Basic Calculus - 1

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Lecture 21 - Part 1

Second Derivative Test - Part 1

This is lecture number 21 of Basic Calculus-1. In the last lecture we had discussed how to use the sign change of the first derivative at a critical point to decide whether that critical point is an extreme point or not. Today, we will go a bit further, we will see how to use the second derivative for the same purpose.

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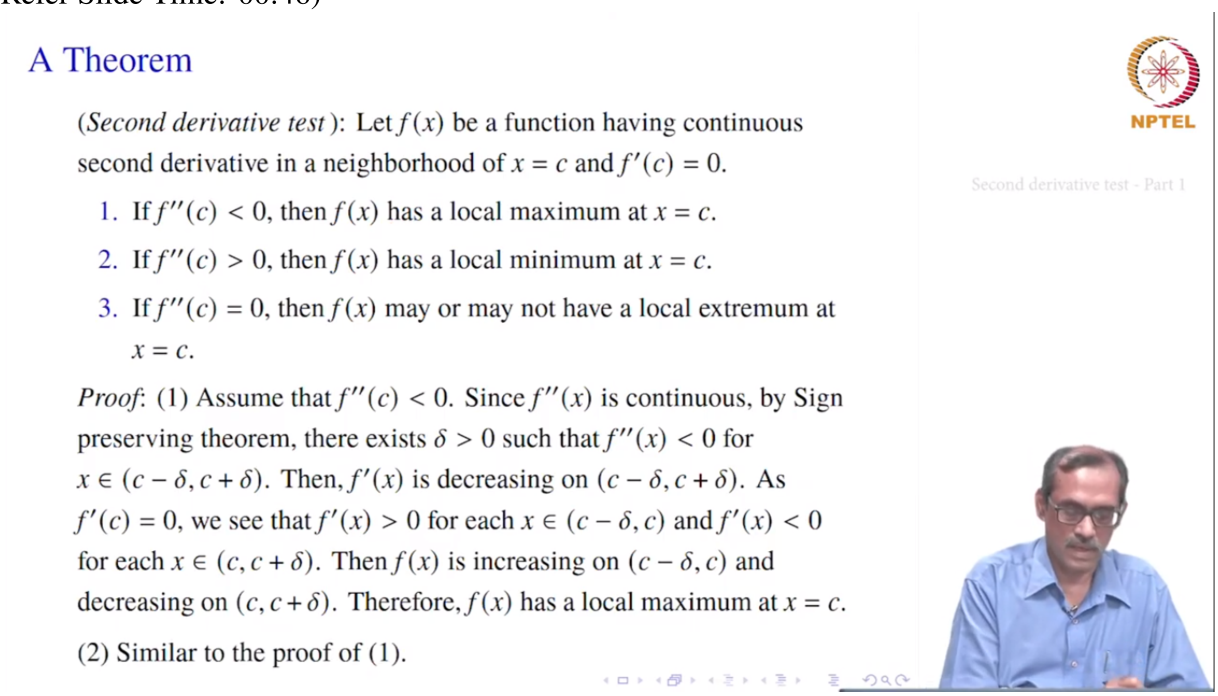

Let us see how far we can go. This is the statement which we want to use. It is called the Second Derivative Test.

Suppose $f(x)$ is a function; it is having the second derivative, and the second derivative is continuous at a critical point. We are taking only one type of critical points since the second derivative exists throughout the domain is assumed or at least it exists in a neighborhood of that point c. So, $f'(c)$ also exists. Thus, c is a critical point means $f'(c) = 0$.

So, we have the function $f(x)$; we have a point $x = c$ in its domain; we assume that $f'(c) = 0$; and the second derivative of f, $f''(x)$ is assumed to be continuous in a neighborhood of $x = c$. With these assumptions, suppose you find that $f''(c) < 0$, then the test says that that critical point c is a point of local maximum. Similarly, if $f''(c) > 0$, it is positive, then that critical point c is a local minimum point. If $f''(c) = 0$, then anything can happen; the test does not say anything. In this case, it may or may not have a local extremum at that point $x = c$.

In a sense, this test will not be conclusive like your change of sign in the first derivative. However, it becomes easier to apply this sometimes. We should discuss how it is applied. We will solve, of course, some problems basing on this. Let us see how to prove this. Out of Parts (1) and (2), we prove only one, and then give some examples to show that when $f''(c) = 0$, then anything can happen.

So, let us assume the conditions in Part (1), that is, $f''(c)$ is negative. We know that f'' is continuous in a neighborhood of c. Then, there is again a neighborhood of c, may be smaller than that, may be larger than the earlier neighborhood, but there is a neighborhood of c, say, $(c-\delta, c+\delta)$ such that f'' preserves its sign in this neighborhood. Since it is negative at c, it remains negative. throughout $(c - \delta, c + \delta)$ for some $\delta > 0$. That is, $f''(x) < 0$ for all $x \in (c - \delta, c + \delta)$. Some such δ exist;, some neighborhood exists where this will happen. That is what our Sign Preserving Theorem says about the continuous functions.

Once f'' is negative, we see that $f'(x)$ is decreasing on that neighborhood $(c - \delta, c + \delta)$. So, what happens? In $c - \delta, c + \delta)$ f' is decreasing. We know that $f'(c) = 0$. So, what happens in the left neighborhood $(c - \delta, c)$ and what happens in the right neighborhood $(c, c + \delta)$?

Of course f' decreasing to $f'(c) = 0$ from $f'(c - \delta)$ in the left neighborhood. It would look something like this, decreasing to 0. That means f' remains positive (non-negative) at all the points in the open interval $(c - \delta, c)$. For each $x \in (c - \delta, c)$, $f'(x)$ is positive.

Next, let us consider the right neighborhood $(c, c + \delta)$. Here also, f' is decreasing. Then, it would look like this. It means f' becomes negative (non-positive) because $f'(c) = 0$. So, $f'(x)$ is negative when $x \in (c, c + \delta)$.

Now you see that the function f', the derivative of f, changes its sign from positive to negative at $x = c$. As we know earlier, this gives rise to the conclusion that $f(x)$ has a local maximum at $x = c$.

It goes from positive to negative. Since f' is changing sign from positive to negative, f' is positive on the left. That means f is increasing on the left side. Next f' is negative on the right. So, f is decreasing on the right side. It is increasing on the left and decreasing on the right. So, at that point c, there must be a local maximum; $f(x)$ has a local maximum at c. That is what we wanted to show.

Similarly, the second one is proved by change of signs and this increasing and decreasing nature of the functions involved. We remember now that $f''(c) < 0$ implies that f has a local maximum, and $f''(c) > 0$ implies that f has a local minimum. We omit the second one since it is similar.

We come to the third one. It says that if $f''(c) = 0$, then we cannot conclude anything. That means there will be some examples where it will be maximum; there will be some, where it is minimum, and there might be some, where nothing can be told; it is neither a maximum nor a minimum. Let us see. It is not difficult to give examples.

Let us take $f(x) = x^4$. Its derivative is $4x^3$ and its second derivative is $12x^2$. Now, $f'(x) = 0$ gives $x = 0$; so 0 is the only critical point for this function. And, f'' is continuous in any neighborhood of 0 here. But we find that $f''(0) = 0$. The second derivative is 0 at the critical point

0. What happens to f?

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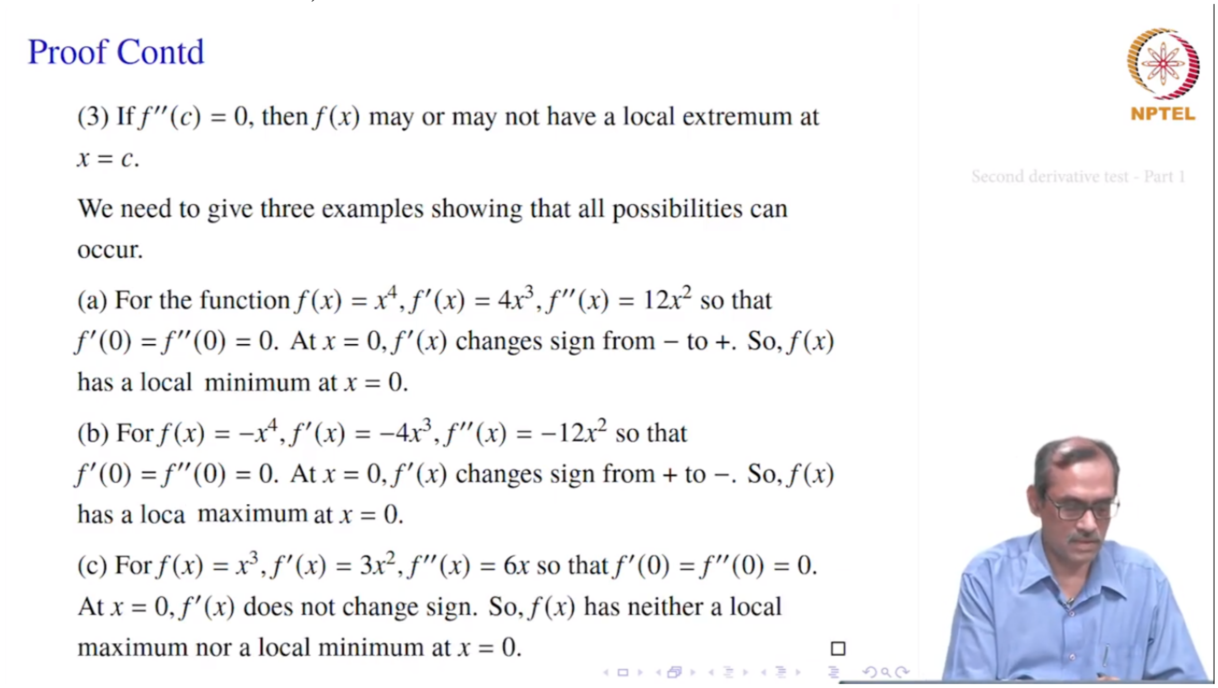

If you look at f'; it is changing sign from $-$ to $+$. If x is negative then $4x^3$ is negative, and if x is positive, then $4x^3$ is positive. So, f' x changes sign from $- +$. Since it is changes sign from $-$ to $+$, $f(x)$ has a local minimum at $x = 0$. Here, we see that $f''(0) = 0$.

Let us take another example, say, $f(x) = -x^4$. A similar thing happens here. Again, $x = 0$ is a critical point and $f''(0) = 0$. Notice that f'' remains continuous in a neighborhood of the critical point $x = 0$. In this case, $f'(x)$ changes sign from $+$ to $-$ though earlier f' was changing sign from $-$ to $+$. Then, there is a local maximum at $x = 0$.

We should give another example where nothing is there. Let us take $f(x) = x^3$. For this function, $f'(x) = 3x^2$. So, $x = 0$ is a critical point. And, $f''(x) = 6x$ so that it remains continuous in the neighborhood of the critical point 0 with $f''(0) = 0$. What happens to the function at $x = 0$? We see that when $x < 0$, $f'(x) = 3x^2$ is positive, and when $x < 0$, $f'(x) = 3x^2$ is also positive. So, $f'(x)$ does not change sign at the critical point $x = 0$. Therefore, there is no local maximum or local minimum of the function at $x = 0$.

So, you see, all three kinds of things can happen at a critical point c if $f''(c) = 0$ That is how this test is not as potent as the change of sign in the first derivative test. But sometimes it becomes easier to apply. We will see its applications soon.

Let us take an example. Find all points of local extrema of the function $f(x) = x^3 - 3x^2 - 24x + 5$. It is a polynomial, a cubic and it defines a function. So, what do we do? We differentiate, and see that it is differentiable everywhere in the domain. Domain of f is of course $\mathbb{R}$. Everywhere the polynomial is defined, and everywhere it is differentiable. The derivative of x^3 is $3x^2$ and so on. Then, $f'(x) = 3x^2 - 6x - 24$. We can factorize it to get the critical points. Now, $f'(x) = 3(x^2 - 2x - 8) = 3(x + 2)(x - 4)$. Therefore, the critical points are $x = -2$ and $x = 4$. These are the critical points. At these critical points we will see what happens to f''. If you differentiate it again, that gives $f''(x) = 6x - 6$.

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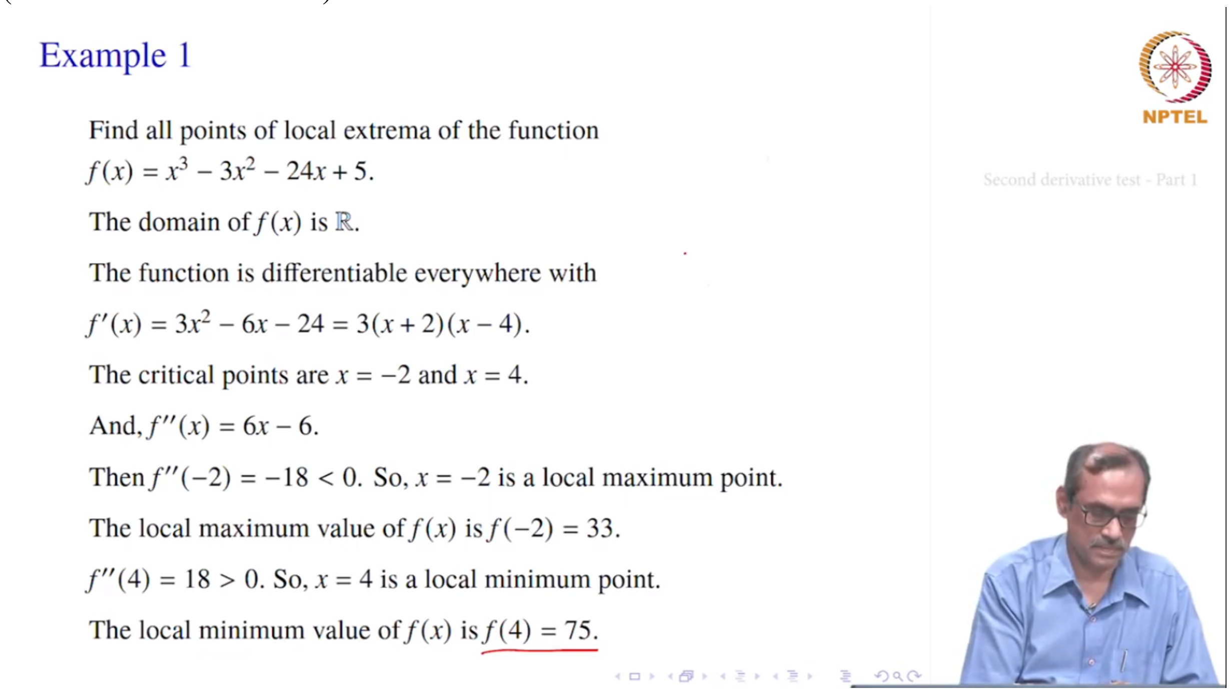

Now, when you take the critical point -2, at that point $f''(-2) = 6(-2) - 6 = -18$, which is less than 0. So, we conclude that $x = -2$ is a local maximum point of the function f$f(x)$. Let us evaluate it at the other critical point, which is 4. There, $f''(4) = 6 \times 4 - 6 = 18$. It is greater than 0. So, $x = 4$ is a local minimum point. That is, -2 is a maximum point and 4 is a minimum point.

To find the local extrema we need the values of the function at a point of extremum. In this problem we have the points. We must find the values also. The local maximum value of the function is $f(-2)$, which is 33, and the local minimum point is $f(4)$, which is 75. You see, the local maximum can be smaller than the local minimum. That is okay, because the curve can go here and there, so smaller maximum and bigger minimum values are all right.

So, a curve can have like this and this one, and they are joined together somewhere. Not like this, because this will not come this way, so we say this one. Now, there is a maximum point here, local maximum, which is smaller than the local minimum. It might happen like that. Fine.

This is how we will be using our principle of applying the second derivative test.

Let us take another example. Here, a box is to be made from a sheet of cardboard that measures 12 by 12, in some units, let us say centimeters. So, 12 centimeter into 12 centimeter is the size of cardboard that is given. It is square sheet. And what is to be done? We want to make a box out of it by cutting a square from each corner of the sheet.

Here, a square is cut; four squares are cut from the four corners of the sheet. It is assumed that they are of the same dimension; the smaller four squares have the same side lengths. The smaller squares are cut and thrown away. And then you fold them up to the sides. After the small squares are cut and thrown away, these sides are folded to get the box. Then, what is the box of greatest

volume that can be constructed by doing this? That means what small squares are cut from the sides: how much we should cut so that the box will give the biggest volume? What we do?
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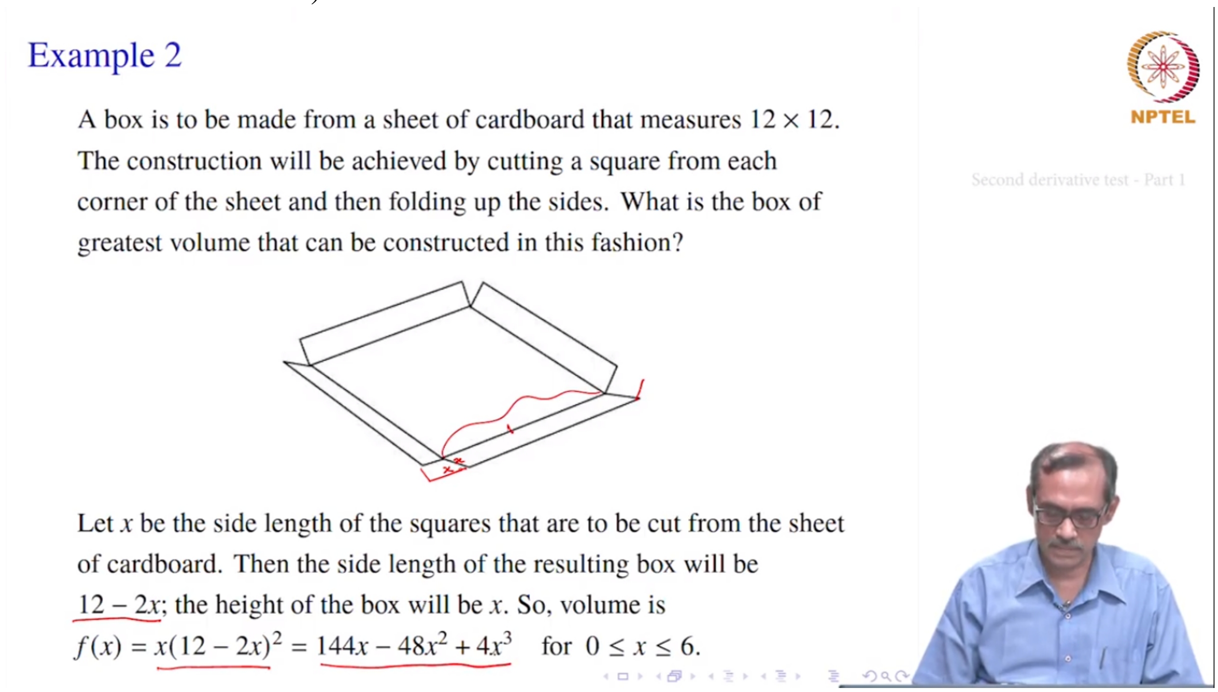

Let x be the side of the length of the square that is to be cut from the sheet of the cardboard. Let us say this is x. That means we are cutting x^2 from the corners. Now, what about the length now? The length was 12 units; this x is cut here, and another x is cut here. So, $2x$ has been cut from each side. That means the remaining side, this is the side, is $12 - 2x$. When it is folded, the height of the box will be this x. That is why the volume of the box will be $x(12-2x)^2$. It is $(12-2x)^2$ times the height x. Is that right? That is our volume; and this is to be maximized. Let us call that as the function $f(x)$.

We want to find x. Wait, what could be the maximum and minimum of this x? Well, you may not cut anything, so x can be 0, or if you cut something then since total is 12, you can cut only maximum of half of it. This will not do, of course because everything will be cut. Here, similarly, if nothing is cut then it will give 0 volume. So, $0 \le x \le 6$. That is what we see. Of course 0 and 6 will not give the extreme values. The volume will be 0 really; so that will not maximize, that will minimize.

But let us keep the problem as it is, and see what happens. Our $f(x) = x(12-2x)^2$ which we expand to $144x - 48x^2 + 4x^3$. That is fine. Now, we want to maximize this function. Here, $f(x)$ equal to this is defined on the domain $[0, 6]$ and it is to be maximized. First of all, we see that inside the interval $[0, 6]$ it is continuous, and it is differentiable on the open interval $(0, 6)$; even it is second time differentiable there.

For the critical points, we put $f'(x) = 0$. That gives $f'(x) = 144 - 96x + 12x^2$. This will be plus $12x^2$ not minus. Now, this is equal to 0 gives, by taking 12 common, $x^2 - 8x + 12 = 0$. That gives $x = 2$ or $x = 6$. That is easy because we can factorize this : $x^2 - 8x + 12 = (x-2)(x-6)$.

So, the critical points are 2 and 6. Each of the critical points should be discussed separately.
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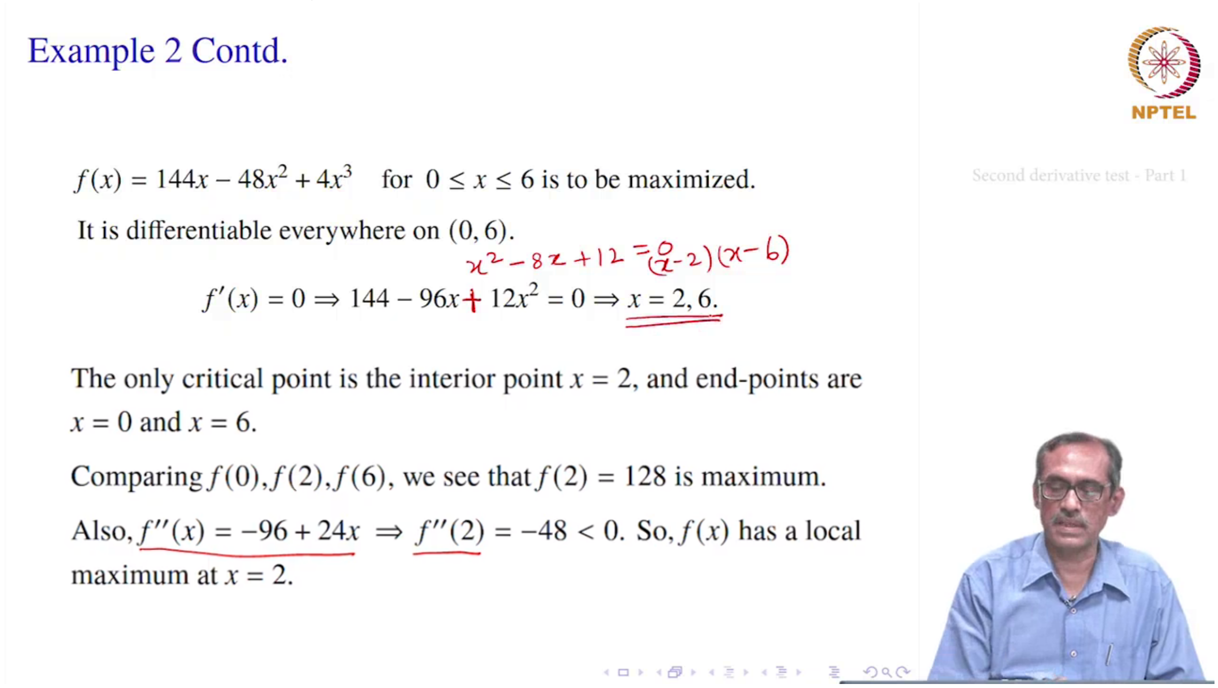

Let us consider $x = 2$. Of course, we may discuss the endpoints 0 and 6. As we see $f'(6) = 0$, but it is only one sided differentiability for 6 since it is an end point. So, the only interior point point is 2, and that is a critical point. The other one which we got from $f'(x) = 0$ will not be taken as a critical point. The function is differentiable there and it is not an interior point, but that is one end point. So, we have the critical point, which is also an interior point is $x = 2$. We also have the endpoints as $x = 0$ and $x = 6$.

Of course, from this if you go without any tests, you see that the function can be having minimum or maximum values only at these points: $x = 0$, $x = 2$ or $x = 6$. Then substitute them in the function, find out which one is maximum. Of course, 0 and 6 will give us 0, so maximum should be equal to $f(2)$.

But we can also apply our second derivative test to see whether this critical point is really a local maximum point or not, though that is not required since we want an absolute maximum here. Let us see, if you compare $f(0)$, $f(2)$ and $f(6)$, you find $f(2) = 120$, which is the maximum, and that settles the problem.

But we are going a bit further. We want to see whether this critical point $x = 2$ is a local maximum or local minimum, or what it is, though absolute maximum we have already got.

If you differentiate, you would get $f''(x) = -96 + 24x$ from this. This is really plus, and that gives f'' at the critical point equal to -48, which is negative. So, $f(x)$ has a local maxima at $x = 2$. It also has another local maximum. In fact, it is 0 at $x = 0$, and 0 at $x = 6$; so it achieves a maximum and comes down again; something like this. As it is a cubic, it is not exactly like this, its maximum point is at 2.

Let us take another problem. Remember again, these are exercises; so you must try yourself first, and then come back to the solution by pausing the video.

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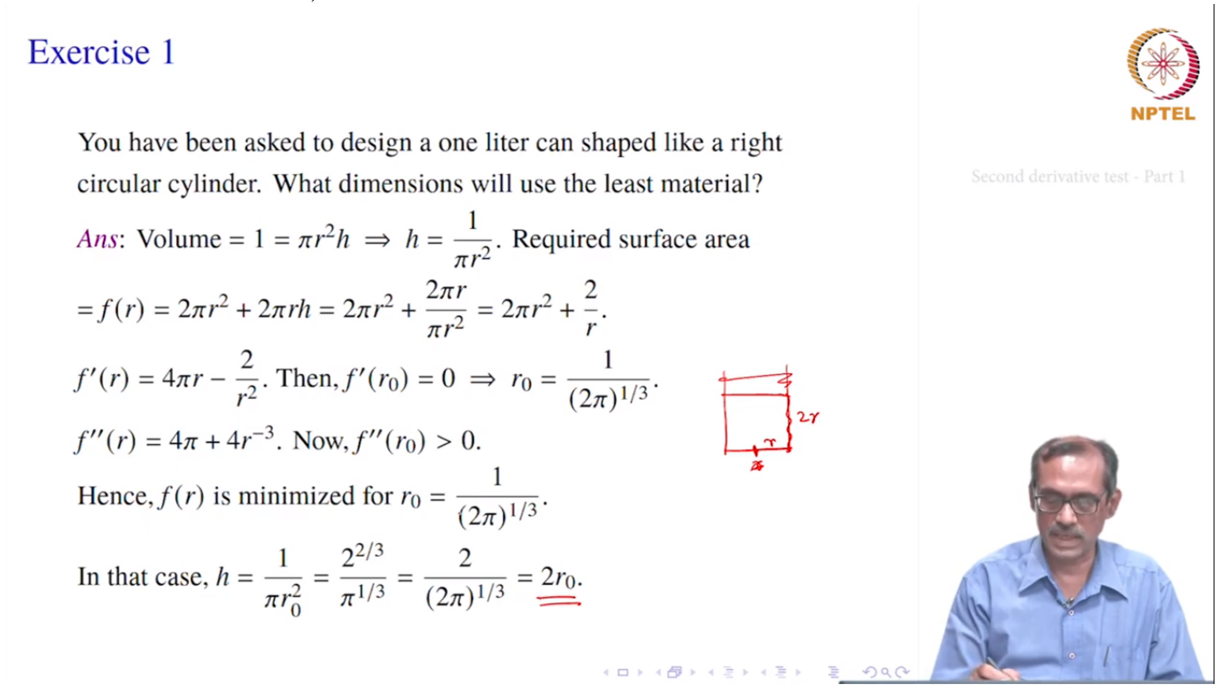

Here the problem is to design a one liter can shaped like a right circular cylinder. It is a one liter can which is a right circular cylinder. Its volume is 1 liter. What dimensions will use the least material? That means we want to keep this volume as 1 liter, and minimize the surface area of the cylinder; that is what it is asking.

Volume is one liter, what should be the minimum of the surface area, that is what we want to find out. Suppose the cylinder has, as usual we keep the usual notation, base radius r, and the height as h. Then, the volume will be $\pi r^2 h$, which is kept as 1 liter; that is given. From this we directly get that h must be equal to $1/(\pi r^2)$. You want to define a function that is the surface area, and express that as a function of r. You can also do otherwise, try that as a function of h. Let us try this way.

Now, the required surface area of this cylinder is: this is πr^2, plus the top is also πr^2, plus with this height, the circular thing gives you $2\pi r h$, totaling to $2\pi r^2 + 2\pi r h$. Now that $h = 1/(\pi r^2)$, we substitute that h, so the area is

$$f(r) = 2\pi r^2 + 2pir/(\pi r^2) = 2\pi r^2 + 2/r.$$

Of course, r is never 0 because that value of r would not give the volume of the the cylinder as 1 liter (unless it becomes infinite). So, our function is $f(r) = 2\pi r^2 + 2/r$. We want to minimize this function. To minimize this, what do we do?

We differentiate, because this is differentiable everywhere. Of course, r is never 0; so this function is defined for $r \neq 0$, and everywhere else it is differentiable. Its derivative is: $2\pi r^2$ with respect to r gives $4\pi r$ and then $2/r$ gives $-2/r^2$. So, $f'(r) = 4\pi r - 2/r^2$. That is equal to 0 at a critical point r_0. Now, $f'(r_0) = 0$ is to be solved. That gives $r_0 = 1/(2\pi)^{1/3}$. How? This equation is $4\pi r_0 = 2/r_0^2$. It gives $4\pi r_0^3 = 2$, or $r_0^3 = 2/(4\pi) = 1/(2\pi)$. So, $r_0 = 1/(2\pi)^{1/3}$. This is the only critical point. Is that right?

At this critical point, we must check that. Of course, geometrically nothing else would be happening; that should give the minimum. That is the answer if you want to guess. But we do not want to guess; we want to solve it. So, we would check that at this point r_0, the function $f(r)$ is really minimized. We need to verify that $f(r) = 2\pi(r^2 + 1/r)$ is minimized at r_0.

To do that, let us take its double derivative. Now, $f''(r)$ is the derivative of $4\pi r - 2/r^2$, which is $4\pi - 2(-2)r^{-3} = 4\pi + 4/r^3$. This is our $f''(r)$. At the critical point $r = r_0$, we have $f''(r_0) = 4\pi + 4/r_0^3$ which is positive. It is bigger than 0. We are not finding it exactly, all that we need is its sign, and it is positive. That means $f(r)$ has a local minimum at this r_0. Since this the only extremum point where $f(r)$ a local minimum, $f(r)$ is minimized at this point. At the other point, that is at $r = 0$, this function is not defined; so we need not consider that.

So, this local minimum becomes an absolute minimum. Here only $f(r)$ is minimized. Now if r is equal to this, then what will be h? We can find h as $1/(\pi r_0^2)$ which simplifies to $2r_0$. All it says is that we have to choose our r and h in such a way that whatever is this r, the height h should be twice of that. That means it will look something like a square. This is your r, half of the diameter. Te diameter of the base is $2r$, which is the height. When this happens, we will be using least material to get the constant volume, whatever is given. That is what it says. Since the volume is 1, we get the exact value of r as $1/(2\pi)^{1/3}$.