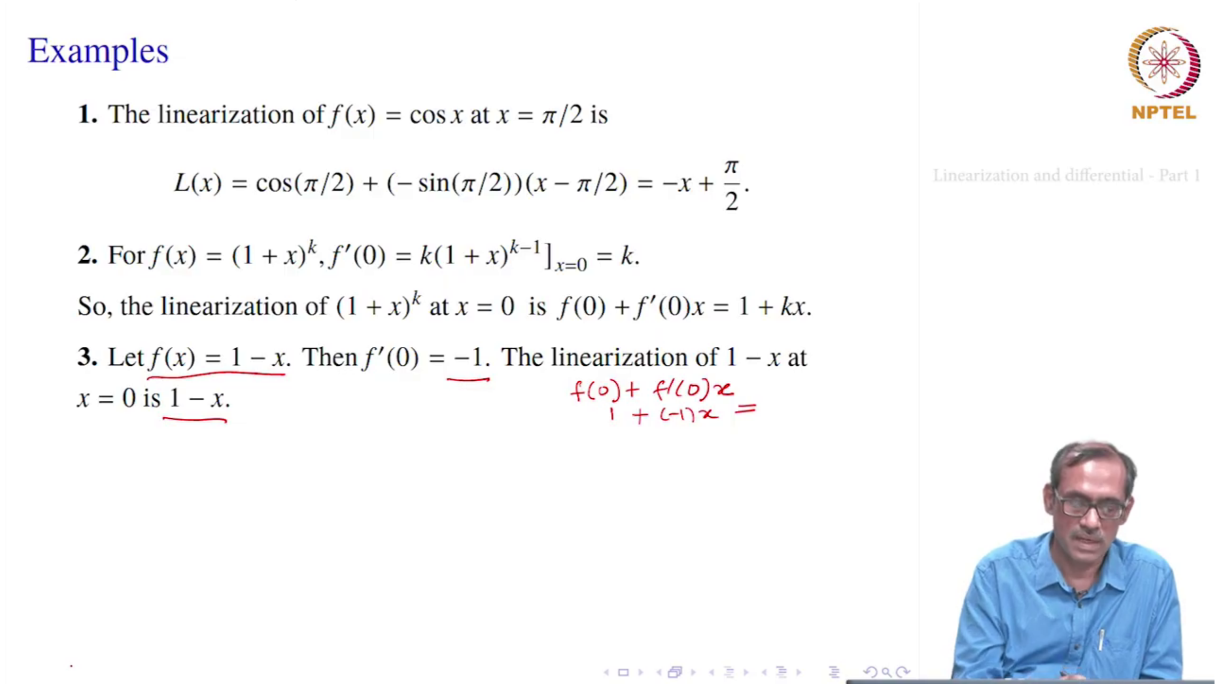

Let us take another example, say, $f(x) = (1+x)^k$, where k is some natural number. (You can also take k to be a rational number.) What is $f'(0)$? Here, $f'(x) = k(1+x)^{k-1}$. You substitute $x = 0$ to get $f'(0) = k$. Since $f'(0) = k$, the linearization at $x = 0$ is $f(0) + f'(0)(x - 0)$. As $f(0) = 1$ it simplifies to $1 + kx$. That is, $1 + kx$ is the linearization of $(1+x)^k$ at $x = 0$.

Of course, if you use the binomial theorem, you would get first two terms as $1 + kx$. And, that is taken as a linearization of this function.
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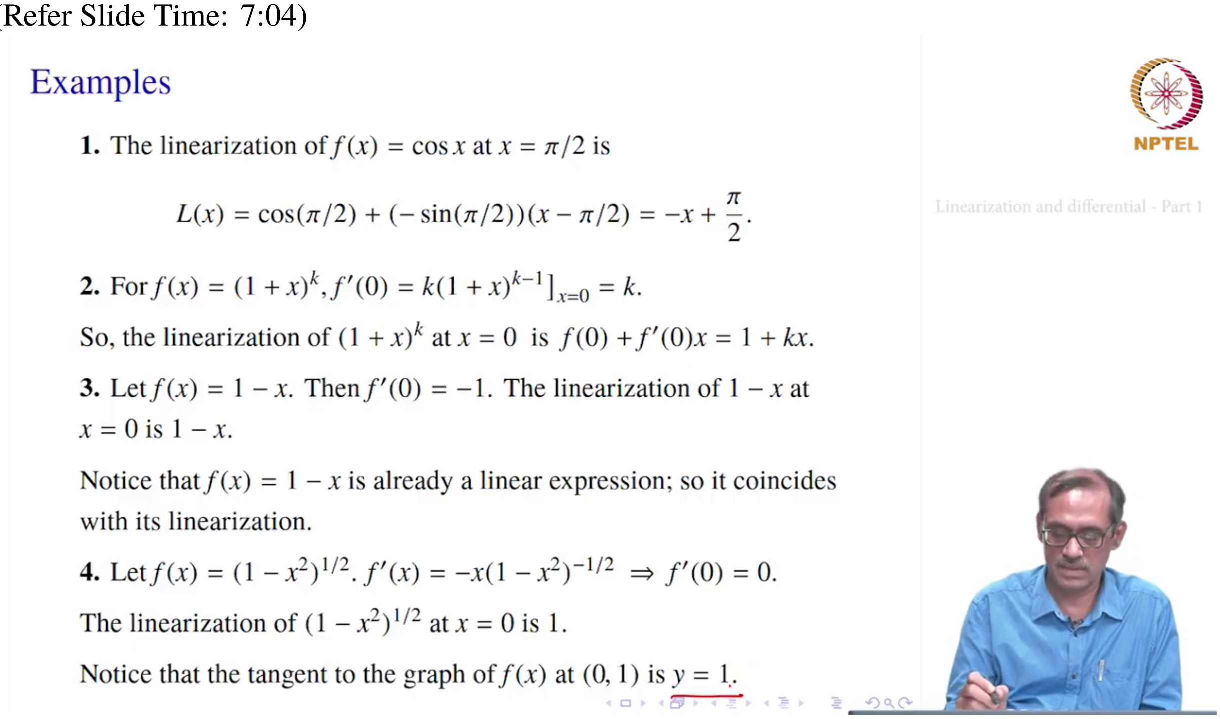

Let us take the third example. Here, $f(x)$ is given as $1-x$. Then, $f'(x) = -1$; so, $f'(0) = -1$. The linearization of $1-x$ at $x = 0$ is $f(0) + f'(0)(x-0)$; where $f(0) = 1-0 = 1$, $f'(0) = -1$. It turns out to be $1-x$. So, that is the linearization of $1-x$. That is, the linearization of $1-x$ is $1-x$ itself; Why is this result? Because, the unction $1-x$ is already linear. So, its linear approximation should be that itself. As $1-x$ is a liear expression, its linearization coincides with itself, as it should.

Let us take another example. Here, $f(x) = (1-x^2)^{1/2}$. We want to linearize this function at $x = 0$. Let us find the derivative. Its derivative is $(1/2)(1-x^2)^{-1/2}$ times th derivative of $1-x^2$, which is $-2x$. These 2s cancel and you get $-x/(1-x^2)^{1/2}$. We substitute $x = 0$ to get $f'(0) = 0$. So, its linearization is $L(x)$ equal to $f(0)$, which is 1 plus $f'(0), whichis0$. So, $L(x) = 1$. That means you get the linear function or the linear expression which approximates $f(x)$ as the constant 1. It does not have x in it; sometimes it can so happen. And, that happens here because $f'(0) = 0$; that is, the tangent is horizontal there. So, you get the constant as the linearization, which is 1, because the tangent of the graph of $f(x)$ at $x = 0$ is $y = 1$. So you do not get any difference between that linearization and the tangent line.

Using this notion of linearization we will introduce another notion which is called the differential. The differential is a very simple notion; the motivating factor here is that you write $f'(x)$ in another notation, namely, as df/dx. Why this notation of division is used? Instead of answering it we go the other way around. We give some meaning to this notation df so that $f'(x)$ will be equal to df/dx. And what is the best thing to do? We just write $df = f'(x) \times dx$. so that you can see $f'(x)$ as a ratio.

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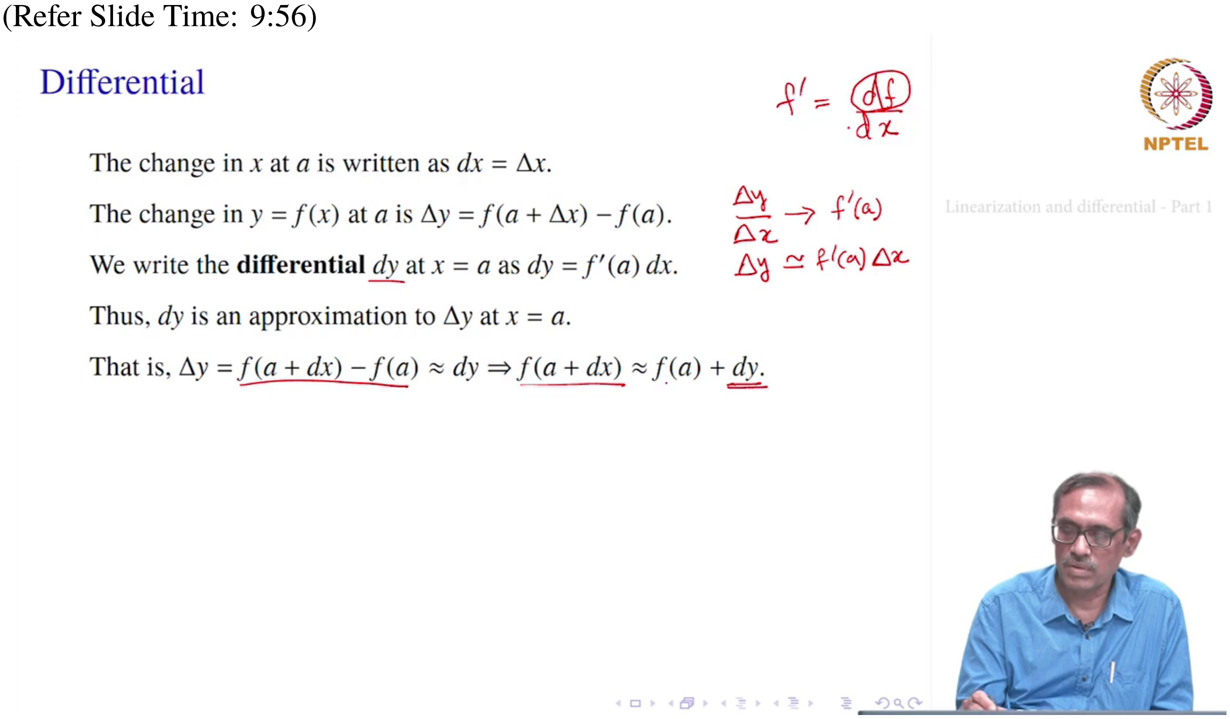

We will connect it to the change in $y = f(x)$ at $x = a$. Suppose we write the change in x as Δx in the neighborhood of a. That means, x can vary anywhere in that neighborhood of a. Let us call the difference of such a point from a as Δx. That is, x has changed from a to $a+\Delta x$. Instead of a we now have a nearby point which is $a+\Delta x$. When x changes from a to $a+\Delta x$, y also changes. How does it change? We do not know. Let us write that change as Δy. Then, Δy is $f(a+\Delta x) - f(a)$. That is the change in y corresponding this change in x, at a. we have now Δx, the change in x and Δy, the change in y. We take the ratio. Why?

When you take the ratio, it is $[f(a+\Delta x) - f(a)]/\Delta x$. Recall that in the limit as $\Delta x \to 0$, this ratio gives you the derivative at a, that is $f'(a)$. So, looking at this ratio, we introduce the notion of a differential. We write the differential of y as dy at $x = a$, and define it $f'(a)dx$. We take this dx equal to Δx. So, $dy = f'(a)\Delta x$; that is the differential.

There are some advantages in introducing this notion. First thing is, you look at the change Δy. This dy which is $f'(a)dx$ or $f'(a)\Delta x$ is really an approximation to Δy. Why is it an approximation? Because $\Delta y = f(a+dx) - f(a)$; that is our definition of increment in y or change in y; and that divided by Δx gives in the limit our $f'(a)$. So, we see that $\Delta y = f(a+\Delta x) - f(a)$ is approximately equal to dy. Or, we say that $f(a+\Delta x)$ is approximately equal to $f(a) + dy$. That is fairly straightforward. All that we say here is that we have the differential dy and we have the increment Δy; if you take $\Delta y/\Delta x$ and take its limit as $\Delta x \to 0$, you would get $f'(a)$. Thus, we say that Δy is approximately equal to $f'(a)\Delta x$ and $\Delta y = f(a+dx) - f(a)$. Then, $f(a+dx) = f(a) + \Delta y$, which is approximately equal to $f(a) + dy$.

So, let us look at how we are going to use this. All that we remember here is that the increment Δy is approximately equal to the differential dy. And the differential dy is equal to $f'(a)\,dx$. Let us see how this approximation helps.

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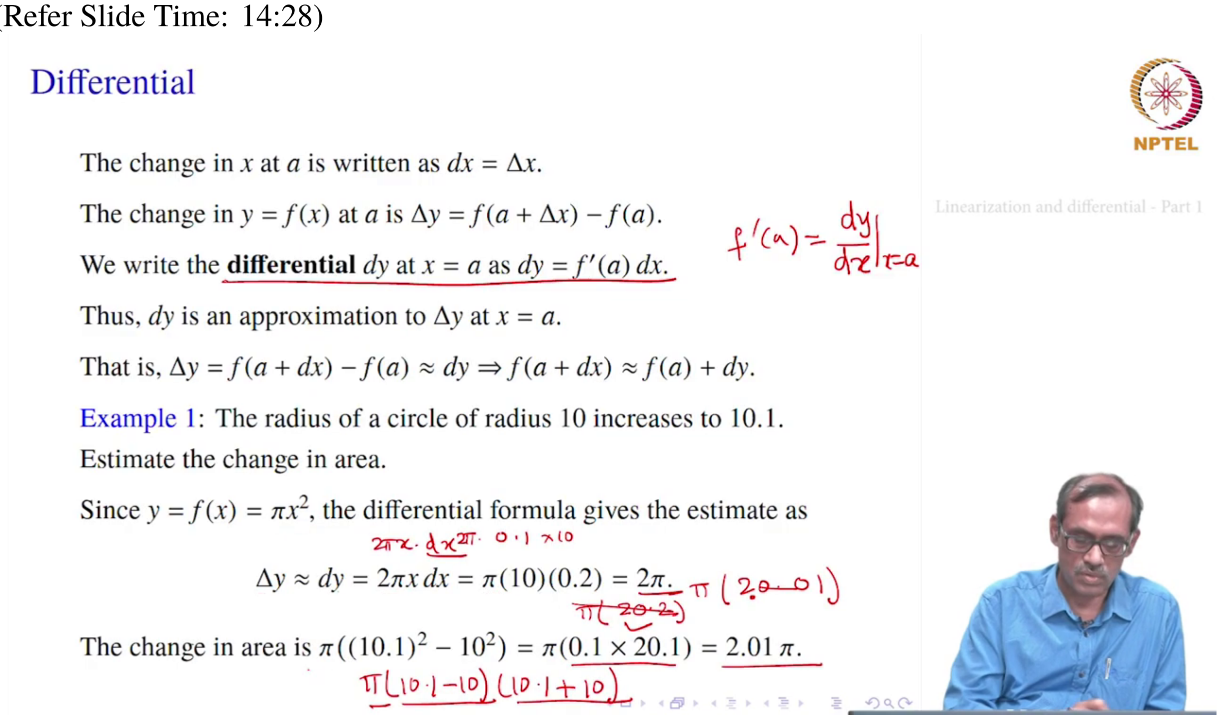

Suppose we have a circle of radius 10. We increase the radius to 10.1. We have a circle of radius 10, and we increase that to another circle so that its radius becomes 10.1. We get a bigger circle.

Then, the area of the circle is changed. We want to see that how the area changes or approximately how much it is changing. That is the question here.

The radius of the circle is 10; it increases to 10.1. How do we estimate the change in the area? Well, if radius is x, then the area of the circle is πx^2. So, we define the function $y = f(x) = \pi x^2$.

Now, the differential formula gives the estimate as the change in y. Thus, the change in area is approximately equal to the differential dy. What is dy? It is $f'(a)\, dx$. Now, $f'(x) = \pi \times (2x)$; so dy at that point is f' at that point times dx at that point. Here, the point is 10; the change in x is $dx = 10.1 - 10 = 0.1$; f' at 10 is $f'(10) = \pi \times 2 \times 10 = 20\pi$. Then, $dy = f'(a)\, dx = 20\pi \times 0.1 = 2\pi$.

But what is exactly the change in y? It is Δy, we can compute directly. We have changed from $x = 10$ to $x = 10.1$. Then, Δy computed directly would give us the area of the circle with radius 10.1 minus the area of the circle with radius 10. So, the changing area is $\pi \times (10.1)^2 - \pi(10^2)$. You can use $a^2 - b^2$ formula to get $\Delta y = \pi(10.1 + 10)(10.1 - 10) = \pi(20.1)(0.1) = 2.01\pi$.

Do you see how much error is committed in this approximation? It is just 0.01π. It gives a feeling that "yes, the increment Δy can be close to the differential dy". That is it.

So, we have introduced the notion of differential, which will be really helpful in some other context also. The main thing here is you can look at $f'(a)$ as the differential divided by the increment evaluated at $x = a$. The notation dy/dx is really given a meaning now. It is something like division provided you know the differential. Of course, this is so because we have defined it in such a manner.