Basic Calculus - 1

Professor. Arindama Singh
Department of Mathematics
Indian Institute of Technology Madras

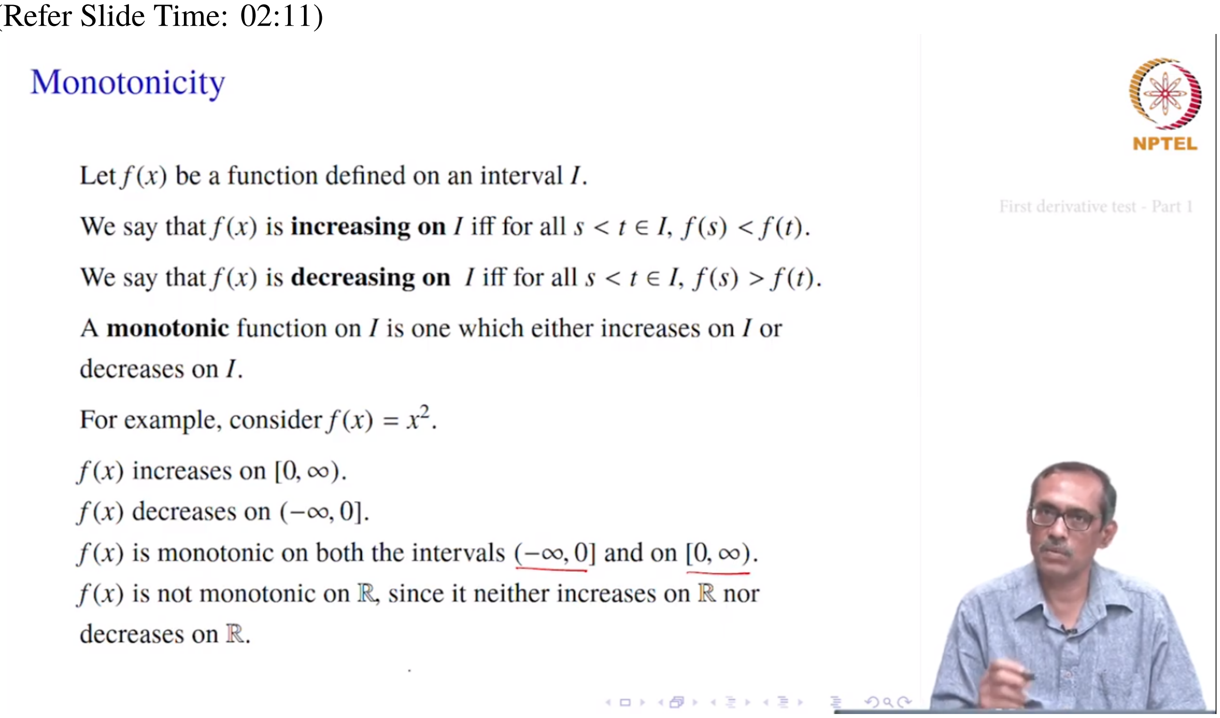
Lecture 20 - Part 1

First Derivative Test - Part 1

This is Basic Calculus-1, lecture 20. If you remember, to compute maxima and minima in a problem, we had found out the critical points and then we referred to the picture of the function or the graph of the function to conclude that at that critical point there is a local maxima. At that point of time, we did not have analytic tools to determine whether a critical point is a point of maximum or a point of minimum.

Today, we will use the first derivative of the function to deduce whether a critical point is an extreme point, and if it is so, then what kind of extreme point it is. Is it a local maximum or a local minimum? The idea is very easy. For example, if you have some curve, where there is a maximum at a point c, then you see that to the left of it, the function is increasing and to the right of it, the function is decreasing. So, you must know first how to use the information on the derivative at that point c to conclude whether it is increasing in the neighborhood of that c or it is decreasing in the neighborhood of c. Specifically, we will need something like on the left side it should be increasing, on the right side it is decreasing, or if it is a minimum point then in the left side it should be decreasing and on the right side it should be increasing. So, you need to know something about increasing and decreasing nature of functions, basing on the idea of the derivative. Let us do that first.

(Refer Slide Time: 02:11)

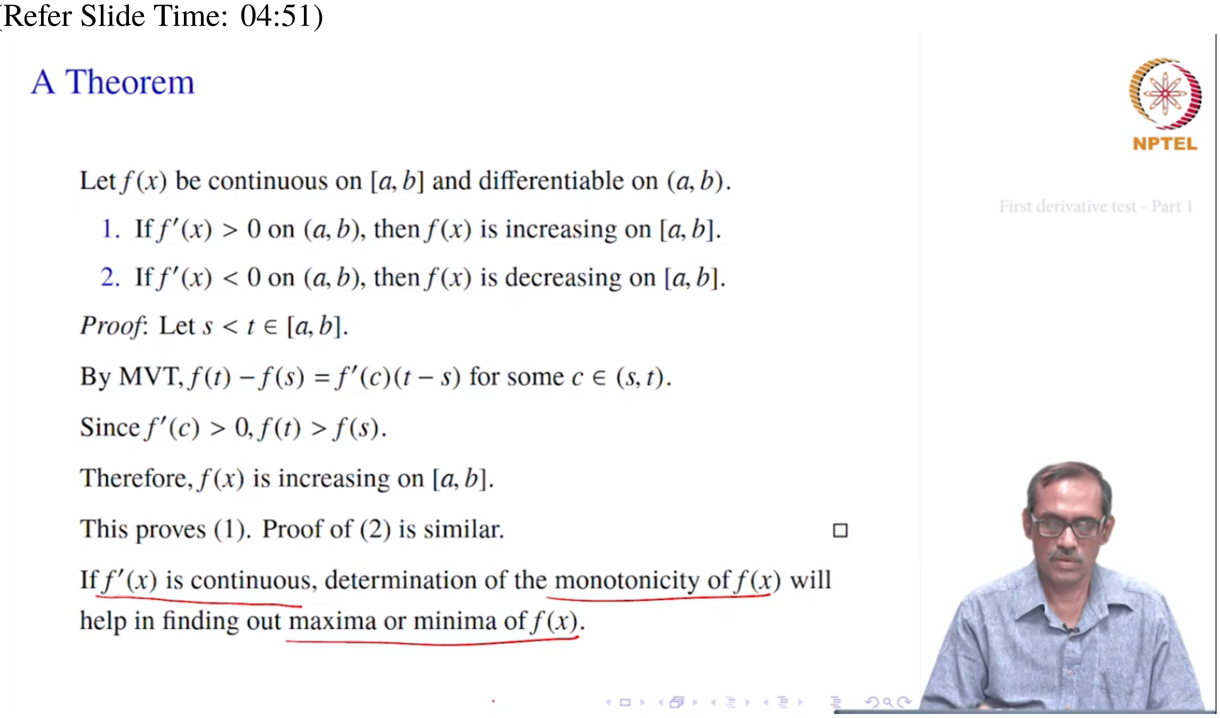


The relevant notion is called monotonocity. Suppose f is a function, defined on an interval I; its domain is I and co-domain is $\mathbb{R}$, of course. For this function, when do we say that it is increasing on I? You could have taken a subset J of that I, and say that it is increasing on J. We are right now taking directly on I. Now, $f(x)$ is increasing on I if for any two points s and t inside that interval I, if $s < t$, then you should have $f(s) < f(t)$. Notice that So, t is to the right of s, and $f(t)$ should be towards the top of $f(s)$. That is how you say f is climbing up or it is increasing.

Similarly, you will say that $f(x)$ is decreasing on I, if $s < t$, that is t is to the right of s, but then you have $f(s) > f(t)$. So, f is climbing down, or it is decreasing. And, if it is either increasing or decreasing (any one of them), then we will say that it is monotonic. That is, it is monotonic on I if either it increases on I or it decreases on I.

For example, take $f(x) = x^2$. It is defined on the whole of $\mathbb{R}$. At 0, it is 0, but when it is negative, this x^2 is positive, and when it is positive, x is positive, x^2 is also positive. What happens when it is on the right side of 0, that is, x is any non-negative number? Then, x^2 would look something like this. It increases. When you choose x from $(-\infty, 0]$, that is x is not positive, it is either negative or 0, then $f(x) = x^2$ looks like this; it is decreasing. So, it is increases on $[0, \infty)$ and decreases on $(-\infty, 0]$. It is monotonic on $(-\infty, 0]$ and also it is monotonic on $[0, \infty)$. Monotonic means, it is one of them, increasing or decreasing. What happens on the whole of $\mathbb{R}$, on the interval $(-\infty, \infty)$? It neither increases nor decreases on the whole of $\mathbb{R}$. So, it is not monotonic on $\mathbb{R}$, but it is monotonic on these two sub-intervals. At one place it is decreasing, at another place it is increasing.

(Refer Slide Time: 04:51)



Our guiding principle will be a result for using the information on derivative to say whether it is increasing or decreasing. Suppose we have a function f which is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f'(x) > 0$ on the open interval

(a,b), (because there only f; is defined), then $f(x)$ is increasing on the whole of $[a,b]$. How come these a and b are cropping up? Because continuity is taking care at a and at b.

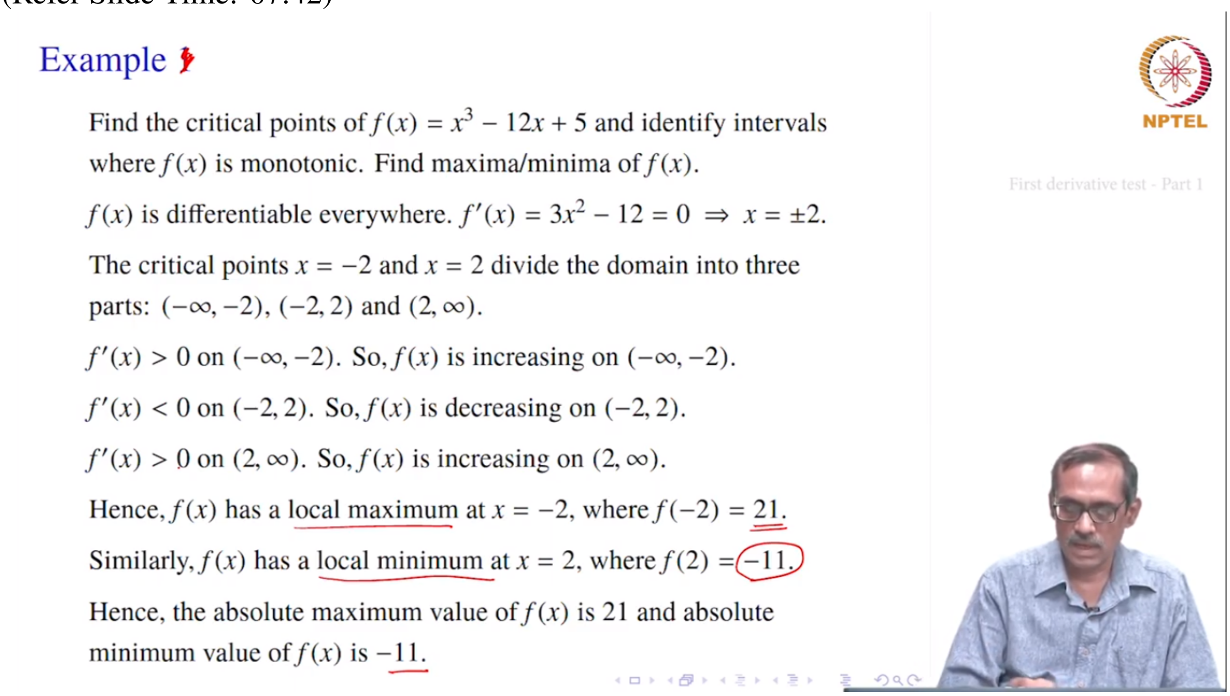
So, if $f'(x) > 0$ on (a,b), it is positive, then $f(x)$ is increasing on $[a,b]$. Of course when f' is positive, its tangent will be something like this. So, the curve will be increasing. At any point you will see, tangents will be this way. That is so intuitively. This is what we will see. And, if $f'(x) < 0$ on (a,b), then $f(x)$ will be decreasing on the closed interval $[a,b]$.

Let us take (1). How do we prove it? Suppose $s < t$, both are in the closed interval $[a,b]$. We will use the Mean Value Theorem. It satisfies the conditions of the Mean Value Theorem. Our t is to the right of s. Its conclusion is $f(t) - f(s) = f'(c) \times (t - s)$, where c is some point between s and t, thus also inside (a,b). We will use the information that $f'(x) > 0$ on (a,b). In particular at c, it is positive. Since this is positive and $t - s$ is positive, we have $f(t) - f(s)$ is positive. That is, $f(t) > f(s)$. That is the end of the proof. It shows that $f(x)$ is increasing on $[a,b]$.

Similarly, if you take the other one: $f'(x) < 0$, then again from the Mean Value Theorem, you will see that $f'(c)$ is negative, this is positive, so the right side is negative. Therefore, $f(t) < f(s)$; and f is decreasing. This is easy.

There is another thing. We have not assumed continuity of f' anywhere. In addition, if f' is continuous, then it will help us in finding out the maxima and minima of the function. We will see that soon.

(Refer Slide Time: 07:42)



Let us take an example for illustrating this result. Consider the function $f(x) = x^3 - 12x + 5$. It is asked to find the critical points of this, find out where the function is increasing, where the function is decreasing; and basing on that, if possible, find the maximum and minimum values of $f(x)$ and the points where they occur. All these things are to be obtained.

First, we find the critical points. This $f(x)$ is defined everywhere on $\mathbb{R}$. So, every point is an interior point. Also, $f(x)$ is differentiable over $\mathbb{R}$. Recall that critical points are those where it is not differentiable or where the derivative is 0. Here, the latter condition is applicable. So, what do we do?

We find out its derivative. It is $3x^2 - 12$; that is it. To find the critical points, we equate this to 0 and compute where it can be 0. You find that at $x = \pm 2$, $f'(x)$ is 0. So, we have now -2 is a critical point, and 2 is also a critical point.

That divides the whole of $\mathbb{R}$ into three sub-intervals. One is $(-\infty, -2)$, another is $(-2, 2)$ and th third is $(2, \infty)$. We are going to find out whether f' has the same sign over all these intervals. If that is so then it will be either decreasing or increasing on those intervals leaving these 2 points of course, the critical points.

Now what happens? On $(-\infty, -2)$, $f'(x) > 0$. Why? $f'(x) = 3x - 12$; once you take anything which is smaller than -2, its square will be bigger than 4, $3x^2 - 12 = 3(x^2 - 4)$ will be positive. That is why $f'(x)$ is positive on $(-\infty, -2)$. That means $f(x)$ is increasing in the interval $(-\infty, -2)$.

Similarly, consider $f'(x)$ for whether it is positive or negative inside the interval $(-2, 2)$. You see that $f'(x) = 3(x - 2)(x + 2)$. If you take any x inside the interval $(-2, 2)$, then one of $x - 2$ or $x + 2$ will be positive, and the other will be negative. In fact, $x + 2$ will be positive, and $x - 2$ will be negative. Therefore, $f'(x)$ is always negative for every point x between -2 to 2. Hence, $f(x)$ is decreasing on $(-2, 2)$. That is clear.

Let us consider the next interval. Suppose we choose any x bigger than 2, x is from $(2, \infty)$. Then x^2 will be again bigger than 4. So, $3x^2 - 12$ will be positive. That is, $f'(x)$ is positive on $(2, \infty)$. Therefore, $f(x)$ is increasing in the interval $(2, \infty)$.

Let us look at the summary. It is increasing on $(-\infty, -2)$, increasing from $-\infty$ to $f(-2) = 21$; it is decreasing on $(-2, 2)$, decreasing from $f(-2) = 21$ to $f(2) = -11$ and it is increasing on $(2, \infty)$, increasing from $f(2) = -11$ to ∞.

To the left of the point -2, $f(x)$ is increasing and to its right $f(x)$ is decreasing. Recall the definition of local maximum: on a left neighborhood it should be increasing, and on a right neighborhood it should be decreasing. That is what happened at -2. Therefore, $f(x)$ has a local maximum at $x = -2$. If we evaluate $f(x)$ at $x = -2$, then it is $-8 + 24 + 5$, that gives you the value 21. So, $f(-2) = 21$ is a local maximum value of $f(x)$. That means it will be something like this, it is increasing to 21. That is how it looks.

Then what about the point $x = 2$? Again, to the left of it is the interval $(-2, 2)$, where it is decreasing; and to the right of it, it is increasing. So, $f(x)$ has a local minimum at $x = 2$. We substitute again to see that the local minimum value is $f(2) = -11$.

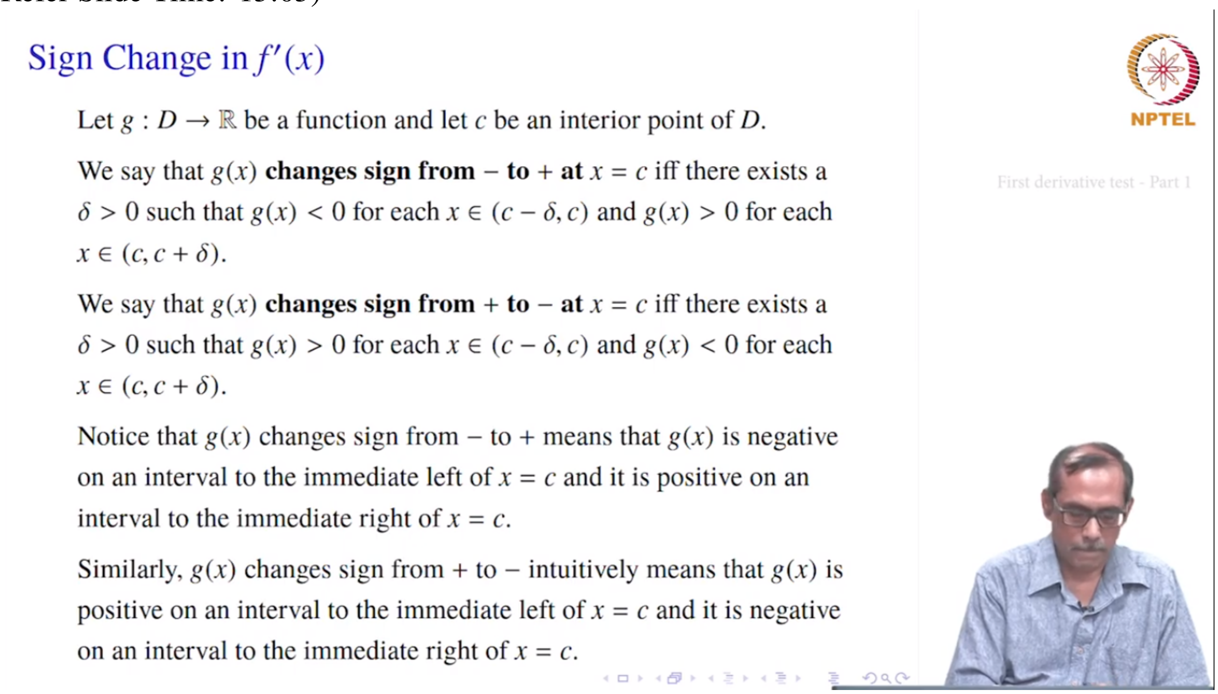
So, from 21, at -2, it decreases to -11 at $x = 2$. It Looks something like this, but not exactly. It is of course a cube, so it comes like this. These are the two critical points: -2 and 2. At one place it has a local maximum, which is at minus 2, and another place it has local minimum, which is at 2.

Now, you can compare the values. Of course, there is nothing to compare here. There is one maximum, and one is minimum. However, these are local minimum and local maximum. Sometimes a local minimum can be larger than a local maximum. But here what happens is the local maximum is $f(-2) = 21$, which is larger of these two values. So, it is the absolute maximum. And, the absolute minimum is $f(2) = -11$.

That settles finding out maxima and minima. we have also identified the intervals where it is monotonic. It has different behavior in different intervals. We also found out the local extrema. The local maximum is also the absolute maximum and the local minimum is also the absolute minimum.

This is how we are going to use the information on the derivatives, but we will make a shortcut soon. Let us try that.

(Refer Slide Time: 15:05)



We have seen that when $f'(x) > 0$, $f(x)$ increases. At a local maximum, f increases on the left and decreases on the right. So, on the left $f'(x) > 0$ and on the right, $f'(x) < 0$. That is what we have found out. That means if f' is positive to the left of the critical point and negative to the right of the critical point, then at the critical point it is 0 of course, and at that critical point some sign change is happening in $f'(x)$. Let us see what does that mean.

Suppose you have a function $g : D \to \mathbb{R}$ and let c be an interior point of D. At the interior points only you are going to consider the critical points. We say that $g(x)$ changes sign from − to + at $x = c$ if there is a left neighborhood of $x = c$, say, $(c - \delta_1, c)$, where $g(x)$ is less than 0, and there is a right neighborhood of $x = c$, say, $(c, c + \delta_2)$, where $g(x)$ is greater than 0. On the left of c it is −, and on the right of c it is +. Then you say that it is changing sign from − to + at $x = c$. Of course, at $x = c$ it is 0 if g is continuous, because of Intermediate Value Theorem.

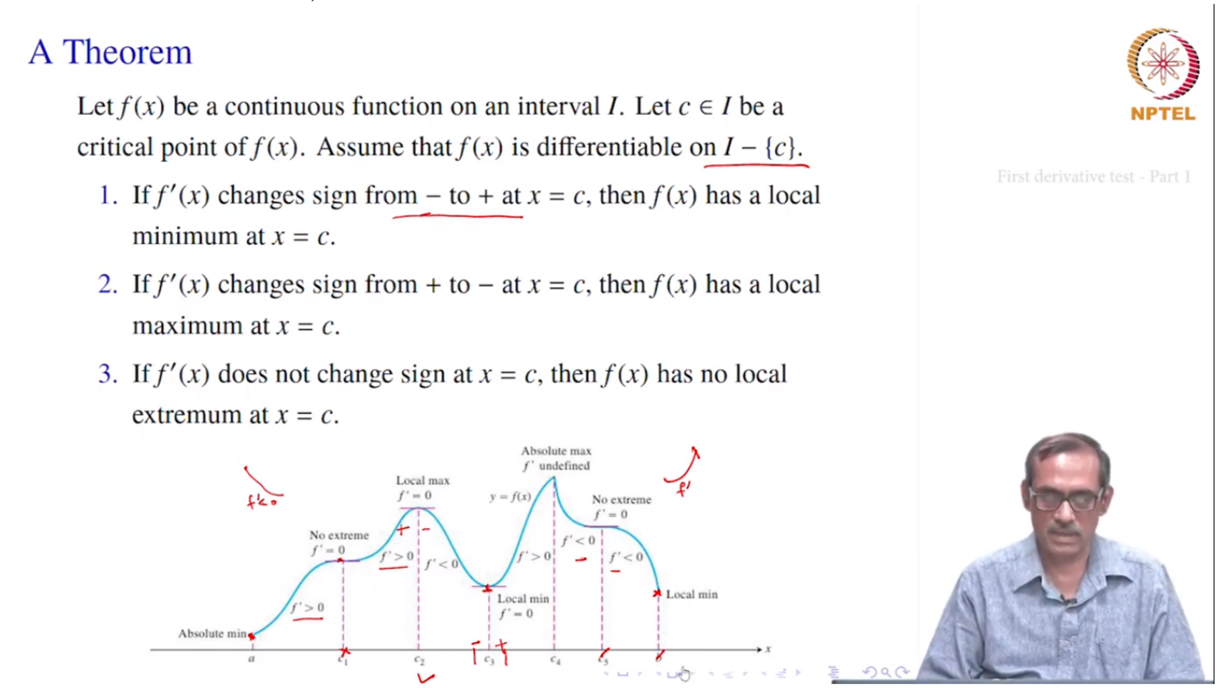
We say that $g(x)$ changes sign from + to − at $x = c$ if the inequalities are changed; that is, in the left neighborhood $(c - \delta, c)$ we have $g(x)$ greater than 0 and in the right neighborhood $(c, c + \delta)$ we have $g(x)$ less than 0. It thus changes sign from positive to negative. That is about g.

We just defined what is the meaning of 'changes sign'. This is intuitively obvious. But this δ

there is not ‘every δ’; it is for some δ; that gives some neighborhood in which this thing happens.

So, $g(x)$ changes sign from $-$ to $+$ means $g(x)$ is negative on the interval that is to the immediate left of c, and it is positive to the immediate right of c. Similarly, it changes sign from $+$ to $-$ means on the immediate left of c it is positive and on the immediate right of c, it is negative.

(Refer Slide Time: 17:41)



How we are going to use this information? Let us look at a function f given like this. Look at the picture. There, f' is equal to 0 at a point, say, $f'(c) = 0$. Here, what happens is f' is less than 0 throughout this. But all that we need is a neighborhood $(c - \delta, c)$. Here, f' is negative on the left of c and f' is positive on the right of c. That means f' is changing sign from negative to positive at c. If that happens, look at point (1) above, f' changes $-$ to $+$ at $x = c$ then $f(x)$ has a local minimum at $x = c$. That is what we see from the picture. So, what are the conditions for this to be satisfied?

Of course, all this analysis will be correct provided we have continuity, differentiability, and so on. So, we should formulate that. Let $f(x)$ be a continuous function on an interval I. Let us say $c \in I$ is a critical point of $f(x)$. Assume that $f(x)$ is differentiable everywhere, except possibly at c. If at c it is not differentiable that is also a critical point. So, at least on $I \setminus \{c\}$ it is differentiable. If $f'(x)$ changes sign from $-$ to $+$, then $f(x)$ has a local minimum at $x = c$. And, if $f'(x)$ changes sign from $+$ to $-$ at $x = c$, then $f(x)$ has a local maximum at $x = c$. Look at the picture. There, f' is changing sign from $+$ to $-$ at $x = c_2$, so, there is a local maxima at $x = c_2$.

Now if f' does not change sign at $x = c$, then $f(x)$ has no local extremum at $x = c$. It is possible. Let us look at the point c_1. Before the point c_1, $f' > 0$, also after it $f' > 0$. So, at that point, there is no local extremum. If it does not change sign at $x = c$, then it does not have a local extremum at that point. Similar thing happens for the other side. Look at the point c_5. There, f' does not change sign. To the left of this point c_5, f' is negative, and to the right also it is negative;

so, there is nothing there.

But we can say something about the endpoints. What happens here? On the endpoints, you do not have the notion of two-sided neighborhood; only one side neighborhood is there. So, on the left endpoint, for example at a, there is no left neighborhood which would be relevant to the domain of the function, but to the right there is a neighborhood, say, $(a, a+\delta)$. Similarly, at the right endpoint, we have a left neighborhood, and we do not have a right neighborhood.

In this case, we would say that there is a local minimum at $x = a$ if f' is positive after a. That means f is increasing after this; so it becomes minimum. Similarly, at b we can say it is a local minimum if f is decreasing on the left of b. The other things can also happen. For example, the function comes like this at a, then this function will be decreasing here and f' will be less than 0 at a. That is also possible. Similarly, at b it is possible that f' is positive, so it is increasing; then a local maximum can happen there. So, for the end points, we will consider only one sided neighborhoods whichever is relevant. That is what it says. Basing on this left side, and right side thing, we should formulate our conclusion about the local maximum, local minimum.