Basic Calculus - 1
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Lecture 4
Absolute Value - Part 2

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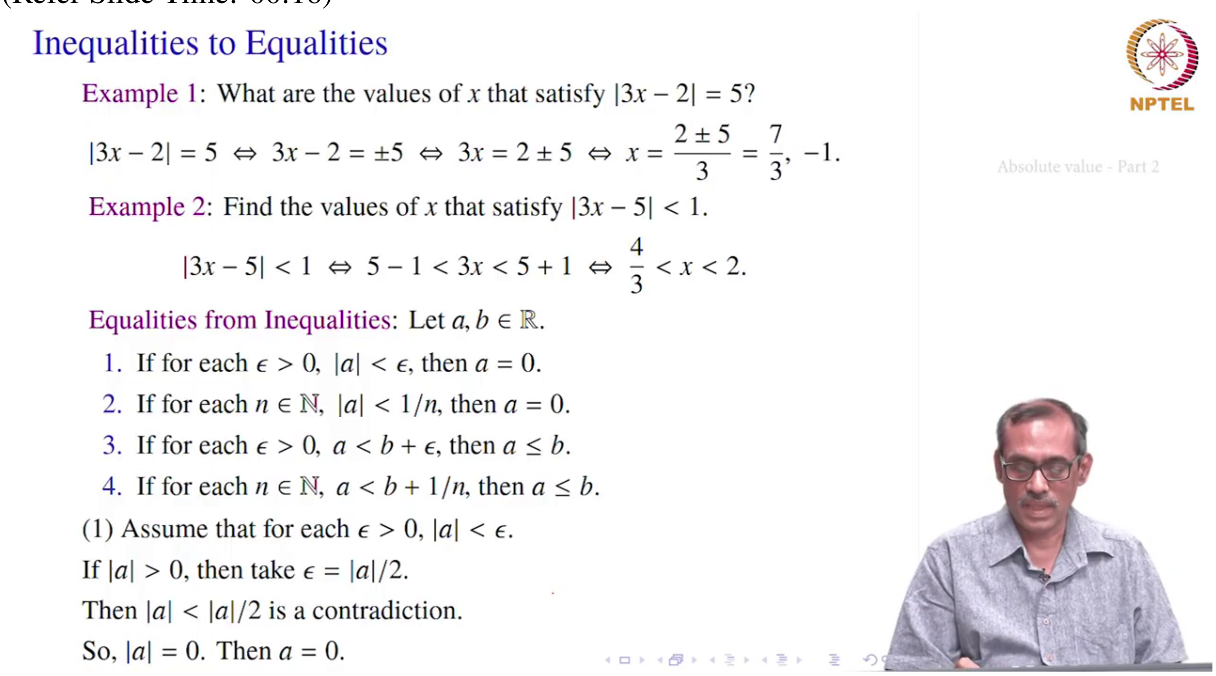

Now let us see, how to use these inequalities to prove equalities. We will see that slowly but let us first find out what does this mean by this equality. Let us find out all values of x that is all real numbers x, which satisfy the equation: $|3x-2| = 5$. Immediately you can think of squaring it : $(3x-2)^2 = 5^2$, and try to solve it. But there can be better ways.

What we do, we know that this modulus is either equal to that or it is equal to minus of that. It means $3x-2$ can be either equal to 5 or equal to -5. That is what we write immediately. Instead of a quadratic we get two linear expressions. Of course, when you solve the quadratic,you would obtain this one, but it is easier directly.

So, $3x-2 = \pm 5$. That will be easier to solve now. It gives two possibilities: $3x = 2+5 = 7$ or $3x = 2-5 = -3$. Then, you divide by 3 to get $x = 7/3$ or $x = -1$.

Similarly, let us find out if instead of equality, we have an inequality. That is, find the values of all x that is all real x, that satisfy $|3x-5| < 1$. Now, you can use your sixth property, which we mentioned earlier: $|x-a| < \delta$ means x belongs to the δ-neighborhood of the point a. We will use that and see what happens. Now, $|3x-5| < 1$ if and only if, $1-5 < 3x < 1+5$. These are all the possibilities. You simplify it to get $4/3 < x < 2$.

These are just simple things telling how we get equalities and how do you get inequalities with modulus. In fact, there is a way in calculus. Sometimes we find it is very difficult to prove equalities but probably easier to prove some inequalities. In those cases you must know how to apply or how to obtain an equality from inequalities. Always we cannot get of course, but there are some cases where this is possible. Let us see those cases.

Suppose for every $\epsilon > 0$, you find that $|a| < \epsilon$. It is a bit difficult to understand. What it says: you have some real number, somewhere, a and you see that its absolute value, absolute value is always on the right side, that is less than ϵ for every ϵ. What will happen to the absolute value instead? Let us consider $|a|$. It is non-negative and that is always less than whatever ϵ you take. So, if this is my $|a|$, now it is less than ϵ. But what is this ϵ? Let me take 1, ϵ is 1 here. I have $|a| < 1$. But I also see that for every ϵ, $|a| < \epsilon$. If I take $\epsilon = 1/2$, $|a|$ cannot lie here; it has to lie here. So this is $|a|$, it can be here. But at $\epsilon = 1/4$ $|a|$ cannot be there; it has to be here. It proceeds that way. It gives a feeling of what might happen to this $|a|$. It says that $|a|$ cannot be anything positive, it has to be 0. If it is anything positive we can always choose another ϵ which is smaller than that; but it should have been smaller than that ϵ. Exactly that is the proof.

So let us see how to proceed; we will come back to this. First let us prove. Assume that for each $\epsilon > 0$, $|a| < \epsilon$. We are now trying to prove 1; and will come back to 2, 3, 4 in a minute. So, assume that for each $\epsilon > 0$, $|a| < \epsilon$. We want to prove that $|a| = 0$. But $|a|$ can be either equal to 0 or can be greater than 0; it cannot be negative. So, the other case remains is $|a| > 0$. Suppose on the contrary that $|a| > 0$. We are really using that geometry we have discussed. We have 0 here, now $|a| > 0$, so $|a|$ is here. We choose our ϵ to be something smaller than $|a|$. Let us say, $\epsilon = |a|/2$. For this ϵ what will happen? $|a|$ has to be less than ϵ; that means, $|a|$ should be less than $|a|/2$. But there is contradiction. Therefore, $|a|$ cannot be greater than 0. So, $|a|$ must be equal or to 0. And $|a| = 0$ gives $a = 0$.

That is how it answers our question; that if $|a| < \epsilon$ for every ϵ, then a has to be equal to 0.

Now let us look at the second statement. Here, we do not have every $\epsilon > 0$, but it says something about natural numbers. It says that $|a| < 1/n$ for every natural number n as we have done there. That means $|a| < 1$, $|a| < 1/2$, $|a| < 1/3$, $|a| < 1/4$ and so on. Then, it says that $|a|$ cannot be positive; it has to be equal to 0.

If I were able to feel it correctly, then all it says that, we are not given the same inequality for every epsilon, but only for some very particular epsilons; they are still infinity in number and they are in the form $1/n$ for natural numbers n. Of course, you can use Archimedean principle. If you have any $1/n$, if you have epsilon positive, then you can choose one n such that $1/n < \epsilon$. So, this really should give rise to the answer. In fact we will prove that through Archimedean principle. Let us see, how to go about it.

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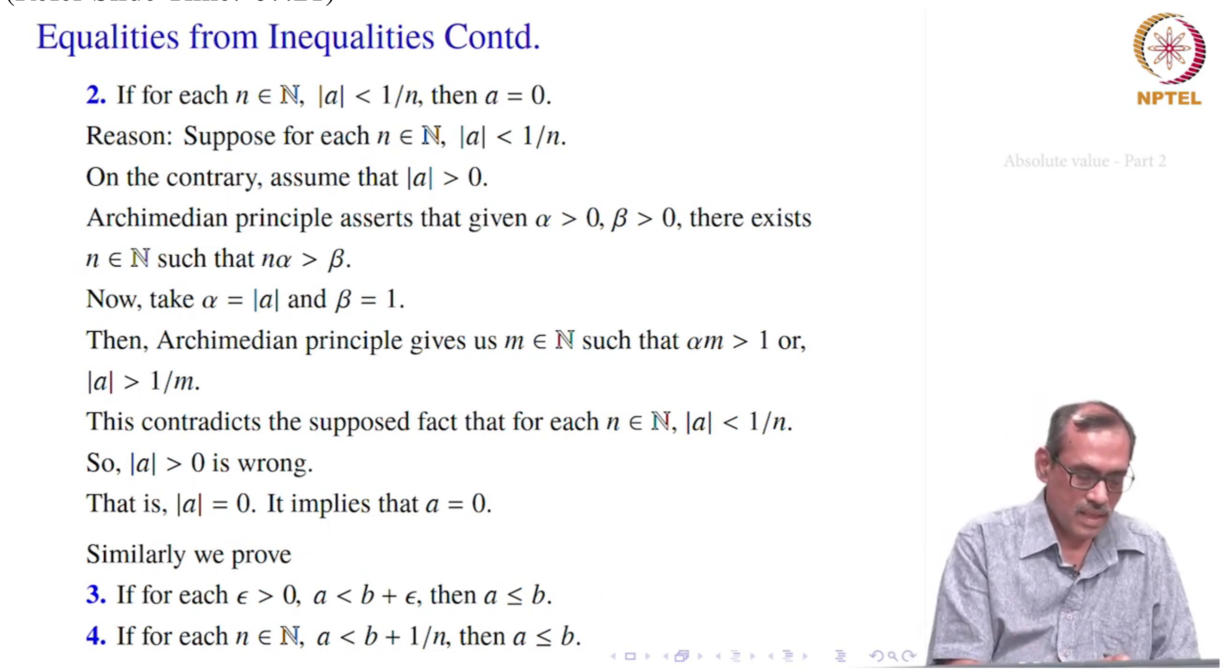

If for each n, $|a| < 1/n$, then we want to show that $a = 0$. We assume that for each n, $|a| < 1/n$. Our aim is to show that $a = 0$. As our experience shows we should try to show $|a| = 0$. But $|a|$ cannot be negative; of course it can be positive. So, let us assume the contrary that $|a|$ is positive. We use the Archimedean principle. Recall what it says: given any two positive numbers say α and β, you can always find one natural number n such that $n\alpha > \beta$. How to use that here? Well, we take $\alpha = |a|$ and $\beta = 1$. That means you will get one natural number m such that $m\alpha > 1$. Or if you write it: $m|a| > 1$, or, $|a| > 1/m$. So, we have a natural number m such that $|a| > 1/m$. But that is a contradiction, because $|a|$ must be less than $1/n$ for every n. How can you find one m for which $|a? > 1/m$? That is the contradiction. This contradiction shows that $|a| > 0$ is wrong. So, $|a| = 0$, and therefore $a = 0$.

Similarly, you can show the other properties, which I am again reproducing it here. You can prove that: for every $\epsilon > 0$, if $a < b + \epsilon$, then $a \leq b$. Now you are not looking at mod but directly real numbers a and b. It is says that $a < b + \epsilon$ for every ϵ. So you as well think of every ϵ very near 0; it is less than everything bigger than b, so it cannot be bigger than b.

If $a > b$, then suppose a is here and b is here. Our condition $a < b + \epsilon$ will not be satisfied for some ϵ. How can you produce such an ϵ? Now that $a < b + \epsilon$ is contradicted; because b is smaller than a, a is bigger. So, what I do: a should be less than $b + \epsilon$. If I take ϵ to be larger than this distance, then a is again smaller or equal to that ϵ. So I take something here: this is my $b + \epsilon$. Now what it says, is $a > b + \epsilon$; but a should be smaller than $b + \epsilon$. So, that is how you have to choose your ϵ, something like $(a - b)/2$, which when added to b will be still smaller than a and that will be the proof.

So, the feeling matters. What happens here is: if $a < b + \epsilon$, then ϵ can be made smaller and smaller so that a has to be less than or equal to b; it cannot surpass b, it cannot be bigger than b, that is what it says. Similarly, the fourth one says that instead of ϵ, take any n; if $a < b + 1/n$ for each natural number n, then $\leq b$.

These are the four things. In one of these forms it will be useful for showing that something equal to something or something is less than or equal to something. Whereas we can produce only less than, a is less than this, $|a|$ is less than this. In that case we can conclude that $a = b$ or $a < b$ and so on. So, these are about equalities and inequalities.

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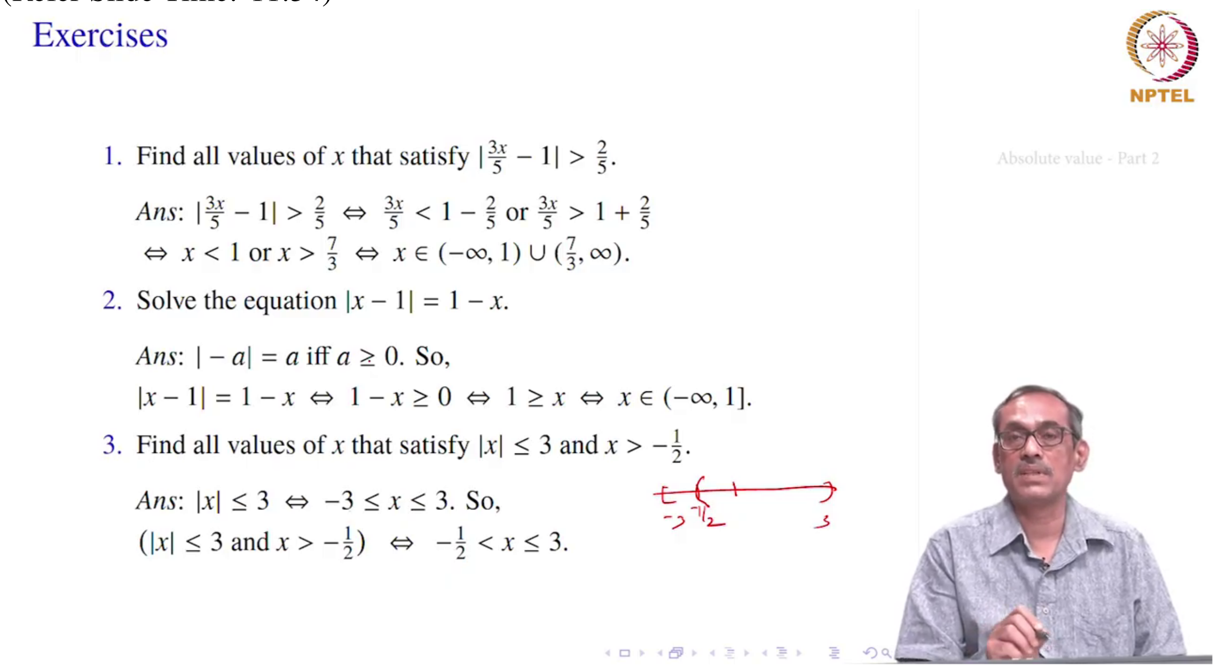

Now, let us use what we have learnt. The question is you have to find all x that satisfies this inequality: $|3x/5 - 1| > 2/5$. You can square both the sides, but we have done better. We know what is the meaning of $|x - a| > \delta$. This means x lies in the complement of the closed δ-neighborhood. That means x has to bigger than $a + \delta$ or x is smaller than $a - \delta$. Let us use that.

Now, $|3x/5 - 1| > 2/5$. We use that. It says $3x/5 < 1 - 2/5$ or $3x/5 > 1 + 2/5$. Now we solve both the things, and go on putting them all. The first one gives $3x/5 < 3/5$, or $3x < 3$, or $x < 1$. The second one gives $3x/5 > 7/5$, or $3x > 7$, or $x > 7/3$. That means all these x satisfy this condition $x < 1$ or $x > 7/3$. We can write that in terms of intervals, of course. That is, $x \in (-\infty, 1) \cup (7/3, \infty)$; that is how it will look.

Now, let us go for the second problem. It asks us to solve the equation $|x - 1| = 1 - x$. Again, you can square it, but do we need it? What we need to use is $|-a|$; because this is $x - 1$ and on the right is $1 - x$. This $1 - x$ is really $-(x - 1)$. It reminds us the equality that $|-a| = a$ if and only if $a \geq 0$. Here, that means $x - 1$ must be greater than equal to 0, or $1 - x$ is greater than 0? Which one we are using? $|-a| = a$ if $a \geq 0$. If you take the other way, you would get $|a| = -a$. Then, $a \leq 0$. In that case, a will be $x - 1$; but we are using this $|-a| = a$ if and only if $a \geq 0$. Here our a is $1 - x$. That gives the inequality that $1 - x$ must be greater than or equal to 0. And that directly gives us $1 \geq x$. So, where is x? That means $x \leq 1$. If this is 0, this is 1, then x can be anywhere

here. That is what we say $x \in (-\infty, 1]$; that is about the equality.

Now let us take the third one. It asks us to find all real numbers x, which satisfy these two conditions. The first condition is $|x| \leq 3$ and the next condition is $x > -1/2$. So, where do we start? One condition is nice, $x > -1/2$, easy to tackle. But the other one comes to $|x|$. We should see first the meaning of $|x| \leq 3$. Again we go back to our earlier thing: $|x - a|$. Here, $|x - 0| \leq 3$. If you have remembered the earlier ones, it directly gives you x belongs to the δ-neighborhood of 0. So, $-3 \leq x \leq 3$. Of course that is clear. Here you have 0 and you have 3, then the distance would be 3. So, the distance is 3 means, on this side is 3 and on this side is -3, so x must be in between this; that is what it says.

Now, you have to combine both. Once we have the other one: $x > -1/2$, we would get $|x| \leq 3$ and $x > 1/2$; that is same thing as telling x is between -3 to 3 and $x > -1/2$. That means it is really this one: x can be here: $-1/2 < x < 3$.

All that we have covered today is about the absolute value, and how absolute value is used in the equalities by shifting the origin. The main concern was that whether $|x - a| = \delta$ or $|x - a| < \delta$ or $|x - a| > \delta$. Then using all these we try to find out the relation between the neighborhoods and the absolute value. So, let us stop here.