**Basic Calculus - 1**
**Professor. Arindama Singh**
**Department of Mathematics**
**Indian Institute of Technology Madras**
**Lecture 19 - Part 1**
**Using Rolle's Theorem and Mean Value Theorem - Part 1**

This is lecture 19 of Basic Calculus-1. Recall that in the last class we discussed Rolle's Theorem and Mean Value Theorem; and also isslustrated those with examples. Today we will have an extension of that. We will not go to any new concepts; we will solve some more problems basing on these results. There will be more examples, and some exercises.

It is better to use the exercises this way. Whenever an exercise or a problem comes, you can read the problem and pause the video for sometime; solve it yourself; and then come back to the solution. That will be better for you. Of course, you can do the same for examples, but you have the freedom; you can read those because they are meant to be examples.

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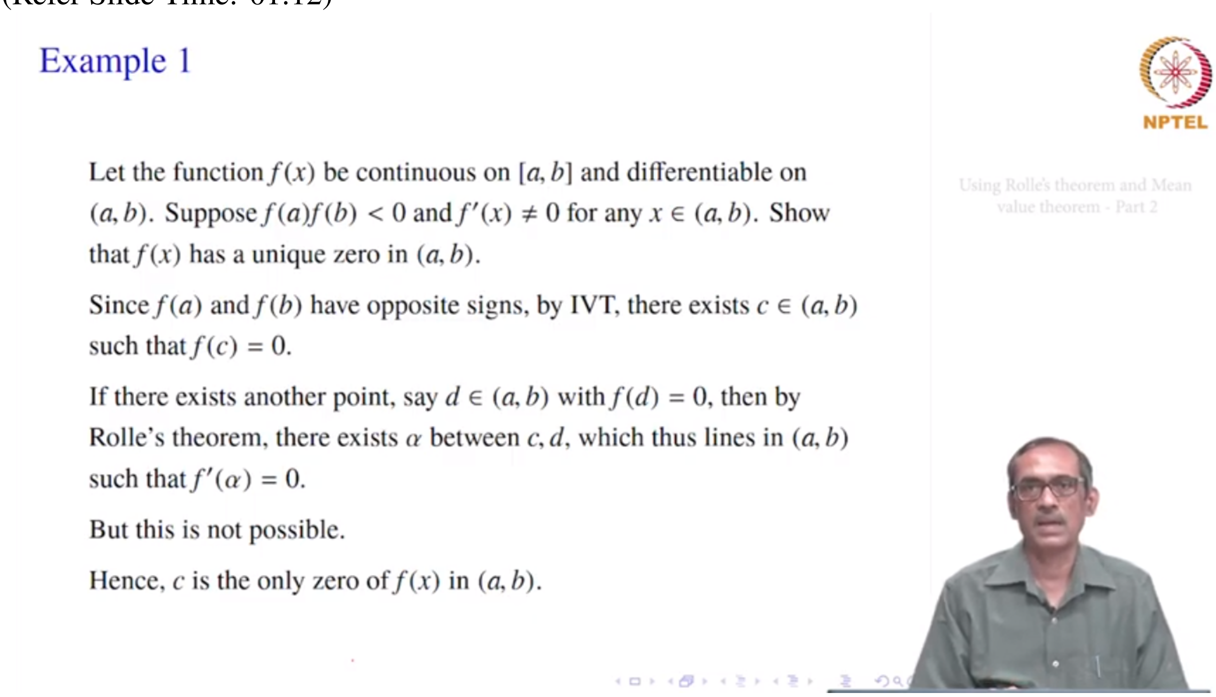


With this in mind, we start with our first example. Let $f(x)$ be a function; it is continuous on $[a, b]$. That is, $f : [a, b] \to \mathbb{R}$; it is continuous. Also, it is given that $f(x)$ is differentiable on the open interval $(a, b)$. These are the assumptions in the Rolle's Theorem, as you remember. But in Rolle's Theorem, we have $f(a) = f(b)$, which is not given here. It is given that $f(a)f(b) < 0$; which means they have opposite signs. If $f(a)$ is plus, $f(b)$ is minus, and if $f(a)$ is minus then $f(b)$ is plus so that that product is less than 0. And, it is also known that $f'(x) \neq 0$ for any $x \in (a, b)$. So, it does not vanish at any point of $(a, b)$.

We need to show that $f(x)$ has a unique zero in the closed interval $[a, b]$. The first thing we have to show is that $f(x)$ has a zero; that is, there is a point $c$ such that $f(c) = 0$. And, the second thing we have to show is that if $d$ is another point, say, $c < d$, then there is something wrong; or with any such $d$, you have to show that $c = d$. One of this you have to show for uniqueness. Let us see.

Now that $f(a)$ and $f(b)$ have opposite signs, we can use Intermediate Value Theorem on the interval $[a, b]$ to get one point $c$ where $f(x)$ is equal to 0. At some point it is negative, at some point it is positive; so in between somewhere 0 must be achieved; that is what the Intermediate Value Theorem says. That shows that there exists a point $c$ such that $f(c) = 0$. Now you must show also uniqueness. For that this condition $f'(x) \neq 0$ will be helpful; and we may use the Mean Value Theorem or Rolle's Theorem. I think, Rolle's Theorem is easier.

So, suppose there exists another point, say $d \in (a, b)$ with $f(d) = 0$. So $c$ and $d$ are two different numbers now; may be, $c < d$ or $d < c$. Now, $f(x)$ satisfies Rolle's Theorem with $f(c) = f(d) = 0$. Think of $f$ on the closed interval $[c, d]$, which is a subset of $(a, b)$. On this closed interval $[c, d]$, $f$ is continuous and $f$ is differentiable on the open interval $(c, d)$. Then, Rolle's Theorem implies that there exists a point $\alpha$ between $c$ and $d$ where $f'(\alpha) = 0$. But we know that $f'(x) \neq 0$ inside $(a, b)$. So, that is the contradiction. Therefore, there does not exist another point. So, $c$ is the only zero of $f(x)$ inside the interval $(a, b)$. See the way we have used Rolle's Theorem.

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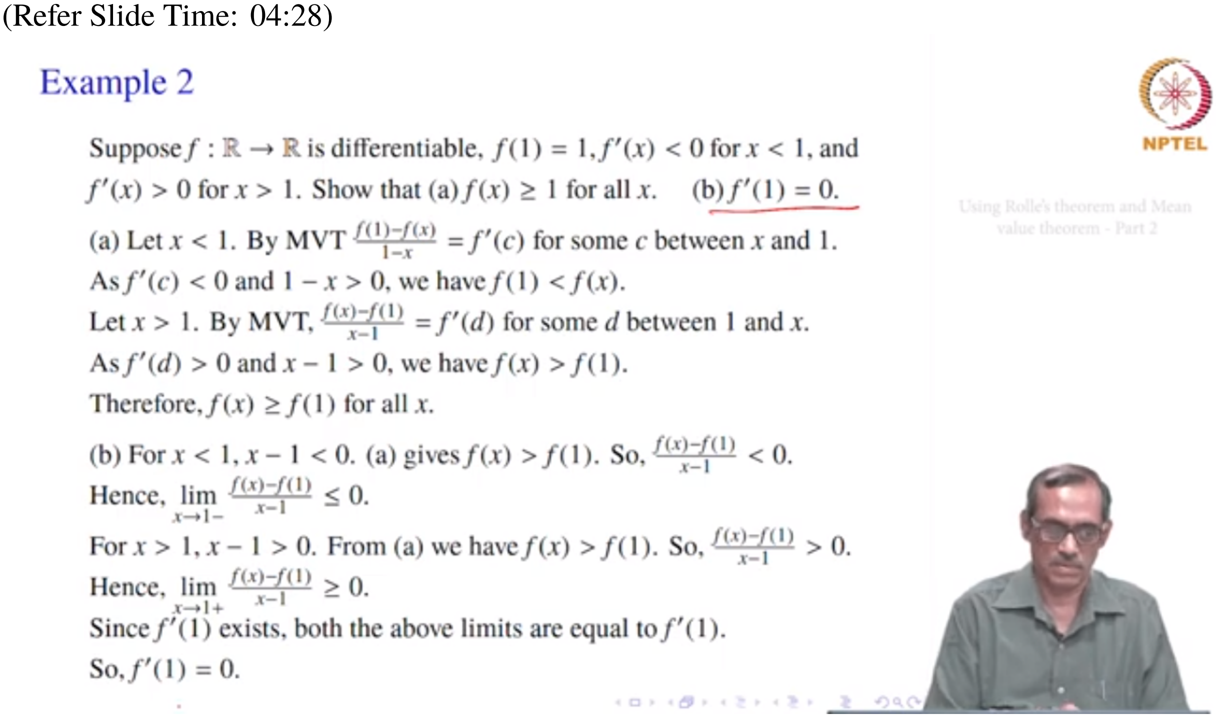


Let us take another example. Here, we have a function which is defined on the whole of $\mathbb{R}$. So, $f$ is a function from $\mathbb{R}$ to $\mathbb{R}$ and it is known to be differentiable. Further, we have some more information: $f(1) = 1$ and $f'(x)$ is 0 for $x < 1$. That is, at $x = 1$, it is equal to 1 and to the left of it $f'$ is 0. See, there are two different information. One information is about $f$, its functional value at 1, and the other one is talking about $f'$, that is, $f'(x) = 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$. It is negative to the left of 1, and positive to the right of 1.

We have to show two things. First, it asks or suggests that $f(x) \geq 1$ for all $x$, and second, $f'(1) = 0$. Now, $f'(1)$ exists because $f$ is differentiable and we have to show that $f'(1) = 0$.

Let us try the first one. The information on $f'$ is given for all points to the left of 1, and again, for all points to the right of 1. Let us consider two different cases.

Suppose $x$ is a point which is less than 1. Now, we use the Mean Value Theorem. See that it satisfies all the conditions in the Mean Value Theorem. Here, suppose $x < 1$. You are considering the closed interval $[x, 1]$, where it will be satisfying the conditions of Mean Value Theorem. Now, it is continuous over that closed interval $[x, 1]$, and differentiable in the open interval $(x, 1)$. So, you can apply the Mean Value Theorem to get a point $c$ between $x$ and 1 such that the the slope of the secant will be equal to the slope of the tangent at $c$. That is, $[f(1) - f(x)]/(1 - x) = f'(c)$ for some point $c$ between $x$ and 1. Now that $f'(x) < 0$ for $x < 1$, this $c$ is between $x$ and 1, this $f'(c)$ must be less than 0. How? We have chosen $x$ to be smaller than 1. So, $1 - x > 0$, it is positive. The ratio $[f(1) - f(x)]/(1 - x)$ is known to be negative. That means the numerator must be less than 0; because $1 - x$ is positive and the ratio is negative. So, numerator must be negative. Now, this gives $f(1) < f(x)$. This is for all those $x$ which are less than 1. We have $f(1) < f(x)$.

Similarly, we will consider $x > 1$. Let us take any point $x$ which is bigger than 1. Again, we apply the Mean Value Theorem on the closed interval $[1, x]$. There, $[f(x) - f(1)]/(x - 1) = f'(d)$ for some $d$ between $x$ and 1, or between 1 and $x$, since $1 < x$. Now again, we consider the same way, which one is negative and which one is positive. Here, we know that $f'(d)$ is positive, and $x - 1$ is positive. Therefore, $f(x) - f(1)$ must be positive. So, it says that $f(x) > f(1)$. As earlier we have got $f(1) < f(x)$, which is same thing as telling $f(x) > f(1)$.

That means at any point $x$, which is smaller than 1 or which is larger than 1, we always have the inequality $f(x) > f(1)$. And, this is about all points $x$ which is not equal to 1. When $x = 1$, what happens to $f(x)$? It is $f(1)$, and that is known to be equal to 1. Therefore, whichever $x$ you may choose from $\mathbb{R}$, $f(x)$ must be greater than or equal to 1. That is our conclusion, so Part (a) is proved.

Now we should go for Part (b). Here, we have to show that $f'(1)$ must be equal to 0. So, what do we do? It is $f'(1)$; it is not given; but $f'$ of smaller than 1 is given, and $f'$ of bigger than 1 is given. Let us choose any point $x$ smaller than 1. As earlier we have seen that $[f(1) - f(x)]/(1 - x)$ is negative. It is same as telling that $[f(x) - f(1)]/(x - 1) < 0$. Now if you take $x > 1$, a similar thing will happen. If you take $x > 1$, then again, we have seen that $[f(x) - f(1)]/(x - 1)$ is positive. So, $[f(x) - f(1)]/(x - 1)$ is again positive or greater than 0 for $x > 1$.

Now, let us look at this. If you take the limit as $x$ goes to $1-$, then $[f(x) - f(1)]/(x - 1)$ is less than or equal to 0, because for all $x$ smaller than 1, the ratio negative. So, in the limit, it can be equal to 0. We thus write: it is less than or equal to 0. If you take the right hand limit, this inequality says that the same limit as $x \to 1+$, from the positive side, the limit of the ratio $[f(x) - f(1)]/(x - 1)$ is greater than or equal to 0.

Now when you say $f'(1)$ exists, it means that both these limits exist and are equal. That is our definition of $f'(1)$. Since they are equal, where one is less than equal to 0, another is greater than equal to 0, we conclude that both of them must be equal to 0. So, $f'(1) = 0$. We need to turn back to our definition of the derivative. It is the limit of $[f(x) - f(1)]/(x - 1)$ as $x$ goes to 1. On the left side, it is less than 0, and on the right side it is greater than 0; limit exists; therefore the limit must be equal to 0. That is the argument we used.

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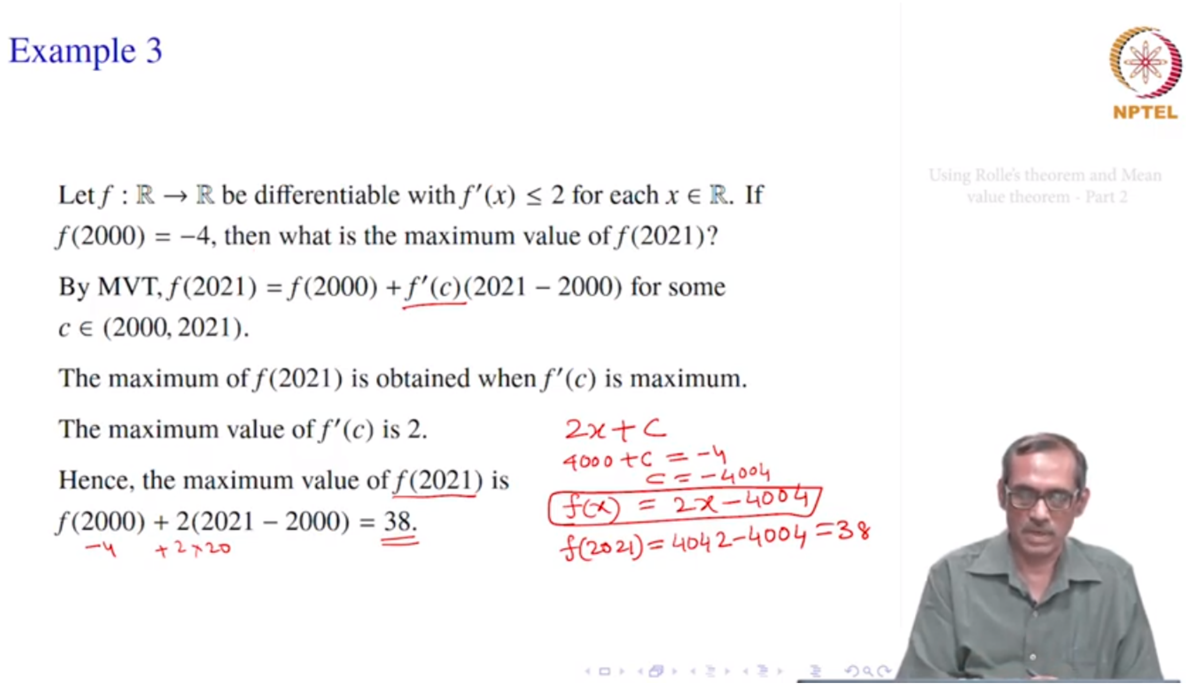


Let us take another example. Here, $f$ is again a real valued function defined over the whole of $\mathbb{R}$; its domain is $\mathbb{R}$. It is given that $f$ is a differentiable function. That is, everywhere it is differentiable. And, its derivative is always less than or equal to 2; that is also given. This is our information. Given that $f'(x) \leq 2$ at every $x$. Also something else is given: $f(2000) = -4$. You want to find out what would be the maximum value of (2021).

Of course the exact maximum value depends on the function. We are really searching for a possible maximum value. So, what is the possible maximum value of $f(2021)$? We can never make it bigger by choosing our $f$, that is the meaning of "what is the maximum value of $f(2021)$"? Given $f$, there is a maximum, but this says something else. It says that whatever $f$ you may take, if $f'(x) \leq 2$ and $f(2000) = -4$, then what could be possibly the largest value of $f(2021)$? How do we proceed?

We apply again the Mean Value Theorem. We want $f(2021)$. Now, $f(2021)$ is equal to $f(2000)$ plus $(2021 - 2000)$ multiplied by $f'$ evaluated at some point $c$ between 2000 to 2021. That is what Mean Value Theorem says. We are applying the Mean Value Theorem for the function $f$, which is defined over the closed interval $[2000, 2021]$. We know it is differentiable in the open interval and it is continuous in the closed interval. So, that is how Mean Value Theorem is applicable. Now that $f'(c) \leq 2$ we can have some estimate for $f(2021)$. You see that $f(2000)$ is some number, $f'(c)$ into (2021-2000) which is a positive number. So, if $f'(c)$ becomes the largest, which is 2 here, then this expression will be largest. So, the maximum is obtained where $f'(c)$

is maximum. Then, maximum means we will take that as 2 itself. With $f'(c) = 2$, we get the maximum value of $f(2021)$. So, we substitute $f'(c) = 2$ here. We get $f(2000)$ plus 2 into this, which is $-4 + 2 \times 21$, which is 38.

But there is some thing to it; it is not yet finished. We say that it is the maximum possible value. Is it really attained? That means whether there exists a function $f$ which satisfies these constraints and that gives $f(2021) = 38$? Of course you can construct an example very quickly because you are taking $f'(x) = 2$ and $f(2000) = -4$. You may see a simple function like $f(x) = 2x$ plus some constant. When differentiated, it gives the value 2. So, let us try $f(x) = 2x + c$. It is just a trial. This would be equal to $-4$ when $x$ is replaced by 2000. Here, this gives $4000 + c$ and that should be equal to $-4$. That gives $c$ equal to $-4004$. That means we take the function $f(x) = 2x - 4004$. Now you see it satisfies all the conditions that $f'(x) = 2$ and $f(2000) = -4$. That is how we construct an example.

Now if I take $f(2021)$, then I would get $2x$ giving 4042 and $-4004$, which is giving 38. You are getting exactly 38. That means this possible maximum value can be achieved by choosing a function suitably, namely, $f(x) = 2x - 4004$. This completes the problem.

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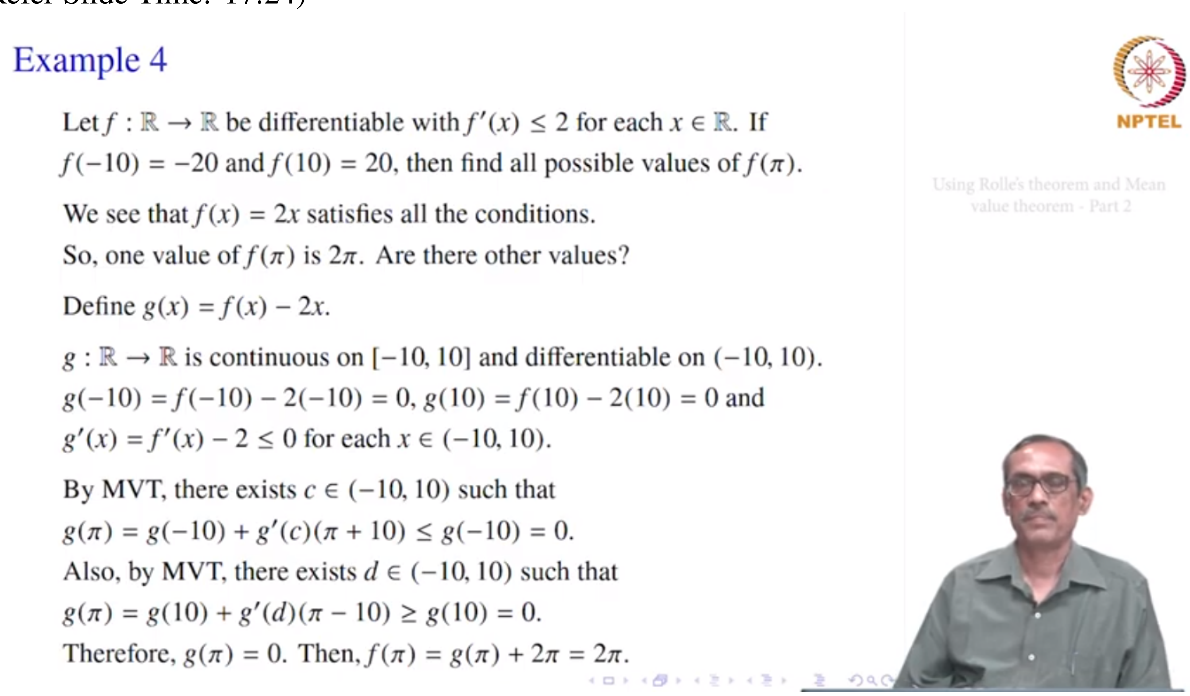


Let us go to the next problem. Here again, the function is defined over the whole of $\mathbb{R}$; it is a real valued function with domain equal to $\mathbb{R}$. It is known, as earlier, that it is differentiable with $f'(x) \leq 2$ for each $x \in \mathbb{R}$. This is exactly the same condition as in Example 3. But we are asked to get something else. It is known that $f(-10) = -20$. This condition is changing. And also, we have $f(10) = 20$. Then we want to find all possible values of $f(\pi)$.

As you have guessed, we can have some $f(x)$, which will satisfy this. It will be equal to $2x$. I think that does. Here, $f'(x) \leq 2$, $f(-10) = -20$ and $f(10) = 20$. So, a possible value of $f(\pi)$ is $2\pi$. It is a possible value; but there can be other possible values. The problem is asking us to find out all possible values of $f(\pi)$. You see that $f'(x) \leq 2$; so it does not look like we will get another value. Let us try and see what happens.

The first thing is, this function $f(x) = 2x$ satisfies all conditions. So, $f(\pi) = 2\pi$ is a possible vale. Now, we want to show that this is the only value. Our guess was $f(x) = 2x$. So, let us define another function $g(x) = f(x) - 2x$. If we can show that $g(x) = 0$ then that would finish the problem. It will conclude that it is the only value. Let us see.

Now, $g(x) = f(x) - 2x$; $g(x)$ is continuous on $[-10, 10]$, and differentiable on $(-10, 10)$. So, you can use the Mean Value Theorem or Rolle's Theorem, whichever one you want. What we have is $g(-10) = 0$, $g(10) = 0$ and $g'(x) = f'(x) - 2 \leq 0$ for each $x \in (-10, 10)$. If you use the Mean Value Theorem, you should get one $c \in (-10, 10)$ such that $g(\pi) = g(-10) + g'(c)(\pi - (-10))$.

See, we have the interval $(-10, 10)$, where $\pi$ lies. By the Mean Value Theorem, we get $c \in (-10, 10)$ such that $g(\pi) = g(-10) + g'(c)(\pi + 10)$. In fact, instead of $c \in (-10, 10)$ we can limit it to $c \in (-10, \pi)$ also. You can think of this closed interval where you apply. So, that gives $g(\pi) = g(-10) + g'(c)(\pi + 10)$. Now that $g(-10) = 0$ and $g'(c) \leq 0$, because $g'(x) \leq 0$ for each x, we have this expression as negative, and $\pi + 10$ is of course positive. So, this part becomes negative. Therefore, this is less than or equal to $g(-10)$, which is 0. That means, $g(\pi) \leq 0$. That is what we get.

Now again, we go to the other side. We have $\pi$ here and 10 here. On this closed interval, if we use the Mean Value Theorem, then you would get $g(\pi) = g(10)$, coming to the other side, plus $g'(d)(\pi - 10)$, where this $d$ is lying between $\pi$ and 10. Now, $\pi - 10$ is negative and $g'(d)$ is also negative. So, this is positive. Then, $g(\pi)$ must be greater than or equal to $g(10)$, which is 0. Here we have $g(\pi) \leq 0$. Here, we get $g(\pi) \geq 0$. Therefore, $g(\pi)$ must be 0.

That is what we wanted. We thought that all possible values boil down to only that value; that $f(\pi) = 2\pi$. We defined $g(x) = f(x) - 2x$ and tried to show that $g(\pi) = 0$. And, that now follows since $g(\pi) \leq 0$ and also $g(\pi) \geq 0$. Therefore, $g(\pi) = 0$. Once $g(\pi) = 0$, we get: $f(\pi) = g(\pi) + 2\pi = 2\pi$.

So, our conclusion is that there is only one such value of $f(\pi)$, which is exactly equal to $2\pi$. Fine?