Basic Calculus - 1

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Lecture 21 - Part 1

Second Derivative Test - Part 2

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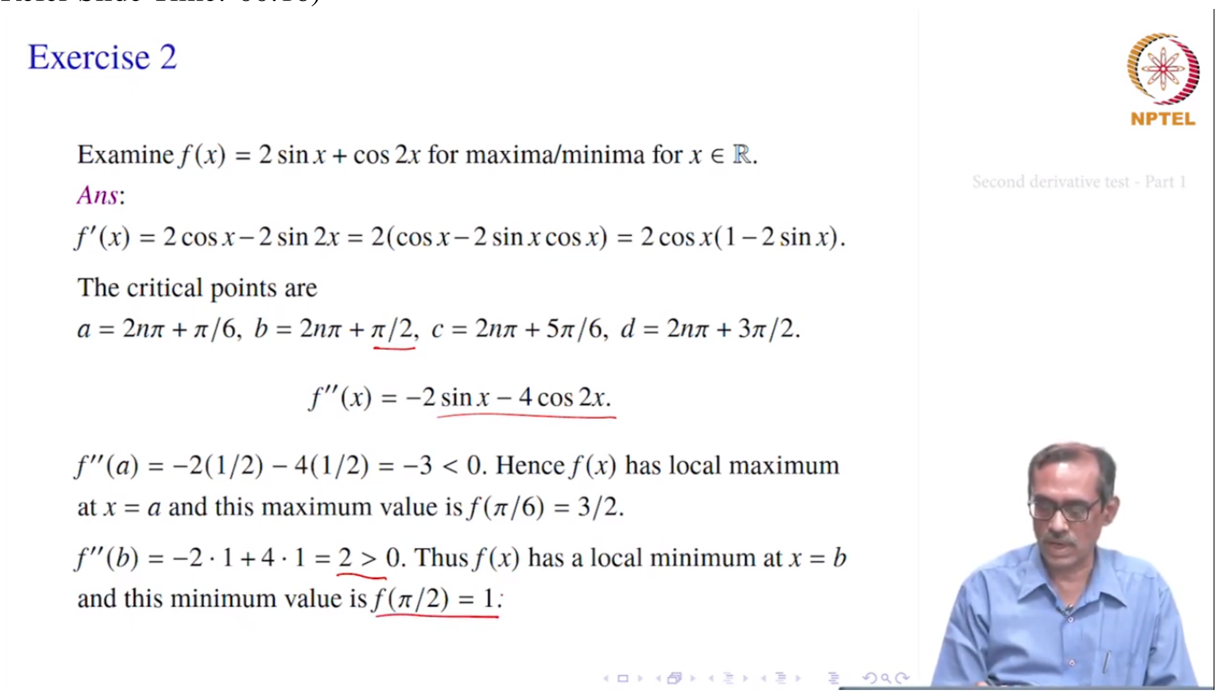

Let us go to next problem. It is asked to examine this function $f(x) = 2\sin x + \cos(2x)$ for its maxima and minima. We need to find all possible local maxima, local minima, absolute maxima, and absolute minima. The function is $2\sin x + \cos(2x)$. It is defined everywhere on $\mathbb{R}$ and also differentiable on $\mathbb{R}$. Its derivative is $f'(x) = 2\cos x - 2\sin(2x)$. If you take 2 common and expand it, it becomes $f'(x) = 2\cos x(1 - 2\sin x)$. That is our $f'(x)$.

We want to find the critical points. This is zero when $\cos x$ is 0 or when $\sin x$ is equal to half. $\cos x = 0$ gives $x = \pi/2$ and $x = 3\pi/2$ added to it $2n\pi$ with varying n, n is any integer. And $\sin x = 1/2$ gives you $\pi/6$ and $5\pi/6$ added to it $2n\pi$ for integers n. So, the critical points are $2n\pi + \pi/6$, $2n\pi + 5\pi/6$, $2n\pi + \pi/2$ and $2n\pi + 3\pi/2$ for integers n. These are all possible critical points over the whole of $\mathbb{R}$, where n is any integer.

At these critical points we must find out maxima and minima. For that purpose, we differentiate it again. From this it will be easier to differentiate. Or, even from the first one it will be easier still. That gives $-2\sin x$ and this gives $4\cos(2x)$. So, We get $f''(x) = -2\sin x - 4\cos(2x)$.

Now we must find the sign of this $f''(x)$ at the critical points a. Here, a a is not a single point; there are many points, depending on this n which changes. At all those points a, $f''(a) = -2\sin a - 4\cos(2a)$.

When $a = 2n\pi + \pi/6$, $-2\sin a$ gives $-2\sin(\pi/6) = -2(1/2) = -1$; and $-4\cos(2x)$ gives $-4\cos(\pi/3) = -4(1/2) = -2$ so that $f''(a) = -3$, which is less than 0. That means all these points are local maxima of the function $f(x)$. What is the local maximum value? Evaluate $f(a)$, which gives $2\sin a + \cos(2a) = 2\sin(\pi/6) + \cos(\pi/3) = 2(1/2) + (1/2) = 3/2$. That is the local maximum.

Similarly we should discuss about the other three types of points, say, b, c and d. If you substitute b, that is $\pi/2$ in $f''(x)$ here, that becomes 0, this becomes π so that gives -2 into 1 plus this minus and -1 comes from cosine. So, that gives 2, which is bigger than 0. Since $f''(b) > 0$, there is a local minimum of $f(x)$ at $b = 2n\pi + \pi/2$. And, the minimum value is $f(\pi/2) = 1$. This is the local minimum at b.

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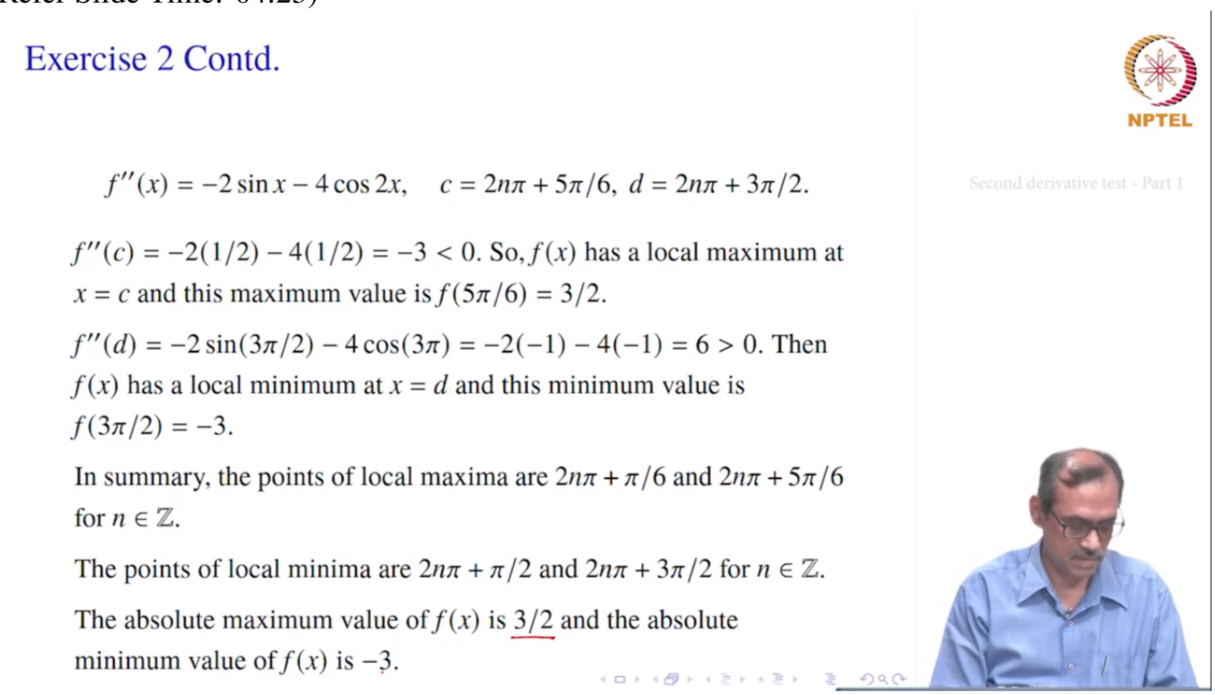

And what about c? Again, at $c = 2n\pi + 5\pi/6$ we will try f''. Evaluating $f''(c)$ we get $-2\sin(5\pi/6) - 4\cos(5\pi/3)$. That gives a half, and this gives half, so the answer is -3, which is less than 0. Therefore, $f(x)$ has a local maximum at $x = 2n\pi + 5\pi/6$, and that maximum value is $f(5\pi/6)$, which is $3/2$ again.

Similarly, if you go for d, we substitute $d = 2n\pi + 3\pi/2$ in $f''(x)$. It is essentially $f(3\pi/2) = 6$, which is bigger than 0. Again, as in the case of b, there is also a local minimum at $x = d$, and this minimum value is $f(3\pi/2)$, which is -3.

Taking care of all these things, what we get is that the points of local maxima are this a and c, that is, at $2n\pi + \pi/6$ and at $2n\pi + 5\pi/6$. Here, the local maximum value is $3/2$. And, $f(x)$ has local minima at b and d, that is, at $2n\pi + \pi/2$ and at $2n\pi + 3\pi/2$. The local minimum values are 1 and -3. If you compare all the values, we get absolute maximum as $3/2$ and absolute minimum is -3. So, that is how we are going to use the Second Derivative Test.

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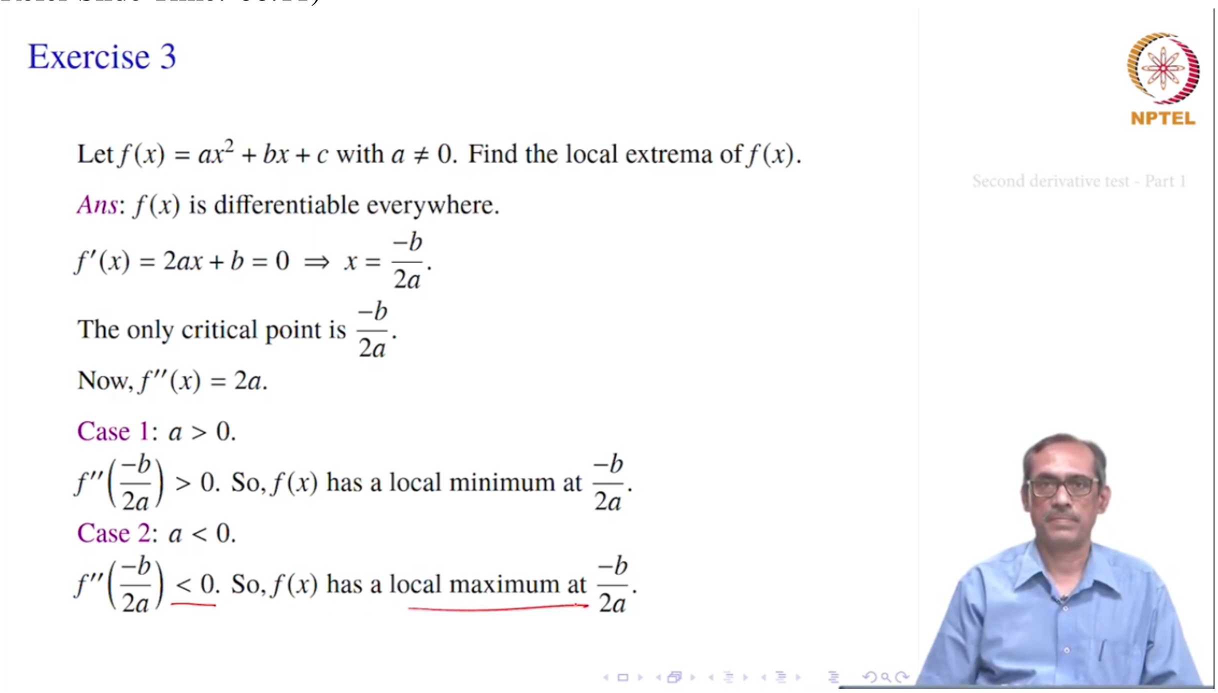

Let us take another problem. We have a quadratic here: $f(x) = ax^2 + bx + c$. It is quadratic means we assume that $a \neq 0$. We want to find the local extrema of $f(x)$, that is, local maximum and local minimum, if they occur. It is very straightforward. What we do is, we see that it is differentiable everywhere, even twice. That is so since it is a polynomial.

Now, $f'(x) = 2ax + b$. To find the critical points we equate that to 0. We get only one critical point here, it is $-b/(2a)$. This is the only point where there can be local maximum or local minimum because $f(x)$ is defined throughout $\mathbb{R}$ so that there are no endpoints. At $-b/(2a)$, we have to find out whether it is local maximum or local minimum. We may need certain conditions since its property will change, because we do not know anything about a, b, and c. All that we know is that a is nonzero. That is why $-b/(2a)$ is well defined.

Okey. We take the double derivative. It is $f''(x) = 2a$; it is straightforward. If $a > 0$, then something happens, and if $a < 0$, then something happens. Specifically, if $a > 0$, then $f''(-b/(2a)) > 0$ as it is $2a$. So, it has a local minimum at $-b/(2a)$. And, if $a < 0$, then $f''(-b/(2a)) < 0$ so that $f(x)$ has a local maximum at $-b/(2a)$. It is quite straightforward.

Let us find what is in the next problem. We want to find the maximum, and minimum values of $1/(1 - x^2)$ in the interval $(0, 1)$. At 1 of course, the function is not defined. Also we have the domain, a resitricted domain, which is the open interval $(0, 1)$. open interval. We need to find maximum and minimum values of the function in the open interval $(0, 1)$, where the function is $1/(1 - x^2)$.

Let us consider first the maximum. Since $f(x)$ is defined in $(0, 1)$, when x goes very close to 1, the term $(1 - x^2)$ becomes very close to 0. So, $1/(1 - x^2)$ can be made as large as possible by choosing our x very close to 1. That means, its limit as x goes to 1 becomes infinity. The limit, of course, is one sided now; it is the left side limit. The limit of $f(x)$ as x goes to $1-$ is infinity. That means it cannot have a maximum value. This is our first conclusion.

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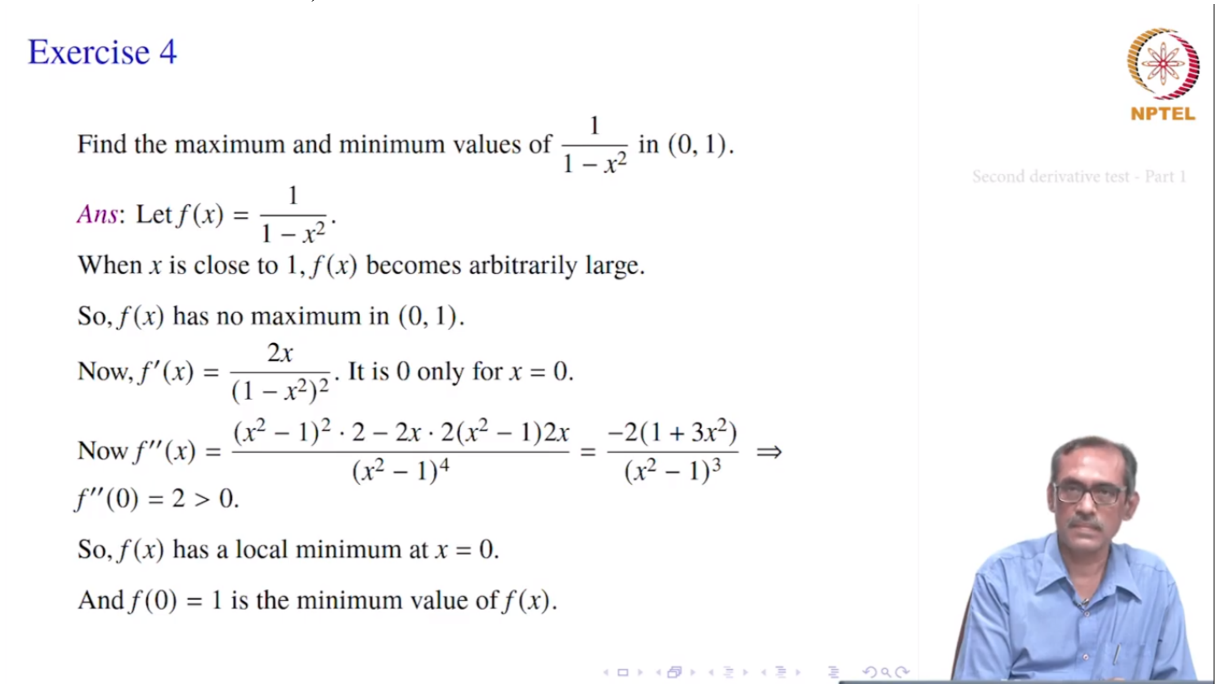

Now, when you differentiate this $(1-x^2)^{-1}$, it gives by the the chain rule that -1 into $(a-x^2)^{-2}$ into the derivative of $1-x^2$ with respect to x. It is $-(1-x^2)^{-2}(-2x)$. So, the minus gets canceled, and you get $2x/(1-x^2)^2$. That is what $f'(x)$ is.

Now it is 0 only for $x = 0$. That means the critical point, the only critical point is $x = 0$. We have to see what happens at 0. For that, let us take the second derivative. Of course, you can go to your sign test, sign of f' from $-$ to $+$ or from $+$ to $-$; but let us use the second derivative. The second derivative is not very difficult to compute here, again, by the chain rule.

You can think of that as $(x^2-1)^2$ if $1-x^2$ is troubling. That gives $(x^2-1)^2$ into the derivative of $2x$, which is 2, minus $2x$ into the derivative of $(x^2-1)^2$, which is 2 into (x^2-1) into $2x$ divided by $g(x)$ square, which is $(x^2-1)^4$. After simplification it gives $-2(1+3x^2)/(x^2-1)^3$.

Now, you want to evaluate this at the critical point 0. At $x = 0$, if we evaluate it, it gives the value as 2. This goes, this gives -1, so 2. And this is bigger than 0. So, $f(x)$ has a local minimum at $x = 0$. In fact, that is the only extremum value; no other local minima or maxima exists. So, that is also the absolute minimum, and its value is 1. Is that fine?

Let us take one more problem. Here, the function is not given, but what is given is its derivative. Its derivative is $(x-1)(x+2)(x-3)$. What is asked is to find the intervals where $f(x)$ is monotonic, that is $f(x)$ is increasing or $f(x)$ is decreasing. It should happen throughout the interval; that is the meaning of increasing or decreasing in an interval, or it is monotonic in the interval. Then, we also are required to find the points of local extrema of $f(x)$, that is, local maximum and/or local minimum.

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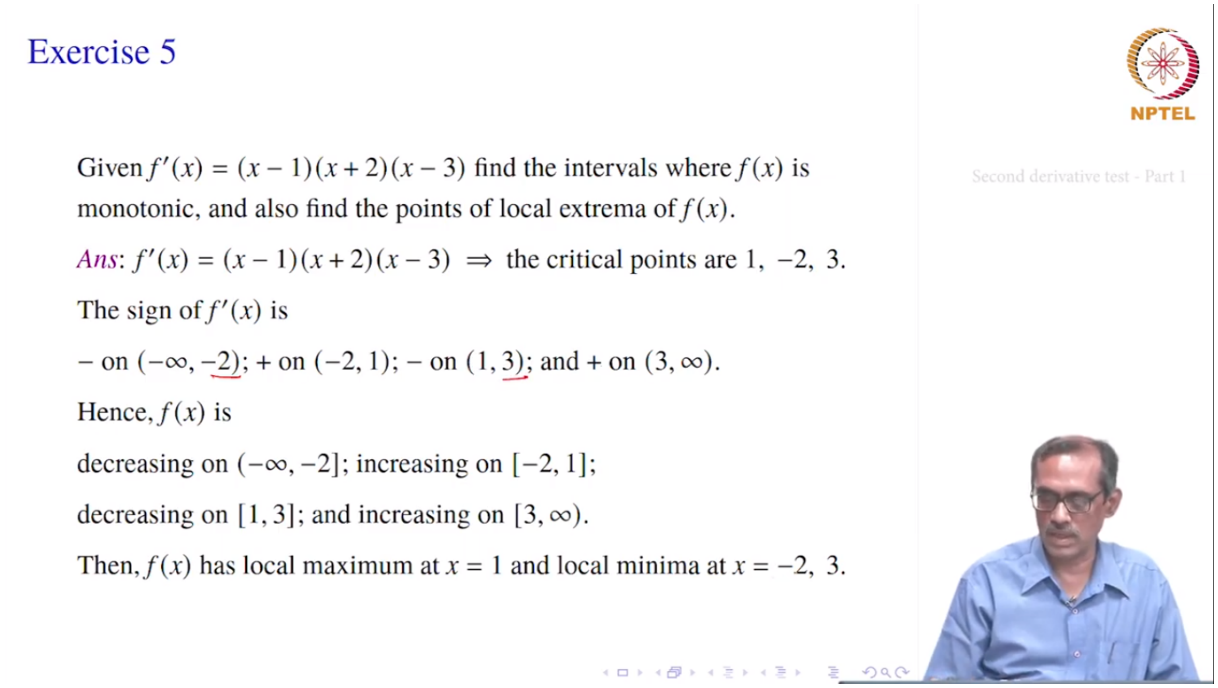

How do we proceed? We have $f'(x)$. That gives us the critical points, which are 1, -2 and 3. These are the critical points. At these critical points, what happens? We want to find the intervals. These three points break the whole of $\mathbb{R}$ into four parts, four different intervals. We have to discuss the sign of $f'(x)$ in all these intervals. First is -2, then comes 1, and then comes 3. It is like this. So, one interval is $(-\infty, -2]$, the next one is $[-2, 1]$, next one is $[1, 3]$ and finally, $[3, \infty)$. These are the four intervals.

On $(-\infty, -2]$, we have $x < -2$. We look at the sign of it. As $x < -2$, it is somewhere here. Now $x - 1$ is negative, $x + 2 = x - (-2)$ is negative, and $x - 3$ is also negative. So, f' negative on $(-\infty, -2)$.

Similarly, you find on $(-2, 1)$. Here, x is positive because it is inside the interval -2 to 1. So, $x - (-2)$ is positive and other two are negative. That gives positive, and the other one is negative again because two of them becomes positive, one is negative. So, it is negative; and on the other side it is becomes positive. So, f' is really changing sign at -2. It is changing sign from $-$ to $+$. It is changing sign at 1 from $+$ to $-$. It is changing sign at 3, which is from $-$ to $+$. Everywhere it is changing sign, but the nature is different.

So, $f(x)$ is decreasing on $(-\infty, -2)$ since it is remaining $-$ on $(-\infty, -2)$. Similarly, it is increasing on $(-2, 1)$ because f' is positive here. And $f(x)$ is again decreasing on $(1, 3)$ as f' is negative here. Further, f' is positive on $(3, \infty)$, so $f(x)$ is increasing on $(3, \infty)$. Accordingly, you get: $f(x)$ has a local maximum at $x = 1$ since $f'(x)$ is changing from $+$ to $-$ at $x = 1$. So, it has a local maximum at $x = 1$, a local minimum at $x = -2$, as f' changes from $-$ to $+$, and also at 3, change is from $-$ to $+$. So, $f(x)$ has a local minimum at $x = -2$ and at $x = 3$.

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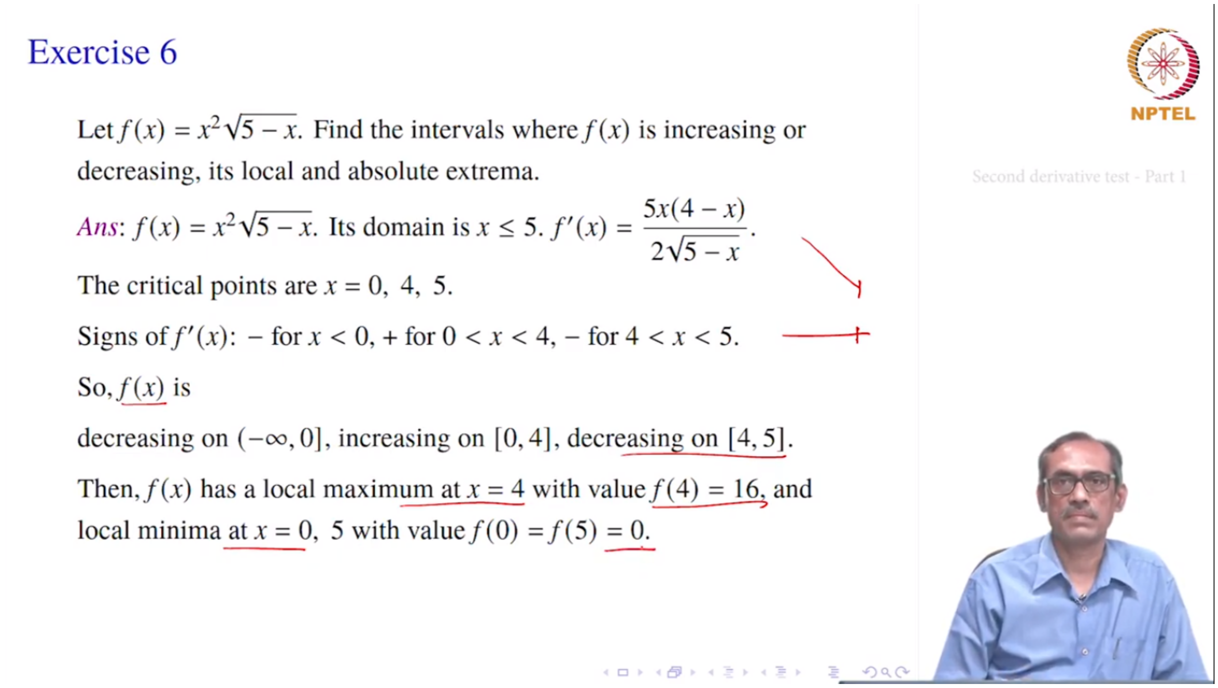

Let us take one more example. Here, the function is given as $f(x) = x^2\sqrt{5-x}$. We want to find the intervals where $f(x)$ is increasing or decreasing, and then its local and absolute extrema. That would be quite straightforward. All that we have to do is differentiate. Well, where is it defined? Of course, $5 - x$ must be greater than or equal to 0. So, x should be less than equal to 5. That is our domain; it is $(-\infty, 5]$.

There, we find $f'(x)$ equal to this. This is the derivative except at $x = 5$. So, $f'(x)$ is defined on $(-\infty, 5)$. At any $x \in (-\infty, 5)$, $f'(x) = 5x(4-x)/(2\sqrt{5-x})$. So, the critical points are $x = 0$, $x = 4$; and $x = 5$, where f' is not defined. Is it okay? Here at $x = 5$, f' is not defined.

We have these as the possible extreme points: 0, 4, or 5. We should discuss extrema at these points. Now, f'' will be a bit complicated here. So, we look at the change of signs in $f'(x)$. That is easy.

We have the four intervals now: $(-\infty, 0]$, $[0, 4]$, $[4, 5]$ and $[5, \infty)$. These are the four intervals where we should consider.

Now f' is negative for $x < 0$, positive for x from 0 to 4, negative for x from 4 to 5, and after 5, we do not have to discuss, because the function is defined only for $x \le 5$. So, there is no question for this, we go only up to 5. That is how the sign of $f'(x)$ are. Therefore, f is decreasing on $(-\infty, 0]$, it is increasing on $[0, 4]$, and it is again decreasing on $[4, 5]$. Accordingly, you get the local maximum, and minimum values.

So, f' changes from negative to positive at $x = 0$. Then, it has a local minimum at $x = 0$. Since f' changes from positive to negative at $x = 4$, $f(x)$ has a local maximum at $x = 4$. What about 5? We do not have any change at $x = 5$. It is an end point. It says is: $f(x)$ is decreasing on $[4, 5]$ and $x = 5$ is the right endpoint. So, $x = 5$ is also a point of local minimum. So, $f(x)$ has a local maximum at $x = 4$ and $f(4)$ evaluated gives 16. Further, $f(x)$ has local minima at $x = 0$ and at $x = 5$ and the minimum value is $f(0) = f(5) = 0$. Is that fine? So, we stop here.