Basic Calculus - 1
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Lecture 22 - Part 2
Concavity - Part 2

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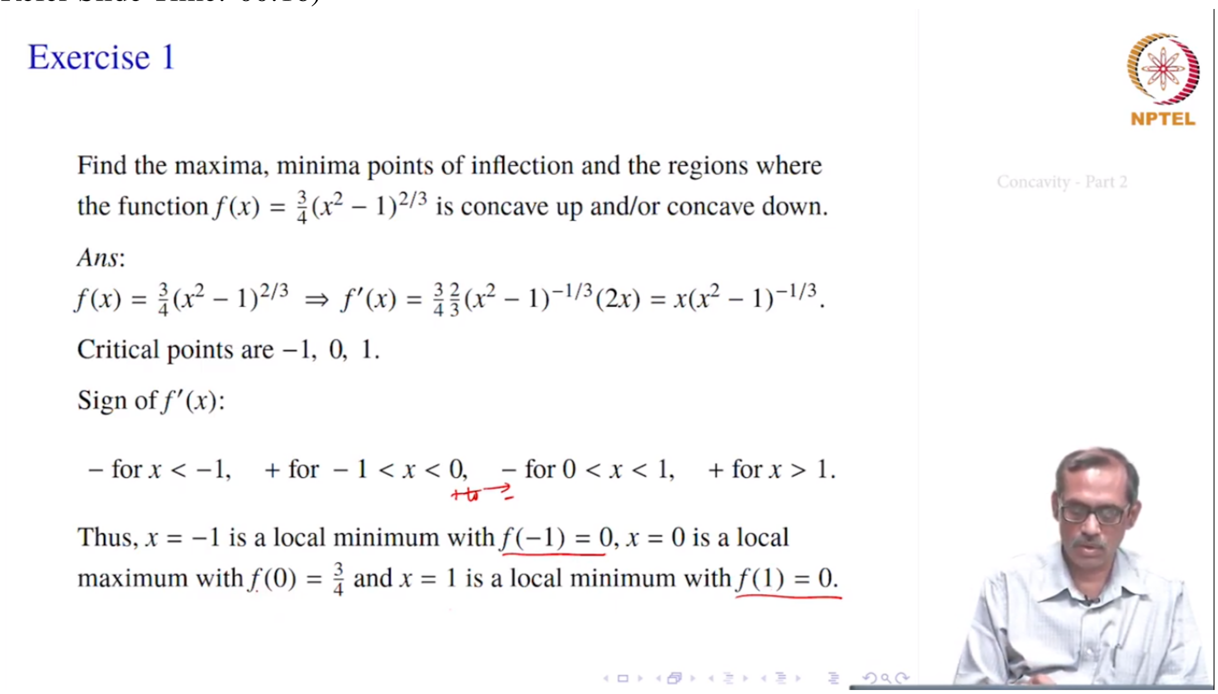

Let us take one more problem. Here, we have to find the maxima, minima, points of inflections, and the regions where the function is concave up or concave down. The function is given as $f(x) = (3/4)(x^2 - 1)^{2/3}$. Again, we have to do some similar thing; we look at the derivatives, and so on. First of all let us take the derivative. The function is differentiable with $f'(x) = (3/4)(2/3)(x^2 - 1)^{2/3-1}(2x)$. It simplifies to $x(x^2 - 1)^{-1/3}$.

There is a problem for the derivative when $x^2 - 1 = 0$, that is, at $x = \pm 1$. The function is not differentiable at these points: 1 and -1. You can check that. For now, leaving those points, we have $f'(x) = x(x^2 - 1)^{-1/3}$. The critical points are those where $f'(x)$ is 0 or where $f(x)$ is not differentiable. So, these are 0, 1 and -1. At 0, the function is differentiable and the derivative is 0; the other two points are those, where the function is not differentiable.

Now we look for the sign of $f'(x)$ for having an idea about the maxima and minima. When x is less than -1, x is negative and $(x^2 - 1)$ is positive, so $f'(x)$ is negative. $f'(x)$ is positive from -1 to 0; it is negative from 0 to 1; and it is positive for $x > 1$. See that these are correct.

So, $f'(x)$ is changing sign at $x = -1$; it is changing sign at $x = 0$; and also it is changing sign at $x = 1$. At $x = -1$, it changes sign from $-$ to $+$, so there is a local minimum at $x = -1$; and its minimum value is $f(-1) = 0$. The local minimum point is $x = -1$ and the minimum value is 0.

At $x = 0$, $f'(x)$ is changing sign from + to −; so, $f(x)$ has a local maximum at $x = 0$; the maximum value is $f(0) = 3/4$.

Similarly, at $x = 1$, $f'(x)$ changes sign from − to + so that it a point of local maximum with the maximum value as $f(1) = 0$.

So, we have now local minimum at −1 and 1, and a local maximum at 0.

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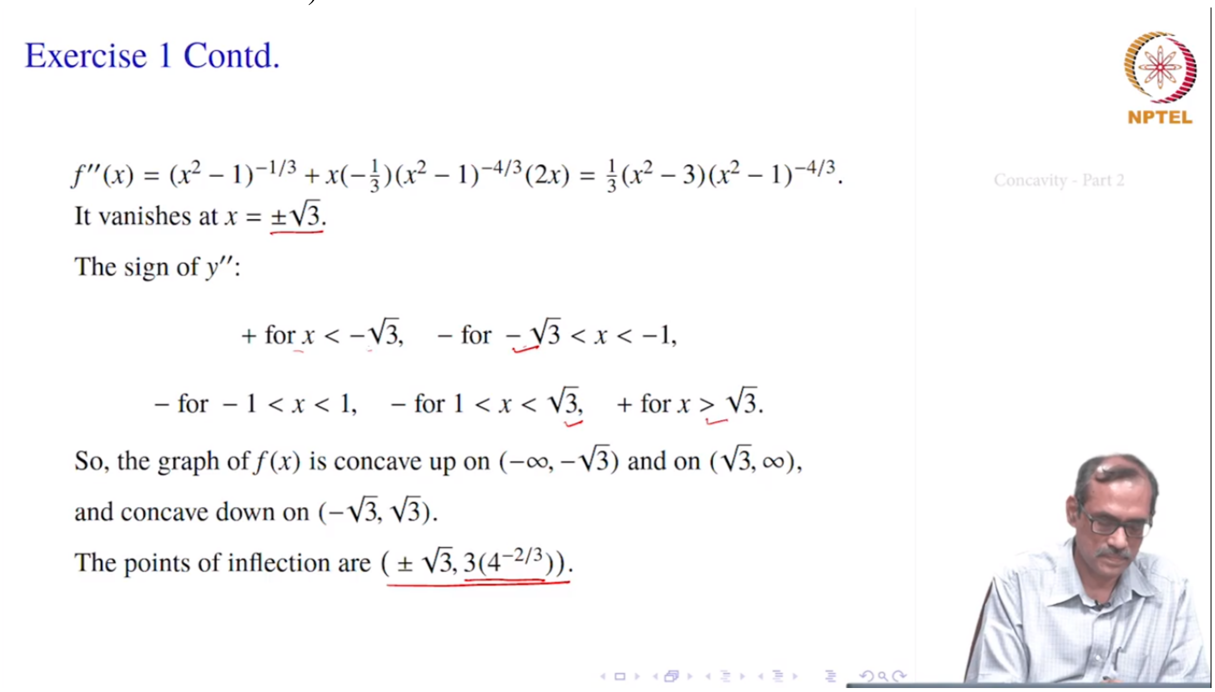

Let us go for the second derivative. If you take the second derivative (you have to work it out), you would get it this form: $f''(x) = (1/3)(x^2-3)(x^2-1)^{-4/3}$. This is so for all x leaving the points ± 1, where f is not even differentiable. So, f'' becomes zero for $x = \pm\sqrt{3}$. These are possible points of inflection. We do not say these are points of inflection because f'' may or may not change sign at those points. We have to discuss this. So, what about the sign of $f''(x)$? We have now two points $-\sqrt{3}$ and $\sqrt{3}$.

If we take $x < -\sqrt{3}$, we will find that $x^2 - 3$ as positive, $x^2 - 1$ as positive; so $f''(x)$ is positive. Similarly, $f''(x)$ is negative when x lies between $-\sqrt{3}$ and -1. Then we go for checking the sign of $f''(x)$ for x between $-\sqrt{3}$ to -1. It is negative here. From -1 to 1, f'' is again negative, and from 1 to $\sqrt{3}$, f'' is also negative. Further, when $x > \sqrt{3}$, $f''(x)$ is positive.

We could have taken minus throughout $(-1, \sqrt{3})$. But $f''(x)$ is not defined at $x = 1$. So, we had to break this into two sub-intervals $(-1, 1)$ and $(1, \sqrt{3})$. We cannot include $x = 1$ here because f'' is not defined at 1.

Now, f'' is negative from -1 to 1, negative for 1 to $\sqrt{3}$, and positive for $x > \sqrt{3}$. So, where do you find the point of inflection? It is changing sign at $-\sqrt{3}$, and then out of these, it is changing sign at $\sqrt{3}$ also; so, both of them are points of inflection.

The graph of $f(x)$ is concave up on $(-\infty, -\sqrt{3})$ because f'' is positive in this interval and also it is positive on $(\sqrt{3}, \infty)$. So, it is cocave up in both these intervals. What about the other intervals?

Where is it changing sign? It is negative from $-\sqrt{3}$ to -1, -1 to 1 and 1 to $\sqrt{3}$; that is, negative on $(-\sqrt{3}, \sqrt{3})$; except at $x = \pm 1$, where it is not defined.

Therefore, it is concave down on $(-\sqrt{3}, -1)$; concave down from -1 to 1; and concave down from 1 to $\sqrt{3}$. (Writing this way is misleading, it should not be.) On these 3 intervals it is concave down, and at those points, -1 and 1, f'' is not defined. So, that one point really does not matter since you can see that the curve is continuous. Leaving those two points, you would say that it is concave down on $(-\sqrt{3}, \sqrt{3})$.

Then the points of inflection are really both $-\sqrt{3}$ and $\sqrt{3}$ because there is a change of sign in f'' at these points. On the left of $-\sqrt{3}$, f'' is positive, and on the right of $\sqrt{3}$, f'' is negative. So, $-\sqrt{3}$ is a point of inflection. On the left side of $\sqrt{3}$, f'' negative and on the right side of $\sqrt{3}$, f'' is positive. So, $\sqrt{3}$ is also a point of inflection. So, these are the two points of inflection of the curve.

This is fairly straightforward, but we have to discuss and see what happens to f'' such as breaking into sub-intervals by looking at where f'' becomes 0, and so on.

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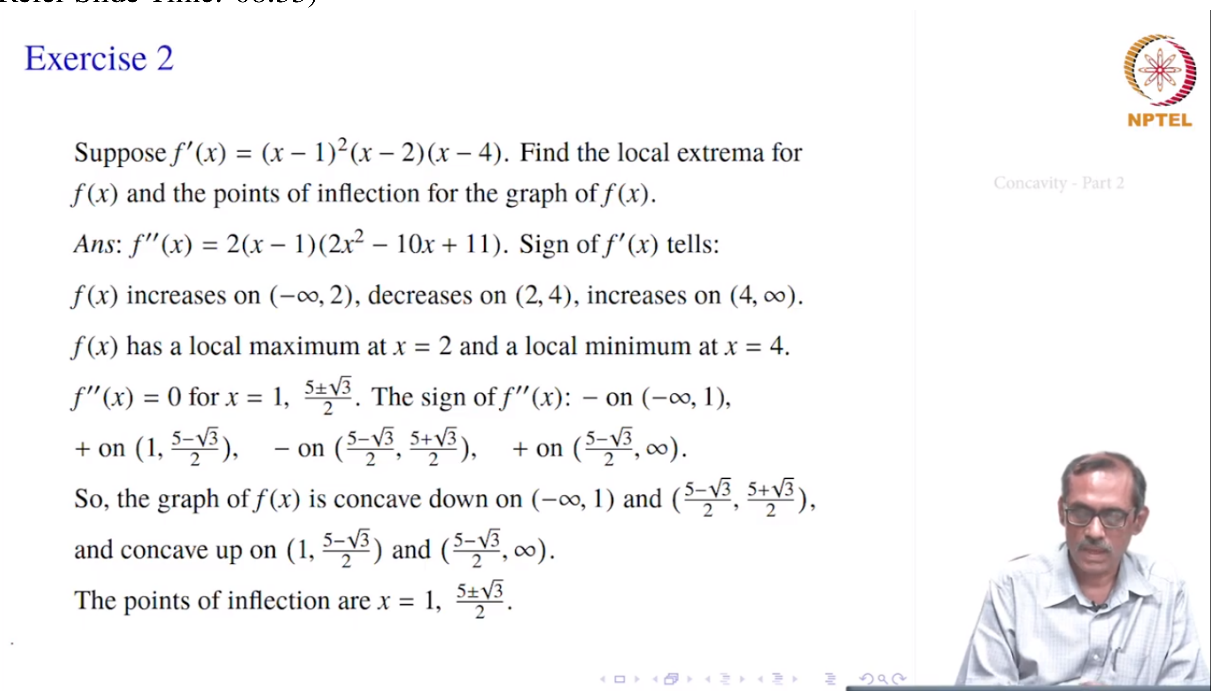

Let us take another problem. Here, we are not given anything about f; but we are given that $f'(x) = (x-1)^2(x-2)(x-4)$. This is the information about f'. We want to find the local extrema for $f(x)$, and the points of inflection for the graph of $f(x)$, though $f(x)$ is not given. However, $f'(x)$ should give rise to $f''(x)$; and we should get the point of inflection from that. Further, this will be true for any function whose derivative is equal to this. That is what the problem assumes. Let us see whether that is going to be true.

Now we go for the derivative of this. If you differentiate, you would get $f''(x) = 2(x-1)(x-2)(x-4) + (x-1)^2(x-4) + (x-1)^2(x-2)$, which simplifies to $2(x-1)(2x^2 - 10x + 11)$. You have to really work it out. Now it is a product; you would think of this as first and expand this; may be that will be easier to apply instead of going for the three products. This is really $x^2 - 6x + 8$. If you differentiate, it gives 2 into $x - 1$ into $x^2 - 6x + 8$ plus $(x - 1)^2$ into $(2x - 6)$. If you simplify, you would get 2 into $x - 1$ into $2x^2 - 10x + 11$.

What about the sign of $f'(x)$? You see that $f'(x) = 0$ at $x = 1$, $x = 2$ and at $x = 4$. So, 1, 2 and 4 are the critical points of this function $f(x)$. So, we should have sub-intervals now. These are $(-\infty, 1)$, $(1, 2)$, $(2, 4)$ and $(4, \infty)$. These are the intervals where we may think of computing the sign of $f'(x)$. You can see the signs directly, but also you can find out.

For $x \in (-\infty, 1)$, we see that $(x - 1)^2$ is a factor of $f'(x)$. So, we may consider directly the bigger interval $(-\infty, 2)$. If you come up to ± 1, of course you will get the same result. It gives same thing between $-\infty$ to 1 and then from 1 to 2. We see that if $x < 2$, then this is positive, and $x - 2$ is negative, $x - 4$ is negative; so, it is really positive. That is, $f'(x)$ is positive on $(-\infty, 2)$. You would say that $f(x)$ is increasing on $(-\infty, 2)$.

Now if you consider a point between 2 and 4, then you will see that this becomes negative, this is anyway positive; so f' becomes negative. Therefore, $f(x)$ decreases on $(2, 4)$. If $x > 4$, this is now positive; so, $f'(x)$ is positive. That is, $f(x)$ increases on $(4, \infty)$. Now, we have some idea about $f(x)$ from this information. It is increasing on the interval $(-\infty, 2)$, it decreases on $(2, 4)$, and again it increases 4 onwards.

Since 2 is a critical point, $f(x)$ is increasing up to 2 and again decreases after 2, there is a local maximum at $x = 2$. Similarly, $f(x)$ is decreasing from 2 to 4, and from 4 again it is increasing, so $x = 4$ is a local minimum point. There is a local minimum at $x = 4$.

Now looking at f'', we see that this is equal to 0 for $x = 1$. If you solve this quadratic, you would get $(5 \pm \sqrt{3})/2$. So, there are three points where $f''(x)$ is equal to 0. That gives us four intervals: $-\infty$ to 1, 1 to $(5 - \sqrt{3})/2$, and then from there to $(5 + \sqrt{3})/2$, and from $(5 + \sqrt{3})/2$ onwards.

We look at the sign of $f''(x)$ on $-\infty$ to 1. It is really negative up to 1; if x is smaller than 1, that makes it negative; and from 1 to $(5 - \sqrt{3})/2$, it is positive, from $(5 - \sqrt{3})/2$ to $(5 + \sqrt{3})/2$, it is negative, and $(5 + \sqrt{3})/2$ onwards it is positive. It is really changing sign at all these points. So, all these points are points of inflection. It says also something else. It is $-$ on $-\infty$ to 1, so it is concave down there, $f(x)$ is concave down there. It is also $-$ on $(5 - \sqrt{3})/2$ to $(5 + \sqrt{3})/2$, so there also it is concave down. And on the other two intervals, that is, on 1 to $(5 + \sqrt{3})/2$ and $(5 + \sqrt{3})/2$ onwards, it is concave up. So, all these three points such as 1, $(5 - \sqrt{3})/2$ and $(5 + \sqrt{3})/2$ are the points of inflection. Now you can try to see how it looks like.

We will try this problem now. It is asking a simple question: is it true that all zeros of $f''(x)$ are points of inflection? We know that it may not; wherever only there is a change of sign, those are the points of inflection. That is what we know. So, suppose $f''(c) = 0$. If there is no change of sign in $f''(x)$ while passing $x = c$, that is f'' has the same sign in the immediate left neighborhood of c and in the immediate right neighborhood of c, then c would not be a point of inflection. Otherwise, it is.

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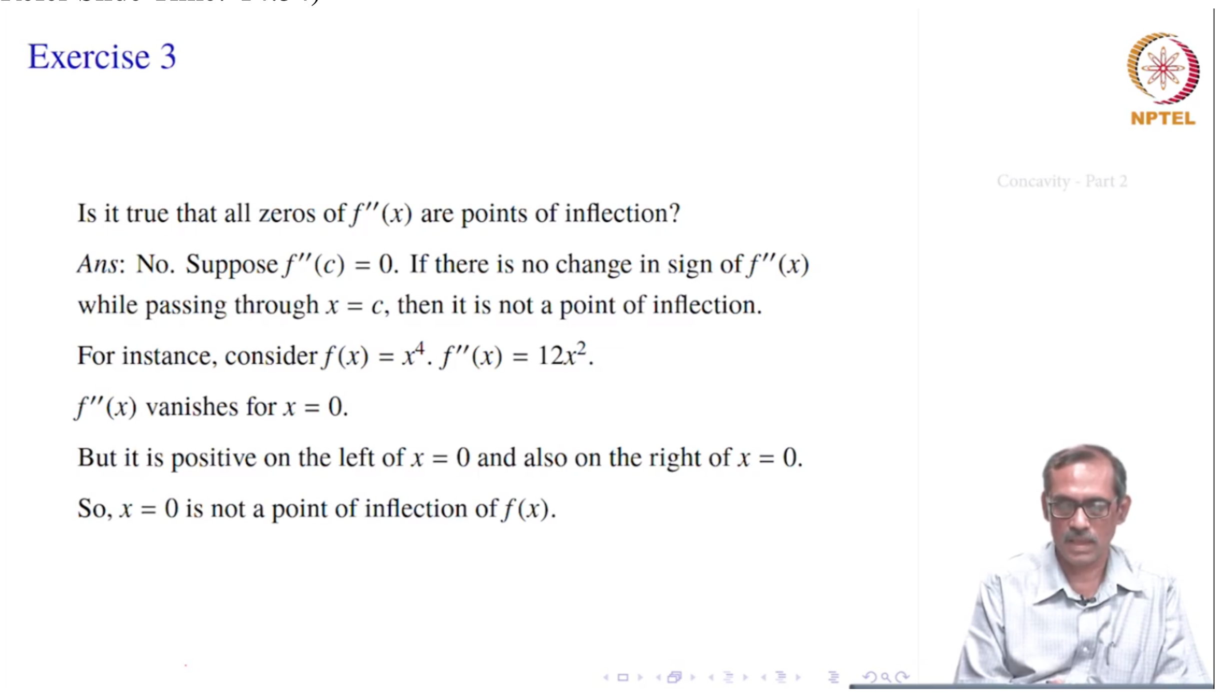

Here is an example for that purpose. Suppose, you take $f(x) = x^4$. Then, $f''(x) = 12x^2$. We know that $f''(x)$ is vanishing at $x = 0$. But $f''(x)$ is positive on the left of $x = 0$; it is also positive on the right of $x = 0$. So, $x = 0$ is not a point of inflection of $f(x)$.

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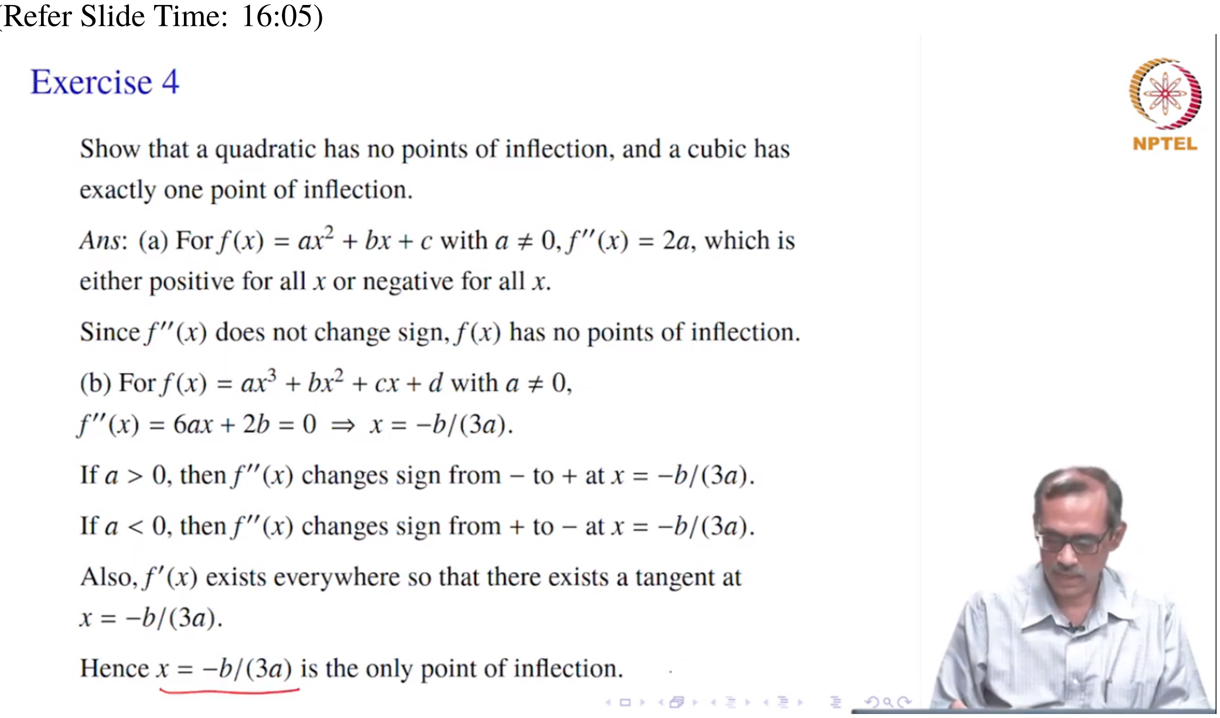

We are getting another problem. Here, we are given a quadratic expression. Suppose it is a function, like $f(x) = ax^2 + bx + c$, where $a \neq 0$. We need to show that it has no point of inflection. Also we need to show that a cubic has exactly one point of inflection. Recall that a cubic means it would look something like $ax^3 + bx^2 + cx + d$, where $a \neq 0$. It claims that a cubic will have exactly one point of inflection. That is fairly easy, if you try these functions.

Suppose we take $f(x) = ax^2 + bx + c$, where $a \neq 0$. We differentiate it twice to get $f''(x) = 2a$. Since $a \neq 0$, it is either positive or negative. So, either $f''(x)$ is positive throughout, or it is negative throughout for all x. Therefore, you will not get any point of inflection there. However, either this curve is concave down or it is concave up. You know it is a parabola, either it looked this way or it would look this way. That is what it says; there is no point of inflection.

Now suppose you take a cube, say, $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$. You find that $f''(x) = 6ax + 2b$. That is 0 for $x = -b/(3a)$. It is possible that at this point $x = -b/(3a)$ there is a point of inflection. It shows that there is at most one point of inflection. But we need to show that there is exactly one point of inflection. So, you must see that $f''(x)$ changes sign at this point. It will have different sign for points which are less than $-b/(3a)$ from that for points which are bigger than $-b/(3a)$. That has to be checked.

So, suppose first that $a > 0$. Let x be smaller than $-b/(3a)$. Then $f''(x) = 6a[x - (-b/(3a))]$ is negative. And if $x > -b/(3a)$, then $f''(x)$ is positive. So, $f''(x)$ changes sign from $-$ to $+$ at this point. Hence, there is a point of inflection at $x = -b/(3a)$.

Similarly, if $a < 0$, then $f''(x)$ changes sign from positive to negative at $x = -b/(3a)$. In this case also, it is a point of inflection. In any case, you would say that it has a point of inflection at $-b/(3a)$.

Now, there is some extra information here. It says that $f'(x)$ exists everywhere so that there exists a tangent at $x = -b/(3a)$. That is to be checked, because our definition says that at point of inflection there should be a tangent and there should be a change of sign. That is clear here. Therefore, $x = -b/(3a)$ is the only point of inflection. Let us stop here.