Basic Calculus - 1
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Lecture 22 - Part 1
Concavity - Part 1

Well, this is lecture 22 of Basic Calculus-1. In the last lecture we had discussed using the second derivative for deciding whether there is a point of extremum at a critical point. We will use the second derivative for something else today. It is about deciding the shape of the curve. It is basically called concavity; that is what we will be discussing today.
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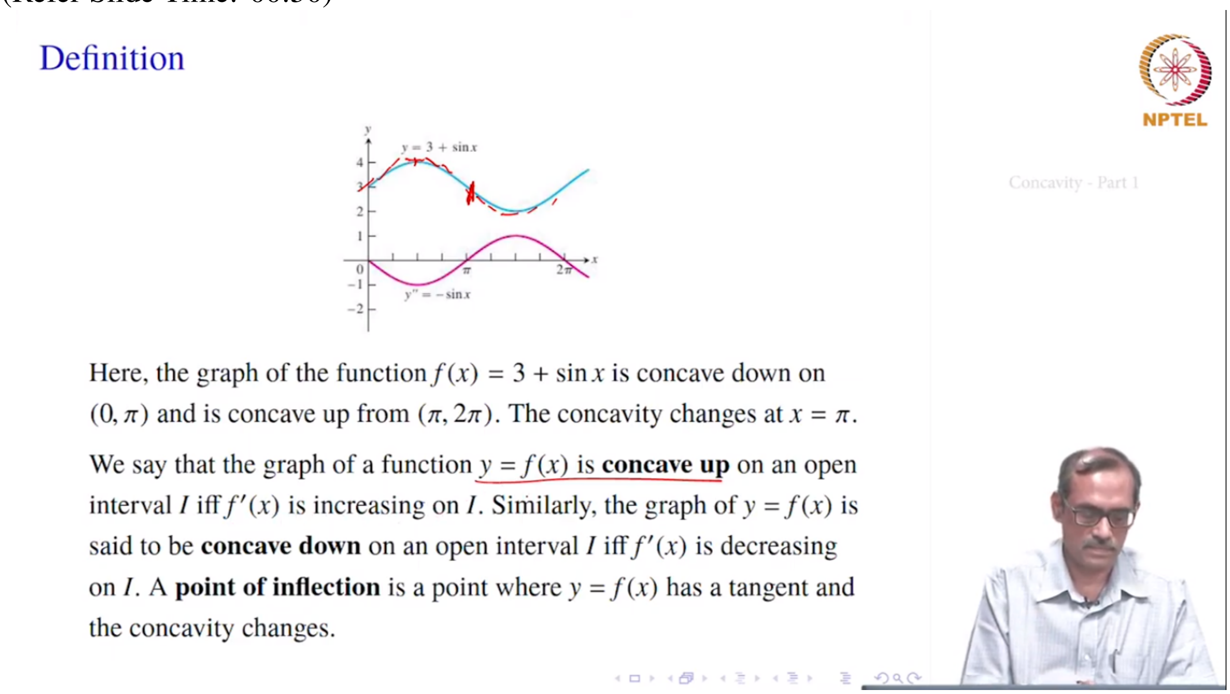

Let us look at this picture. Here is a plot in blue. It is a plot of $y = 3 + \sin x$. Of course, $y'' = -\sin x$ is plotted in pink. Look at this blue one, which is our function $f(x)$, it is $3 + \sin x$. When a curve looks something like this, looked from the x-axis side, we will be calling that as concave down, and when it is looking something like this, we will say it is concave up. Of course we should define these rigorously; but this is just an intuitive idea that this one is called concave down, and this one is called concave up.

Now in this plot of $y = 3 + \sin x$, over the interval $(0, \pi)$, the curve is concave down, and from $(\pi, 2\pi)$, the curve is concave up. You would say that concavity changes at $x = \pi$. This is just an intuitive idea. Soon we will be telling how to decide about that.

Look at this when it is concave down. If you look at the tangents, they will be something like this. All these tangents are having positive slope, and at a point inside when it is maximum the tangent is having slope 0. It is a horizontal tangent. Then you go to the right side; you will see that they come this way so that they are negative; it happens up to π. That means, it is positive, going to

0, then from here it is negative. From 0 to π, the slope of the tangent is really going from positive to 0 to negative, that is, it is decreasing. And on the other side, you will see that it is negative, then it becomes 0, and then it becomes positive. It is increasing from π to 2π. Looking at that, we will be defining our concave up, or concave down.

We will say that $y = f(x)$ is concave up on an open interval I, here, it is the open interval $(0, \pi)$, if $f'(x)$, which is slope of the tangent, is increasing on I. So, it is up, concave up, since it is increasing on I. Similarly, we will say that $y = f(x)$ is concave down on an open interval I if $f'(x)$ is decreasing on I. Notice that at that point π, the concavity changes. We will say that such a point is a point of inflection. That is, a point of inflection is a point where $y = f(x)$ has a tangent and the concavity changes.

These are the three notions we introduce with concavity: concave up when $f'(x)$ is increasing, concave down when $f'(x)$ is decreasing, and a point of inflection where a tangent exists and the concavity changes.

We will be applying these ideas to some other functions, not for this only. The main idea is to see how the derivative of a function give the information on the shape of a curve.

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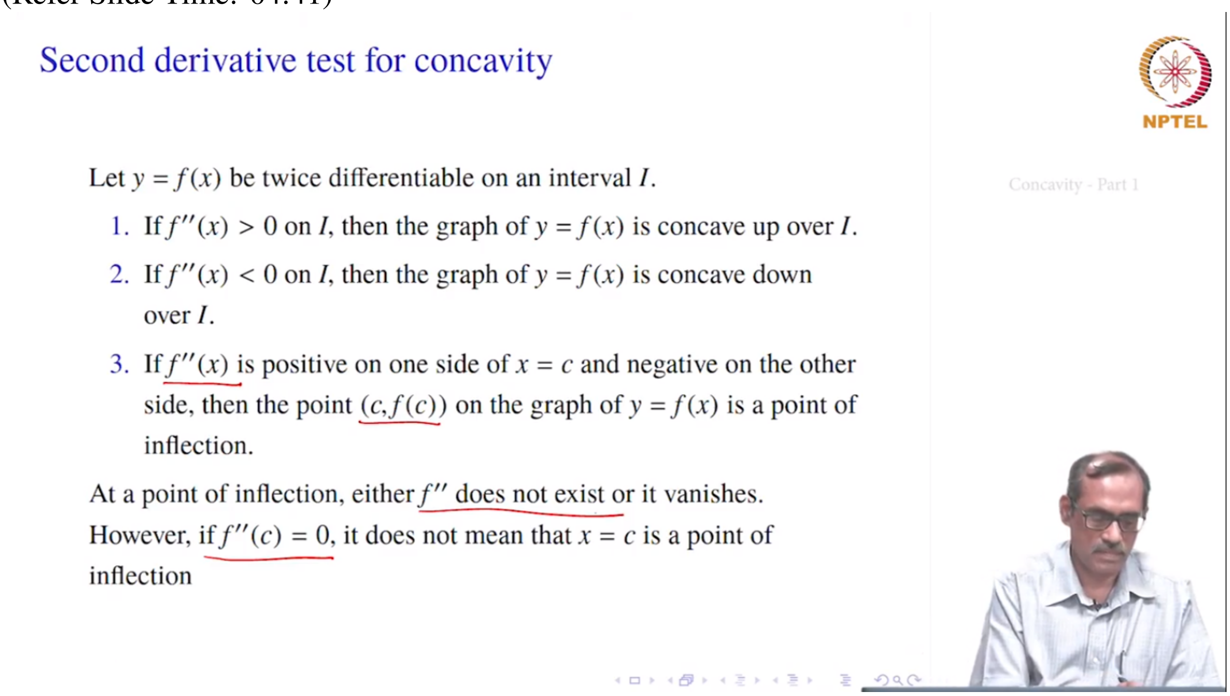

For that instead of looking at whether f' is increasing or decreasing, we can look at the second derivative. If $f'(x)$ is increasing or decreasing, then there will be some sign change or some specific sign of the derivative. See, if $f''(x) > 0$ on I, then $f'(x)$ is increasing on I so that $y = f(x)$ is concave up over I. Similarly, if $f''(x) < 0$ on I, then $f'(x)$ is decreasing so that $y = f(x)$ is concave down on I.

What about points of inflection? Of course, if $f''(x)$ is positive on one side of $x = c$ it is negative on the other side, then this point $(c, f(c))$ on the graph is a point of inflection. We will not say that $x = c$ is a point of inflection; we would say that the point on the graph that is, $(c, f(c))$

is a point of inflection. The concavity changes either from up to down, or from down to up at that point. For instance, look at the point $\pi, f(\pi))$ on our earlier curve.

It may happen that at a point, either f'' does not exist or it vanishes. Suppose that $f''(x)$ exists and is continuous. Now, on one side of the point of inflection, it is positive and on the other side it is negative. So, it must be zero there due to the Intermediate Value Theorem. But if $f''(c) = 0$, it does no meant that $(c, f(c))$ is a point of inflection. Because it may not have changed sign at that point. So, changing of sign is important for the point of inflection. For example, you take a constant function. For this, you would not say that every point is a point of inflection since f'' is zero everywhere. We want to avoid that. You will say that f'' does not exist or it vanishes, and that happens at a point of inflection, but any of this will not imply that it is a point of inflection. To check that it is a point of inflection, we should find out what happens to $f''(x)$ on its left and on its right. It should change; that is what it is.

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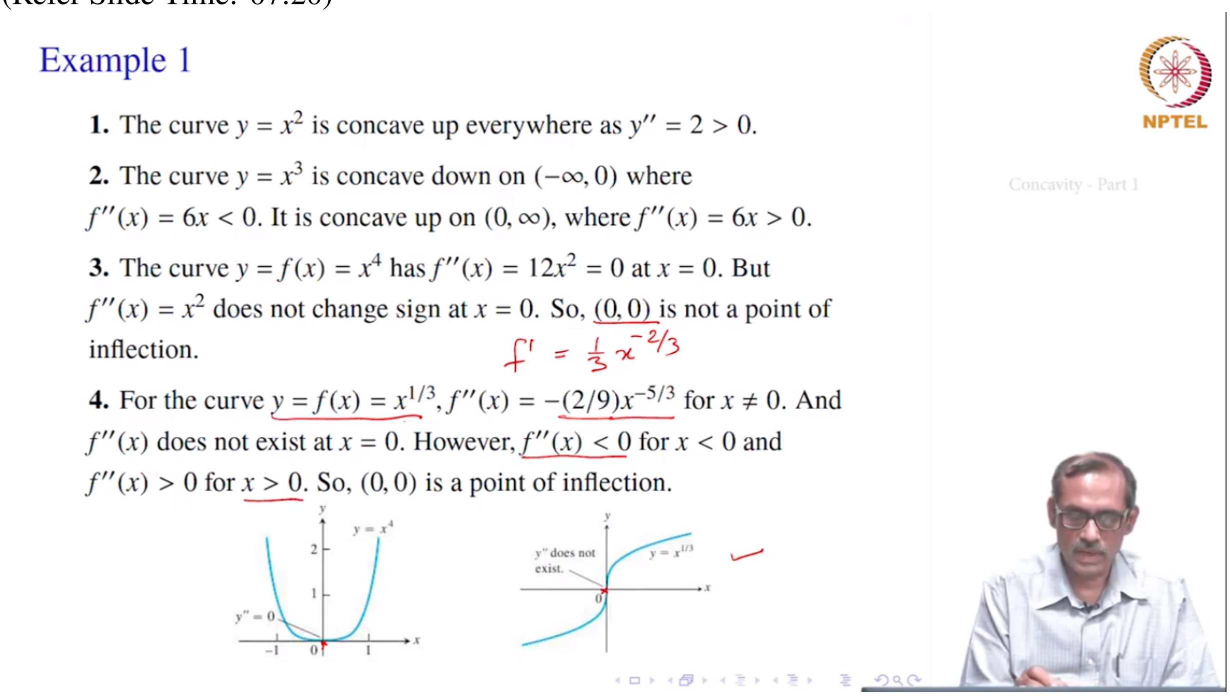

Let us apply it on a problem. Consider the curve $y = x^2$. It is a simple one, formula type; it is concave up everywhere because $y'' = 2$ which is greater than 0. It is concave up; it looks something like this.

If you take the curve $y = x^3$, and go for its derivative, that gives $3x^2$. Again its differentiation gives $6x$. So, $f''(x) = 6x$. Where is it greater than 0, and where is it less than 0? From $(-\infty, 0)$, x is negative. In that case you get $6x$ to be negative. So, $f''(x)$ is negative on $(-\infty, 0)$. Therefore, the curve $y = x^3$ is concave down on $(-\infty, 0)$. If you take x to be positive, then $f''(x) = 6x$ is bigger than 0. In that case, whenever x is positive, that is, on the interval $(0, \infty)$, the curve $y = x^3$ is concave up.

Let us take another example. Say, $y = x^4$. In this case, $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Where does it change sign? It is at $x = 0$ of course. If you take $x < 0$ or $x > 0$, everywhere $f''(x)$ is positive. At $x = 0$, $f''(x)$ is 0. Once $x \neq 0$, $f''(x) = 12x^2$ does not change sign near $x = 0$. That is, when you transit from a left neighborhood of 0 to a right neighborhood, you do not find any change of sign in $f''(x)$. It is always positive. So, the point $(0, 0)$ on the graph of $y = x^4$ is not a point of inflection even though $f''(0) = 0$. The reason is f'' is always positive for $x \neq 0$; so it is always concave up at any point; of course, not at 0.

Now for the curve $y = x^{1/3}$, we see that $f'(x) = (1/3)x^{-2/3}$. Differentiate once more to get $f''(x) = (1/3)(-2/3)x^{-5/3} = -(2/9)x^{-5/3}$. Now we are considering the point at $x = 0$. Of course $f''(x)$ is not defined at $x = 0$. So, we are trying to see what happens at $x = 0$. For $x < 0$, when x is negative, $(1/x)^{5/3}$ is negative. That gives $f''(x)$ to be negative. And, if $x > 0$, then that is positive so that $f''(x)$ is positive. So, from $-\infty$ to 0, $f''(x) < 0$ so that $f(x)$ is concave down, $f''(x) > 0$ from 0 to ∞ so that $f(x)$ is concave up, and $(0, 0)$ is a point of inflection. You can look at the graph how does it look. This is $x^{1/3}$. At $(0, 0)$ there is a point of inflection, since concavity changes. This is concave up; this is concave down as you see. However, $(0, 0)$ is not a point of inflection for the curve $y = x^4$ as we have seen there since there is no change in concavity at that point. That is how we will be talking about concavity.

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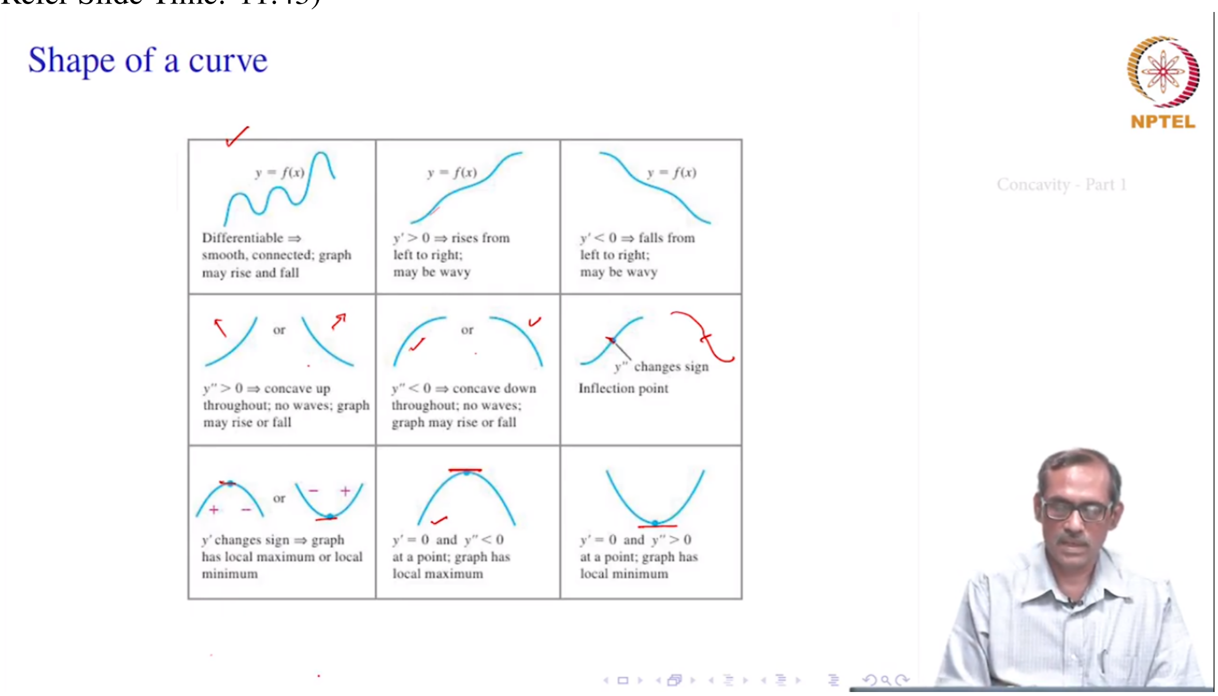

Now we have some idea about the shapes of curves. Suppose, the curve is $y = f(x)$. Look at the first picture here. It is differentiable everywhere as it looks, there are no corners and it is smooth, you can differentiate one more time, and there is no break anywhere, but the graph may rise or fall, something can happen.

Once you say it is differentiable, there are no corners and graph may go up and may come down, and so on. In general, it might look something like this.

But once you say that $y' > 0$, that is, $f'(x) > 0$, then $f(x)$ is increasing. So, it will be only climbing up. The curve may look something like this. Still, it can be wavy somewhere. But on the whole it will be rising up. It will be climbing up. That is how it is an increasing function.

And, if $y' < 0$, then it will fall; but it can still be wavy, having rise and fall; but on the whole it will be falling. So, that is what happens when $f'(x) < 0$.

Now, you come to the double derivative. Suppose the double derivative $y'' > 0$. Then y' itself will be increasing. So, the curve would look something like this, it is concave up throughout, no waves; but the graph may rise or fall; it may look something like this, or it may look something like this. Both are really concave up. So, it may be like this, one of these; but there are no wavy nature, there is no rising and falling in between.

See, if $y'' < 0$, then it will be concave down. So, it will look like this, or like this. It may increase, it may decrease, but its concavity is downwards. There are no waves again; but throughout it will be concave down. It may rise, it may fall, it may be increasing, it may be decreasing, but it would look something like this. So, concavity basically says something about the shape of the curve.

If the curve has an inflection point somewhere, then it may be concave up and then concave down, or it can be concave down and then concave up. It may look like this where y'' changes its sign. That can happen, but not in a form like the other two, which we discussed earlier in fourth and fifth parts. If y' changes sign, then we do not know whether y'' changes sign or not. If y' changes sign from positive to negative, then there will be a maximum point, and if it changes sign from minus to plus there will be minimum point. It gives rise to local maxima or local minima. Then, that is how it would look; the curve would look something like this.

If y' is equal to 0 and y'' is less than 0 at a point, then there is a horizontal tangent, and y'' is less than 0. So, it will be concave down, looking something like this. Again, it will have a local maxima.

Similarly, if y' is equal to 0 at a point and y'' is greater than 0, then it will be concave up there, and it would look something like this. Again, it will have a local minimum point.

These are some classification of curves basing on the information from the derivatives. That is how the shapes of curves will be determined.

Let us solve one example basing on this idea. Really we can try to sketch a curve, that is, by determining how the curve looks like in which intervals. We will not consider every point in $\mathbb{R}$, but we will discuss wherever there is an inflection point or there is a maximum point, minimum point, and so on. That is what we will be discussing.

We take the function $f(x) = (1+x)^2/(1+x^2)$. It is defined everywhere as x^2 is never equal to -1. Let us see how to use the information on the derivatives. First thing is, there are no symmetry here, whether symmetry around 0, symmetry about x-axis or symmetry about y-axis, nothing is there. Its domain is the whole of $\mathbb{R}$.

Now let us look at the derivatives. $f(x) = (1+x)^2/(1+x^2)$. So, it will give you $(1+x^2)$ times the derivative of $(1+x)^2$, which is $2(1+x)$, and then minus $(1+x)^2$ into the derivative of $(1+x^2)$, which is $2x$; and this is divided by $(1+x^2)^2$. That will be the derivative. If you simplify, that would give $f'(x) = 2(1-x^2)/(1+x^2)^2$. Similarly, you go for the second derivative, differentiate it again;

you would reach here. It will be $f''(x) = 4x(x^2 - 3)/(1 + x^2)^3$. So, these are the first derivative and the second derivative.

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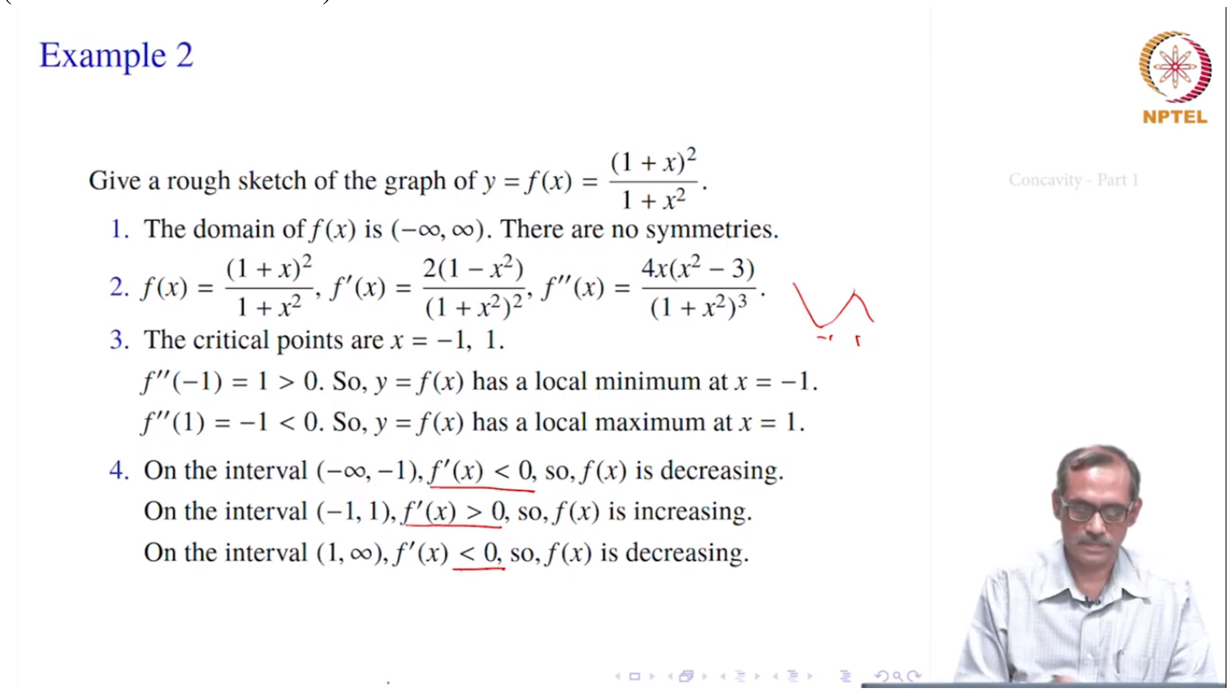

It is differentiable everywhere; so we just equate $f'(x)$ to 0 to get the critical points. Equating this to 0, we get the critical points as -1 and 1. These are the critical points. We want to see whether there is a possibility of maxima, minima, and so on at these points. Since we already have the second derivative, we use the second derivative test for finding extreme points.

Let us look at $x = -1$. What about f'' there? Substituting -1 there, we would get one $-$ sign; so that is negative, down is positive anyway. Let us calculate. It gives $f''(-1) = 1$. It is correct, and this is bigger than 0. Since this is bigger than 0, and it is a critical point, we have a local minimum at $x = -1$. So, at -1, it would look something like this as there is a local minimum. Fine.

Since $x = 1$ is a critical point, we evaluate f'' at 1. That gives $f''(1) = -1$. It is less than 0. Therefore, $f(x)$ has a local maximum at $x = 1$.

This is about the extreme points. Then let us look at where f' is increasing, decreasing and so on for finding the shape of the curve. In the interval $(-\infty, -1)$, let us evaluate f'. When $x < -1$, it is 1 minus, it is something less than -1, so x^2 will be bigger, and this will be negative. So, $f'(x) < 0$. That means $f(x)$ is decreasing when x varies from $-\infty$ to -1.

On the interval $(-1, 1)$, $x > -1$ but smaller than 1; then this will change its sign. You would get $f'(x) > 0$; that is, $f(x)$ is increasing on $(-1, 1)$.

Then let us take $(1, \infty)$, which comes from the other critical point 1. So, $x > 1$ means this is again bigger, so it becomes smaller again; $f'(x) < 0$. Therefore, f is decreasing.

So, on $(-\infty, -1)$, $f(x)$ is decreasing. It is something like this at -1. In $(-1, 1)$, it is increasing, and again in $(1, \infty)$, it is decreasing. Roughly, it would look something like this.

Now, let us look at where f'' vanishes. As you see, $f'' = 4x(x^2 - 3)/(1 + x^2)^3$. This is equal to

0 when $x = 0$ or $x = \pm\sqrt{3}$. These are the points where $f''(x)$ is 0. We should look at the interval these points create. We have now four intervals.

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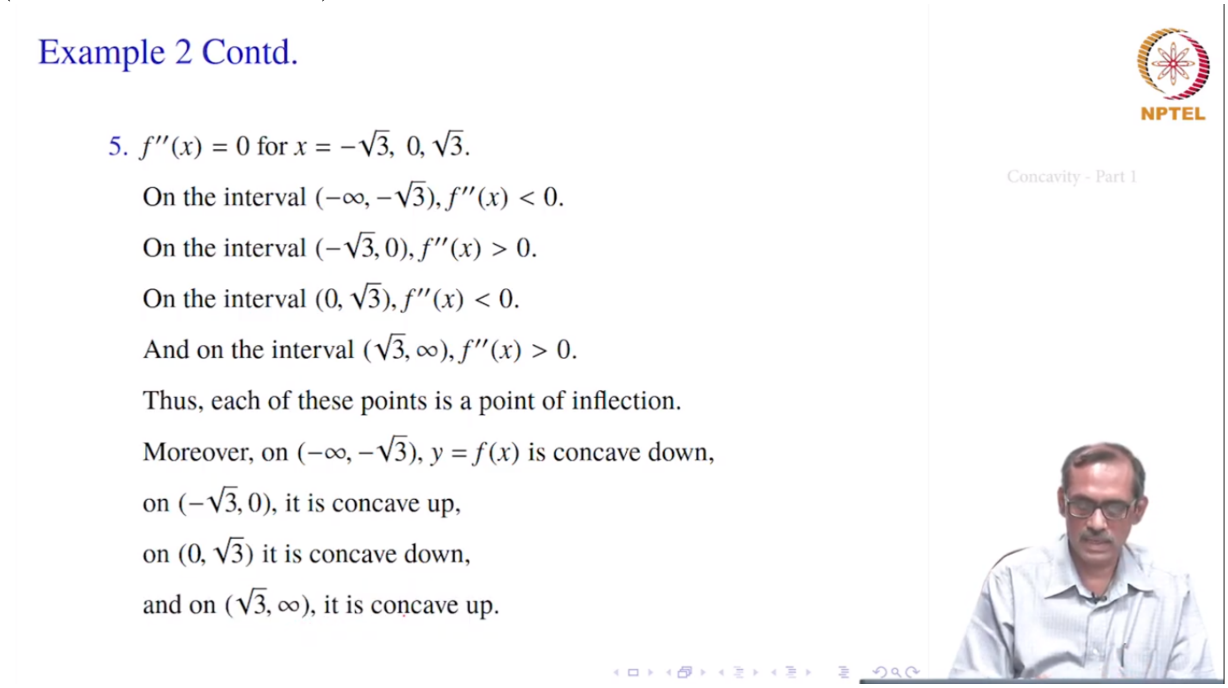

If you evaluate $f''(x)$ for $x \in (-\infty, -\sqrt{3})$, that is less than 0. If you look at $(-\sqrt{3}, 0)$, $f''(x) > 0$ there. If you look at $(0, \sqrt{3})$, $f f(x) < 0$ there. If $x > \sqrt{3}$, then $f''(x) > 0$. In these four intervals $f''(x)$ is really changing sign. At these three points, $f''(x)$ is 0. Since it is changing sign around each of these points, all these points are the points of inflection. So, there are three points of inflection. The concavity or the shape of the curve is changing at those points. And what happens in these intervals? Since $f''(x) < 0$ in $(-\infty, -\sqrt{3})$, $f(x)$ will be concave down there. Similarly, it is concave up in $(-\sqrt{3}, 0)$, concave down in $(0, \sqrt{3})$ and concave up in $(\sqrt{3}, \infty)$.

What about other points? At other points curve will be almost similar. But then we should find out where some straight line touches the curve, like the asymptotes. Or if x goes to ∞, becomes very large, what happens to the function? Or if x goes to $-\infty$, then what happens to the function? That would give roughly an idea about what happens for large x.

First, let us look at x going to ∞. When x goes to infinity, x is becoming larger but staying positive. So, $2/x$ becomes close to 0, but it is staying positive; $1/x^2$ remains positive but near 0; and this one remains near but greater than 1; and we write 1+ because of that. It means as x becomes larger and larger, and becomes infinity or approaches infinity, $f(x)$ will approach 1, but always remain greater than 1. That means it is really decreasing to 1. That is, as x goes to infinity, $f(x)$ is decreasing to 1. If $y = 1$ is this line, it will go something like this at ∞. It will not touch, but can touch at ∞ only. That is how it behaves for large x.

When x goes to $-\infty$, this remains negative, this remains positive, this remains positive; so it will be looking smaller because it is $1/x^2$ plus 1. So, it is minus something. It will be smaller than 1, but it is close to 1. That we write as 1−. So, it stays near 1 but smaller than 1. On this side it will be near 1 as x goes to minus infinity. Let us take the line $y = 1$. It may look something like this, going like this, and when x goes to $-\infty$, it will be staying near 1, but smaller than 1.

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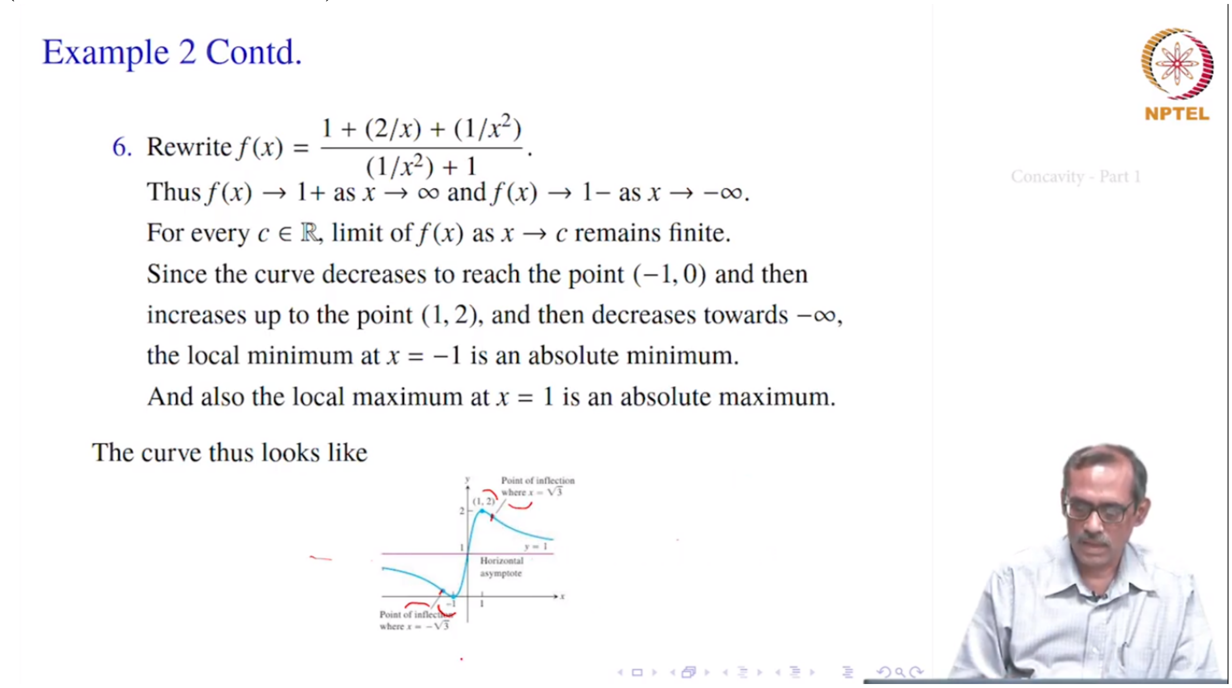

That gives some idea about when x is becoming very large or very small. And, if you take x goes to any other c, then the limit also remains finite. It is not blowing up anywhere. There is a limit of the function everywhere. It is not unbounded. You see that the curve decreases to reach the point $(-1, 0)$. At -1, its value is 0. It increases up to the point $(1, 2)$ and then it decreases towards $-\infty$. The local minimum occurs at $x = -1$; and local maximum occurs at $x = 1$. Comparing the values, we find that the local minimum is an absolute minimum and the local maximum is an absolute maximum.

To summarize all this information, we would say that the curve might look something like this. The curve meets the line $y = 1$ at ∞. It decreases until $x = -1$, where it has a local minimum; then it increases until $x = 1$, where it has a local maximum; its points of inflection occur at $x = \pm\sqrt{3}$; so, it is concave up here and concave down here. It goes like this, but not going beyond this, but this way. That is how the curve would look like.

You see how the derivatives are used to get an idea of the about the shape of the curve.