Basic Calculus - 1
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Lecture 20 - Part 2
First Derivative Test - Part 2

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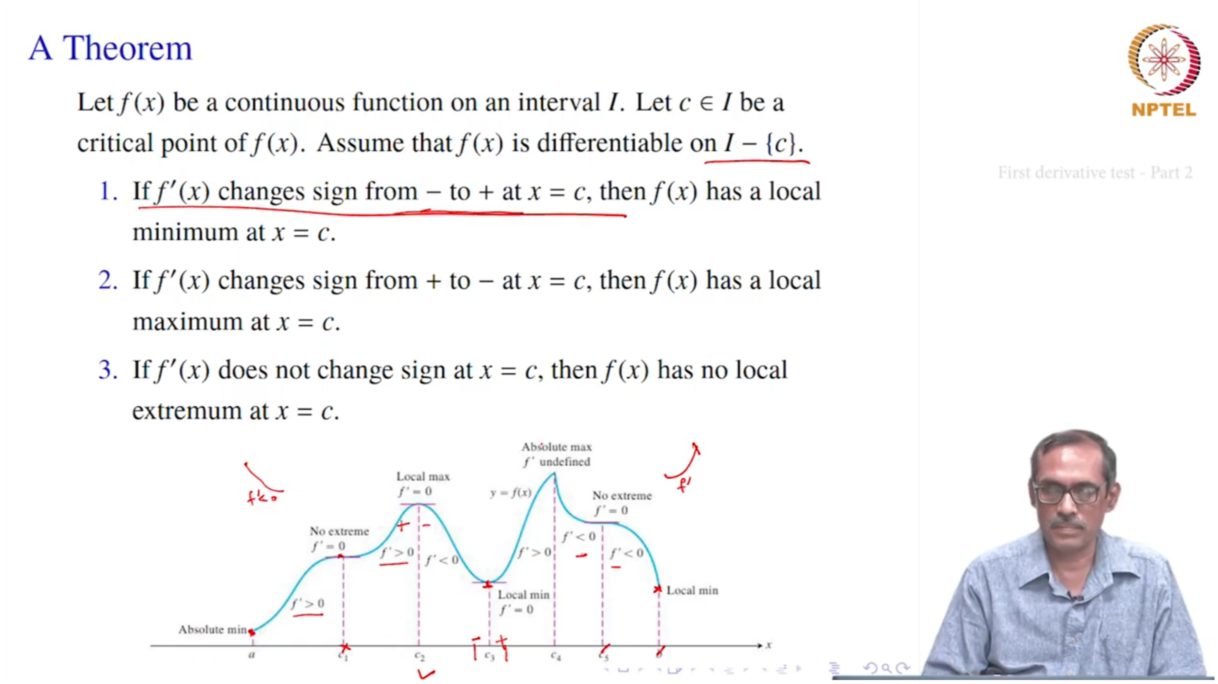

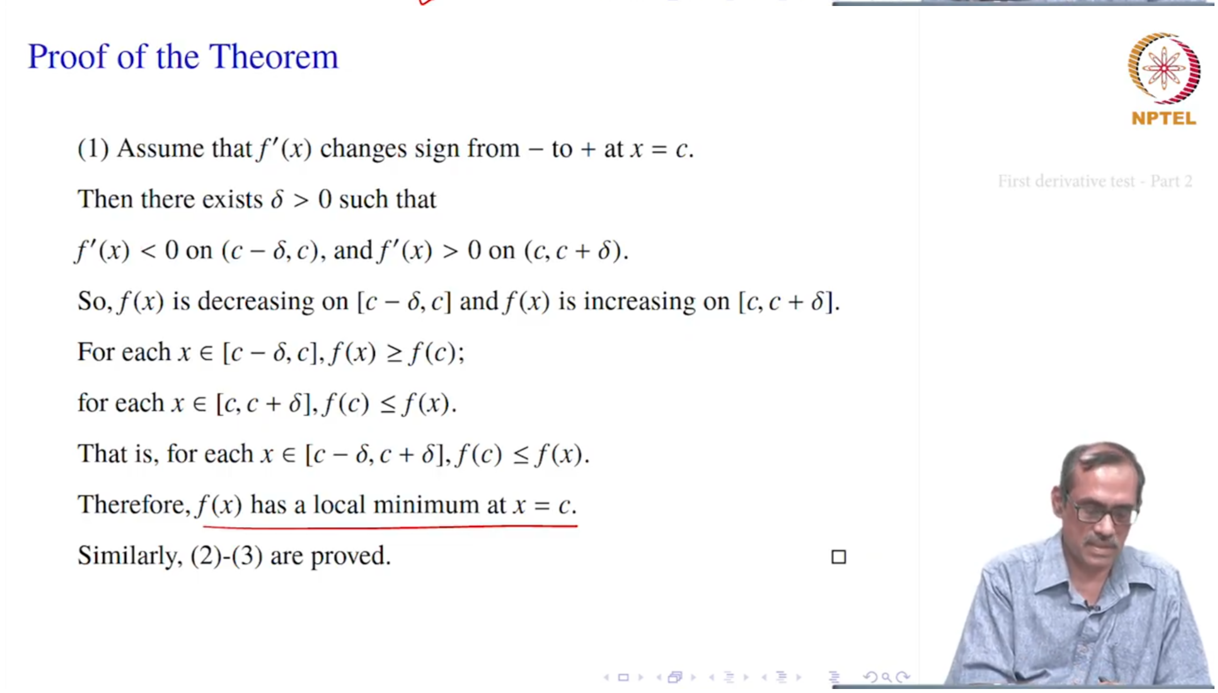

The proof is easy; it is not difficult. Let us consider Case 1 first. Assume that f' changes sign from $-$ to $+$. We have seen informally how does it happen. We want to show that $f(x)$ has a local minimum at $x = c$. So, suppose it changes sign from $-$ to $+$. That means there exists a left side δ-neighborhood, $(c-\delta, c)$ where f' is less than 0; and in $(c, c+\delta)$, the right side neighborhood, f' is bigger than 0. That is the meaning of "changes sign from minus to plus". On the left it is minus, and on the right it is plus. Now, f' is less than 0 means $f(x)$ is decreasing on $(c-\delta, c)$; and this is greater than 0 implies that $f(x)$ is increasing on $(c, c+\delta)$.

That shows that $f(x) \geq f(c)$ for every $x \in (c-\delta, c)$ because it is decreasing to $f(c)$. Also, it is increasing from $f(c)$; it is increasing after $f(c)$. So, it is decreasing on the left and it is increasing on the right, and c is there. Now, $f(x) \geq f(c)$ for for every x in that δ-neighborhood. That means $f(x)$ has a local minimum at $x = c$, That is clear. Similarly we can prove parts (2) and (3). There, only the sign changes will occur so that the inequalities will change.

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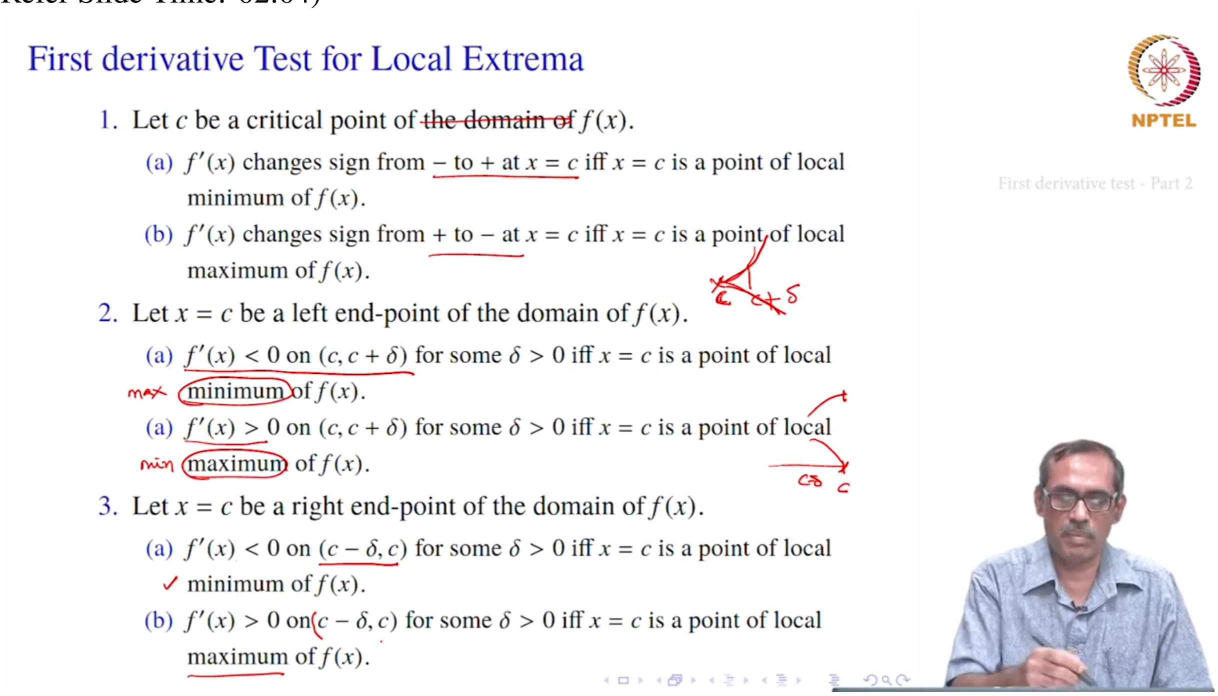

We will formulate how to use this information for testing of local maxima and minima; but only at the critical points. Let c be a critical point of $f(x)$. Of course the critical points lie inside the domain of the function; but we these need not be interior points; these can be endpoints also. For the interior points that theorem is applicable. Prior to that we have already discussed the left and right endpoints. So, we formulate as follows.

Let c be a critical point of $f(x)$. If $f'(x)$ changes sign from $-$ to $+$ at $x = c$, then $x = c$ is a point of local minimum of $f(x)$. If $f'(x)$ changes sign from $+$ to $-$ at $x = c$, then $x = c$ is a point of local maximum.

Let $x = c$ be the left endpoint of the domain of $f(x)$. If $f'(x) < 0$ on $(c, c+\delta)$ for some $\delta > 0$, then $x = c$ is a point of local maximum. If $f'(x) > 0$ on $(c, c+\delta)$ for some $\delta > 0$, then $x = c$ is a point of local minimum.

Let $x = c$ be the right endpoint of the domain of $f(x)$. If $f'(x) < 0$ on $(c-\delta, c)$ for some $\delta > 0$, then $x = c$ is a point of local minimum. If $f'(x) > 0$ on $(c-\delta, c)$ for some $\delta > 0$, then

$x = c$ is a point of local maximum.

Let us see the statements about the endpoints. Suppose $x = c$ is a left endpoint. Then, only a right hand side neighborhood is relevant. Assume that $f'(x) < 0$ on a right hand side neighborhood, say, on $(c, c + \delta)$ for some $\delta > 0$. This means $f(x)$ is decreasing on $(c, c + \delta)$. So, $f(c)$ is a local maximum value. Similarly, if $f'(x) > 0$ on $(c, c + \delta)$, then $f(x)$ is increasing on $(c, c + \delta)$. It implies that $f(c)$ is a local minimum value.

Now, when $x = c$ is the right endpoint of the domain of $f(x)$, only a left hand side neighborhood is relevant. So, suppose $f'(x) < 0$ on $(c - \delta, c)$ for some $\delta > 0$. Then, $f(x)$ is decreasing on $(c - \delta, c)$. It implies that $f(c)$ is a local minimum value. Similarly, if $f'(x) > 0$ on $(c - \delta, c)$ for some $\delta > 0$, then $f(x)$ is increasing on $(c - \delta, c)$. It follows that $f(c)$ is a local maximum value.

Look at the picture. Here, you have the left endpoint a, and f' is bigger than 0 on a right neighborhood. Once f' is bigger than 0, $f(x)$ is increasing from $f(a)$ to some value on the right of $x = a$. Then, there is a local minimum at $x = a$. So, positive to the right neighborhood means it should be local minimum; is that clear? And if f' is less than 0 on the right of a, then it is this way; $f(x)$ is decreasing from $f(a)$ to some value on its right; then, $x = a$ is a point of local maximum. Exactly opposite happens for the right end-point.

You have to really remember these things. But as you see, there is nothing much to remember. We can simply resort back to the geometrical notions and set it right. So, let us apply this test on some problems.

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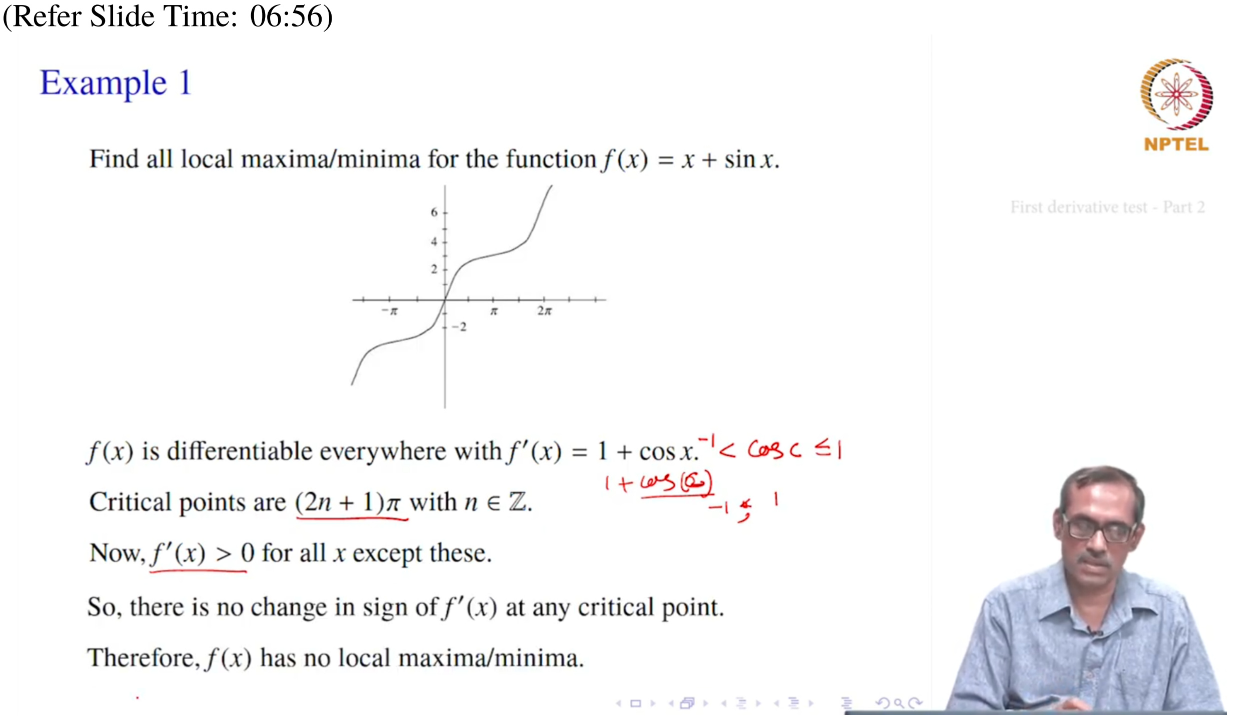

Let us take the first example. Here, we are asked to find all local maxima and minima for the function $f(x) = x + \sin x$. If you plot the graph, it would be something like this. But sometimes it is misleading; because at these points the curve looks horizontal, but it really may not be so. We should out analytically what happens.

Now, this function is differentiable on the whole of $\mathbb{R}$; and its derivative is $1 + \cos x$. Then we should find its critical points. So, when is it 0? $1 + \cos x = 0$ implies $\cos x = -1$. One solution of this is $x = \pi$. Due to periodicity there will be $2n\pi$ added to it. So, all critical points are here: $(2n + 1)\pi$ for $n \in \mathbb{Z}$; that is for any integer n. This will have your π, 3π, 5π and so on on the one side; and on the other side you have $-\pi$, -3π, -5π and so on. All these points are the critical points of this function.

We must find out whether there is a maxima or minima at these critical points. So, you have to look for the change of sign in f'. What happens for f'? You take f' at any of these points, that is 0. Now, suppose you take any x which is not of the form $(2n + 1)\pi$. Then it will be 1 plus cosine of that something c, which is not equal to $(2n + 1)\pi$. But $\cos c$ always remains between -1 to 1; this lies between -1 to 1. And this is equal to $1 + \cos c$; it will lie between $1 + (-1)$ to $1 + 1$; that is, between 0 and 2. So, $1 + \cos c$ is always bigger than 0. Then, $f(x)$ is increasing throughout. I it is increasing throughout, it does not have a local maximum or a local minimum. So, even these critical points are not points of extremum, that is what we conclude. That is, this function has neither local maxima nor local minima.

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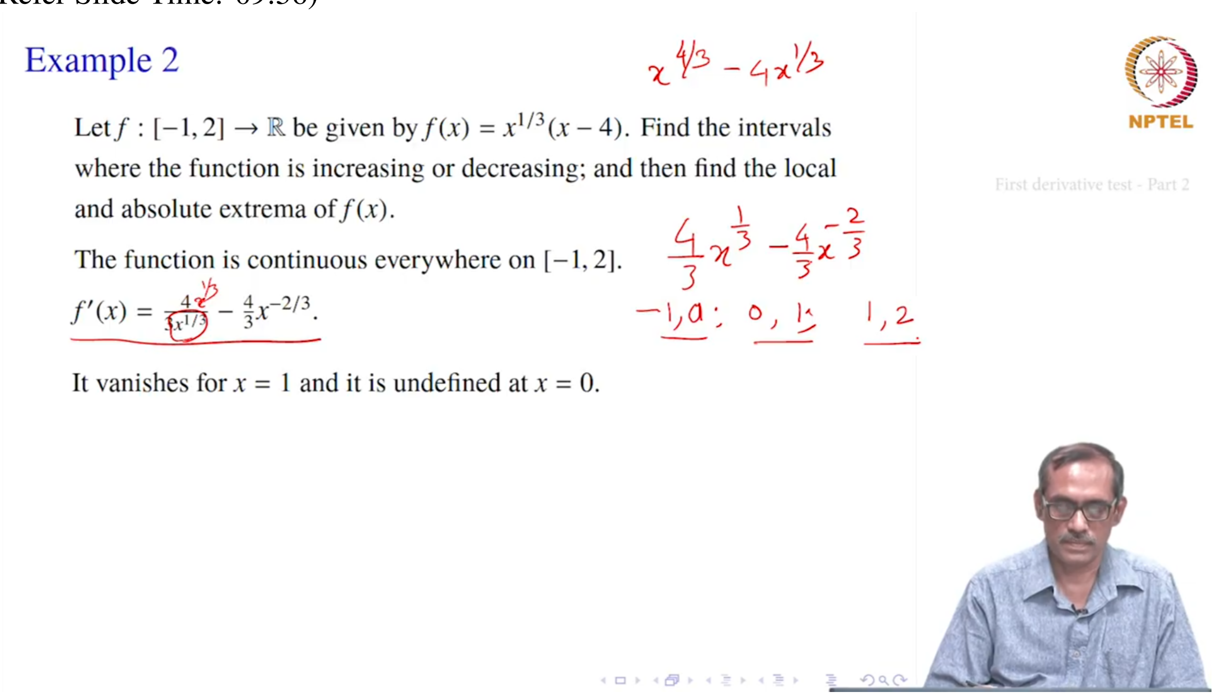

Let us take another example. Suppose f is a function defined on the closed interval $[-1, 2]$ given by $f(x) = x^{1/3}(x - 4)$, which is $x^4/3 - 4x^{1/3}$. This is also defined everywhere. (Negative also it is negative, positive it will be positive, $1/3$ is there; and $x - 4$ takes care of the other factor, further it is less than or greater than 0.) Now, we are asked to find the intervals where the function is increasing or decreasing; and basing on that we need to find the local maxima/minima and the absolute maxima/minima of this function $f(x)$.

First, we take the derivative. It is $f'(x) = (x^{4/3} - 4x^{1/3})'$. The first term gives $(4/3)x^{4/3-1} = (4/3)x^{1/3}$. This $1/3$ should be here; and it is $4/3$ times $x^{1/3}$. The second term gives $(1/3)x^{1/3-1} =$

$(1/3)x^{-2/3}$. So $f'(x) = (4/3)[x^{1/3} - x^{-2/3}]$; that is how it looks. Where is it equal to 0; what are the critical points? You see that this is on the top. It vanishes for $x = 1$ and it is not defined at $x = 0$.

So, the critical points are now two, not only one. These are $x = 0$, where $f(x)$ is not differentiable; and $x = 1$, where $f'(x)$ is zero. We have these two critical points. Of course, the function is continuous on the closed interval $[-1, 2]$; so, we can apply our earlier reasoning. These two critical points divide the interval $[-1, 2]$ into three sub-intervals: $[-1, 1]$, $[1, 0]$ and $[0, 2]$. Since there are two critical points, you get three sub-intervals.

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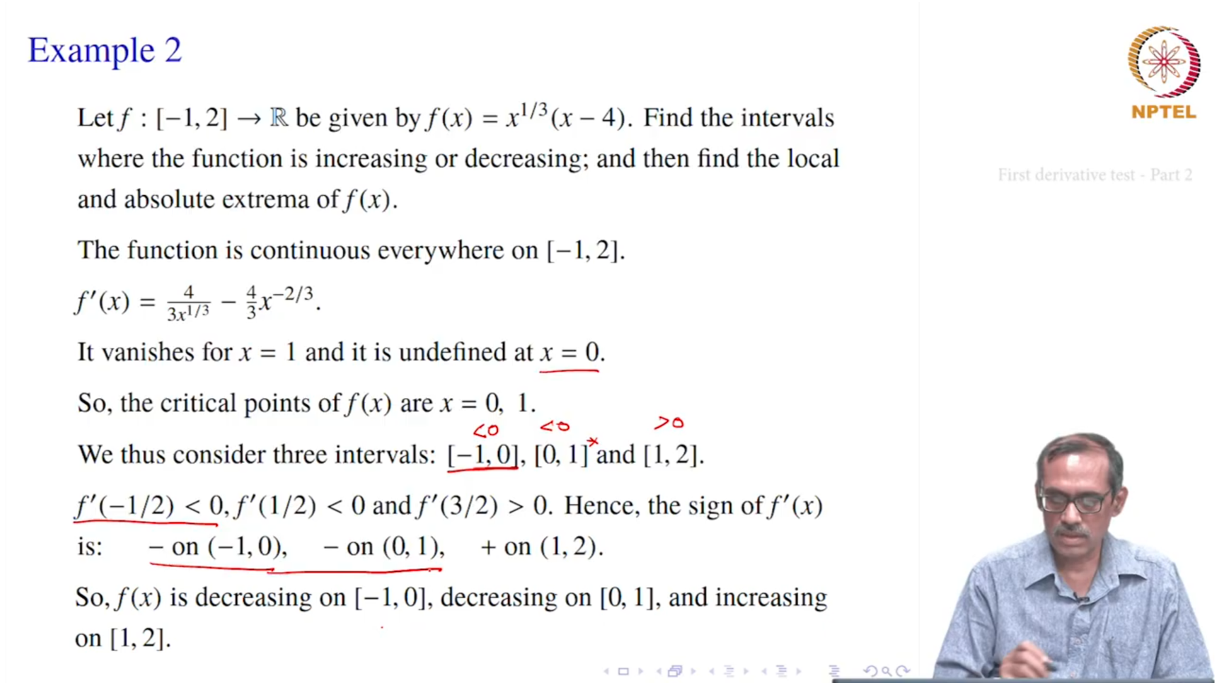

You consider these three intervals and find out whether the function is increasing or decreasing in these intervals. In $[-1, 0]$ the function has the same behavior; f' is continuous except at that point 0, where it is not defined. If $f(x)$ is increasing, then it is increasing throughout the interval; if it is decreasing, it is decreasing throughout the interval. That behavior can change from intervals to intervals. Inside $(-1, 0)$, we find that $f'(-1/2) < 0$. That gives us that f' is less than 0; it is negative in this interval $(-1, 0)$. Of course, you can directly argue now. Similarly, $f'(1/2) < 0$; so f' is less than 0 on $(0, 1)$. But, $f'(3/2) > 0$; so f' is greater than 0 on $(1, 2)$. There is only change of sign; it is at 1; there is no change of sign at 0. This change of sign happens like this; f' is $-$ on $(-1, 0)$, $-$ on $(0, 1)$, and $+$ on $(1, 2)$; whereas f is not defined at 0.

So, $f(x)$ is decreasing on $(-1, 0)$, decreasing on $(0, 1)$, and increasing on $(1, 2)$. That means $f(x)$ is decreasing throughout $(-1, 1)$, (f' is not defined at 0), and increasing on $(1, 2)$. So, it does not have a local extremum at $x = 0$ because there is no change of sign at $x = 0$. There is no change of sign here; it is same minus throughout, from -1 to 1 itself, except that 0; neither from minus to plus nor from plus to minus, when you go from left side of 0 to right side of 0. But, at $x = 1$ there is some change; f' is changing sign from $-$ to $+$ at $x = 1$. So, there is a point of local minimum of $f(x)$ at $x = 1$ since f is decreasing on the left of 1, and increasing on the right of 1.

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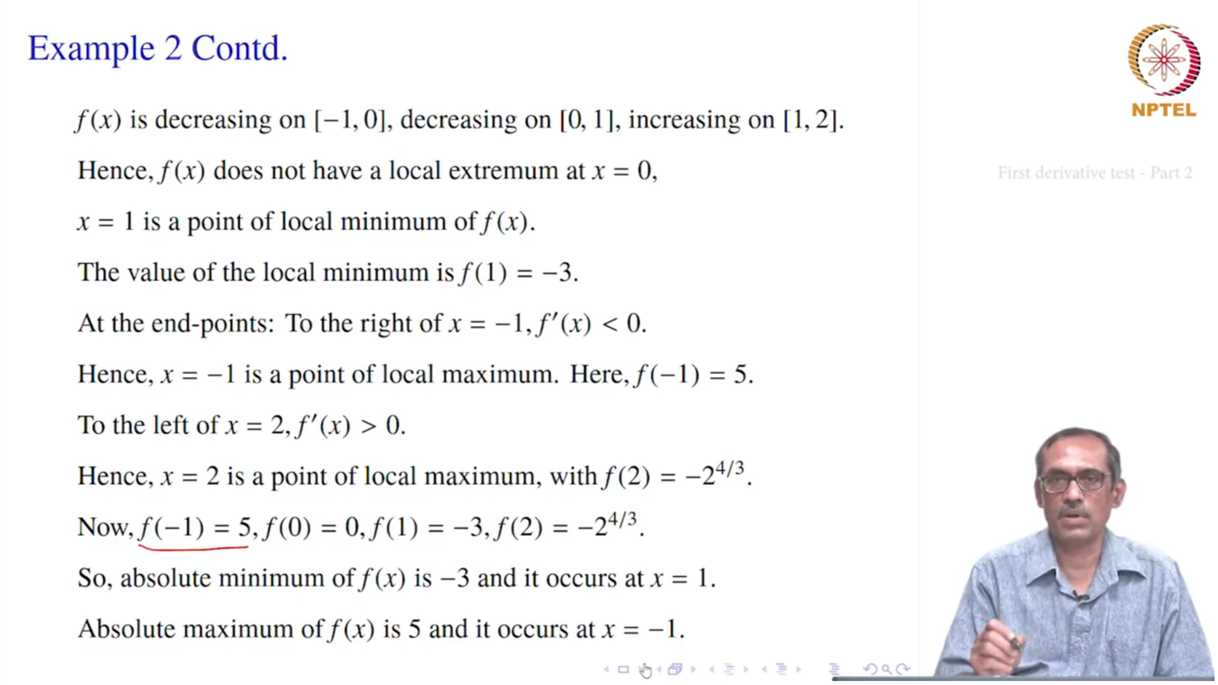

So, f has a local minimum at $x = 1$, and the value of this local minimum is of course $f(1)$, which is -3. We also have to consider the endpoints. It is a closed interval; its endpoints are -1 and 2. At -1, only a right neighborhood is relevant; left neighborhood is not relevant. So, what we do? We look to the right of $x = -1$. To the right of it, it is $(-1, 0)$, where the function is decreasing; that is f' is less than 0. Since f' is less than 0, it is decreasing. So, f is decreasing from its value at $x = -1$. That is, $f(x)$ has a local maximum at $x = -1$; that is what it says. And the local maximum value is $f(-1) = 5$. You just substitute -1 there to get it.

What about the other endpoint? We have the other endpoint as 2. At $x = 2$, only a left neighborhood is relevant. And on the left neighborhood $(1, 2)$, f' is bigger than 0; so f is increasing. It is increasing like this to $f(2)$. Again, there is a maximum there, a local maximum at $x = 2$. And its value is, of course it is not simple as earlier; it is $-2^{4/3}$.

Now, we can compare all these values, whatever maximum or minimum values we have got; and then we find out their absolute maximum or absolute minimum. For comparison, let us list them out. They are

$$f(-1) = 5, \quad f(0) = 0, \quad f(1) = -3, \quad f(2) = -2^{4/3}.$$

Notice that $f(0)$ is also listed for finding absolute extremum since $x = 0$ is a critical point. Had you just taken the critical points without finding out whether they are local maximum or minimum; you would have also got the same thing for the absolute maximum or minimum. But, the endpoints are not critical points; and they will have to be taken in the list and evaluated. Here, the critical points are 0 and 1, and endpoints are -1 and 2. We evaluate $f(x)$ at all these points and compare them. We find that the absolute minimum is -3, which occurs at $x = 1$; and the absolute maximum is 5, which occurs at -1.

That is how we will be proceeding to solve problems basing on information of the first derivative; coming to increasing decreasing nature of the function; and then thinking about maxima and minima. Let us stop here.