Basic Calculus - 1
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Lecture 18 - Part 2
Rolle's Theorem and Mean Value Theorem - Part 2

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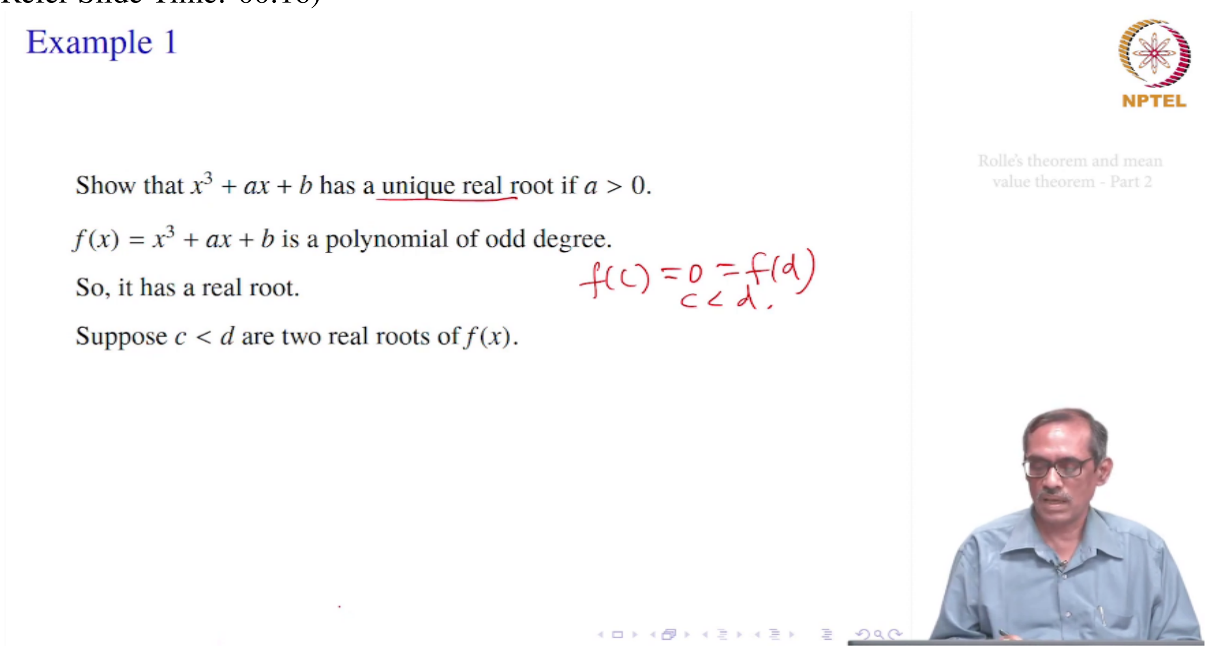

Our first example is this. Suppose you take a function which is defined as $f(x) = x^3 + ax + b$. But we do not pose it as a function now; we pose it slightly differently. We say, that the cubic polynomial $x^3 + ax + b$ has a unique real root, if $a > 0$. That means we consider the polynomial $x^3 + ax + b$, where $a > 0$, and show that it has a unique real root.

Of course, we know by the fundamental theorem of algebra that there will be only maximum of three roots. There is a real root because it is of odd degree; that also we know. But it says there is a unique root. There cannot be two roots, if $a > 0$. So, how do we proceed?

We define a function, say, $f(x) = x^3 + ax + b$. Now, this is a polynomial of odd degree. Therefore, it has a real root. Suppose that there are two roots. We want that root to be unique. So, suppose there are two roots, say, c is a root, d is a root, and $c < d$. That means $f(c) = 0$, $f(d) = 0$ and $c < d$.

Now, we concentrate on the closed interval $[c, d]$; where the function is continuous, and on the open interval (c, d), it is differentiable. So, we can apply our Rolle's theorem. Rolle's theorem says that there exists a point r inside (c, d) where $f'(r) = 0$. We compute $f'(r)$. This says that $f'(x) = 3x^2 + a$. So, $f'(r) = 3r^2 + a$. Now, r^2 is non negative, a is given to be greater than 0; so, $f'(r)$ is always greater than 0. That is, $f'(r)$ cannot be equal to 0. This is a contradiction to Rolle's theorem. That means our assumption that there are two real roots is wrong. That proves that it has unique real root.

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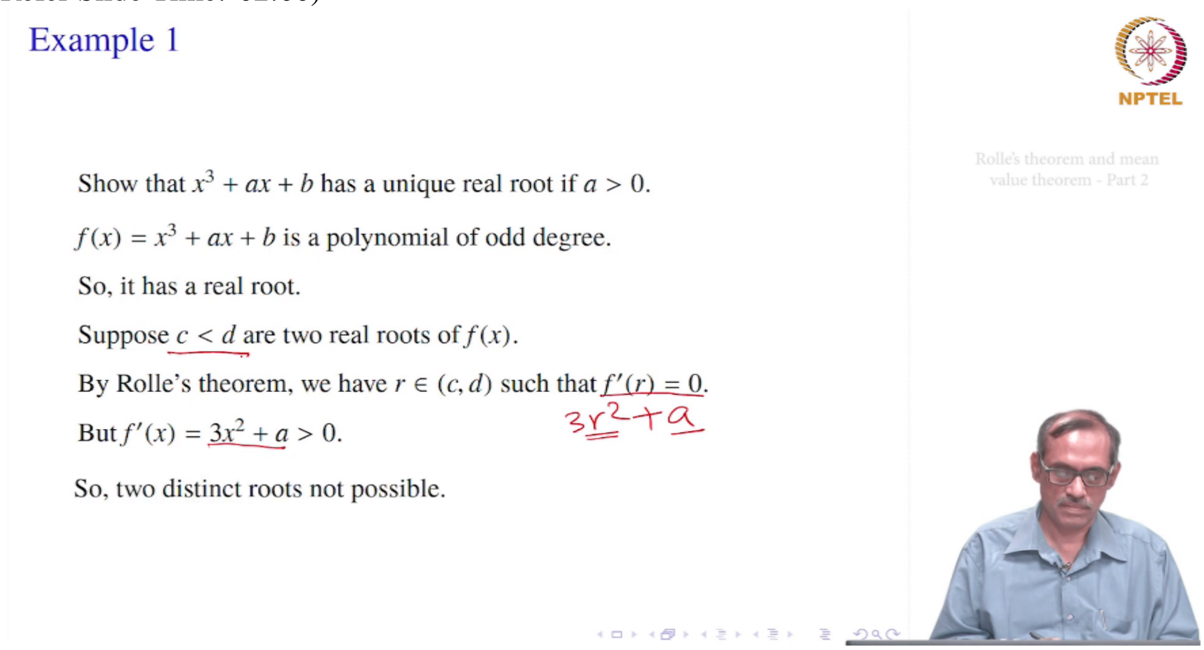

Let us take the second example. It is given here that $f(x)$ has three zeros in the interval $[a, b]$. So, roots and zeros are used synonymously. And that gives a root of an equation and zero of a polynomial or a function. That is why we write $f(x)$ has three zeros in the interval, the closed interval $[a, b]$. That means there are three points, say, $x_1, x_2, x_3 \in [a, b]$ such that $f(x_1) = 0$, $f(x_2) = 0$ and $f(x_3) = 0$.

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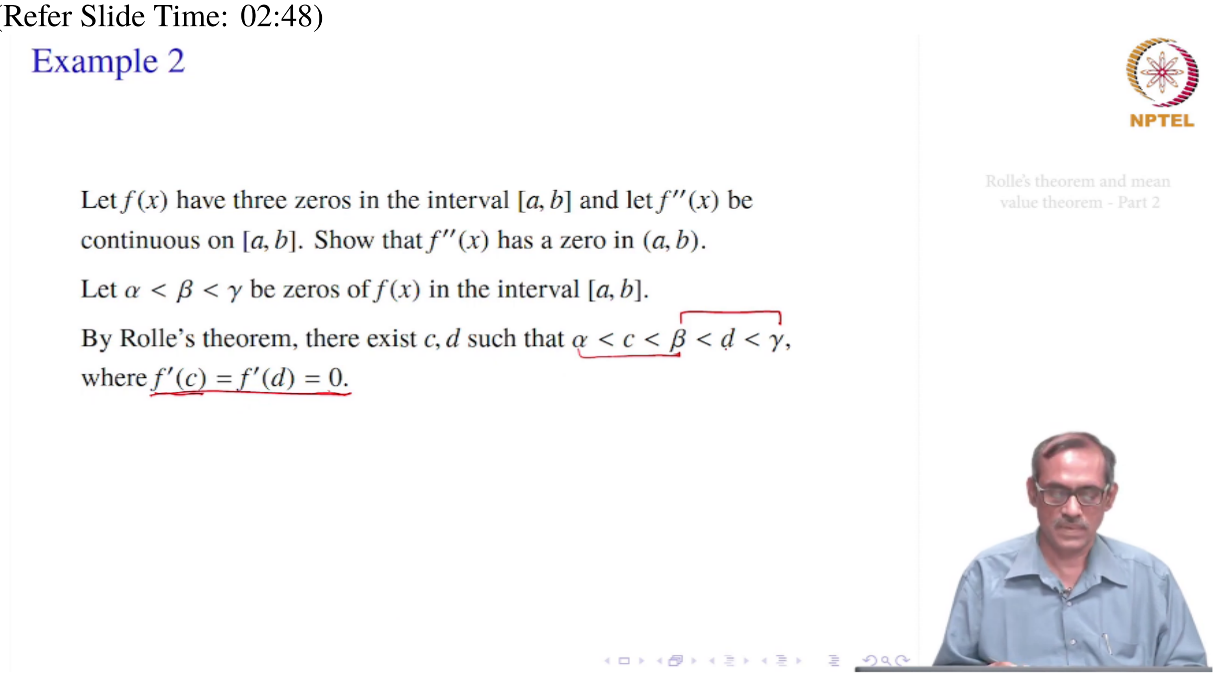

Let us assume that $f''(x)$ is continuous on the closed interval $[a, b]$. We are not taking f', but we are taking f'' and that is continuous on the closed interval $[a, b]$. We want to show that $f''(x)$ has a zero in the open interval (a, b). So, $f''(x)$ is continuous and $f(x)$ has three zeros will imply that f'' becomes 0 at some point inside (a, b). So, how do we proceed?

We may have to apply Rolle's theorem successively. Suppose the three zeros are α, β and γ. They are three zeros. Without loss of generality, we can say that $\alpha < \beta < \gamma$. These are the three zeros inside the interval $[a, b]$. That means, $a \leq \alpha < \beta < \gamma \leq b$.

Now, let us look at Rolle's theorem. It says $f(\alpha) = 0$, $f(\beta) = 0$ and f is continuous in the closed interval $[\alpha, \beta]$, and it is differentiable in the open interval (α, β); so, by Rolle's theorem there is a point c such that $f'(c) = 0$. Similarly, you look at the two roots β and γ. With the same argument, there exists a point d such that $f'(d) = 0$. You have now got two points c and d, $c < d$ inside the open interval (a, b) such that $f'(c) = f'(d) = 0$.

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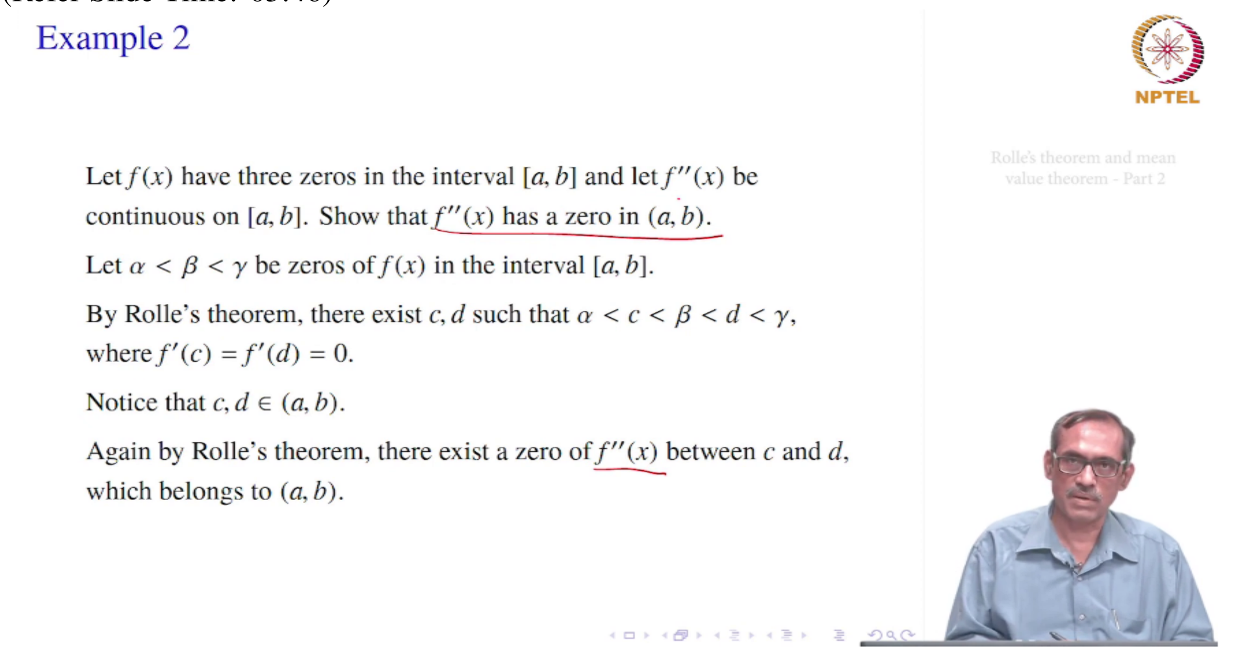

Now, look at f'. It is given that f'' is continuous. So, f' is also continuous on the closed interval $[c, d]$ and f' is differentiable in the open interval (c, d). So, you can again apply Rolle's theorem to conclude that there is a point between c and d, such that f'' becomes 0 at that point, and that point also belongs to the open interval (a, b) because $z < c < d < b$. That proves that $f''(x)$ has a zero in (a, b). That is how we may have to use Rolle's theorem many more times; specially when the order of the derivatives are more, this might be required.

Let us take the next example. Here we are asked to show that there exists a unique number c between -1 and 1 such that this equation is satisfied for that number c; that is, $(1 - c)^{-1} + \sqrt{1 + c} - 3.1 = 0$. Here we may have to compute something. How do you show that there is a point c in between -1 to 1 such that this equation is satisfied? We will go for uniqueness later.

For that what we do is, we think of the function $f(x) = (1 - x)^{-1} + \sqrt{1 + x} - 3.1$, whatever is given there. If I take a point say -0.99, which is inside $(-1, 1)$, and compute $f(-0.99)$, we would get something like, say, it is greater than -2.4. There will be some other points which are left behind here, we may write greater than -2.4. It will not be exactly equal, it may be greater than -2.4. Similarly, if you compute at 0.99, which again belongs to the open interval $(-1, 1)$, we would get that $f(0.99) < 98.2$.

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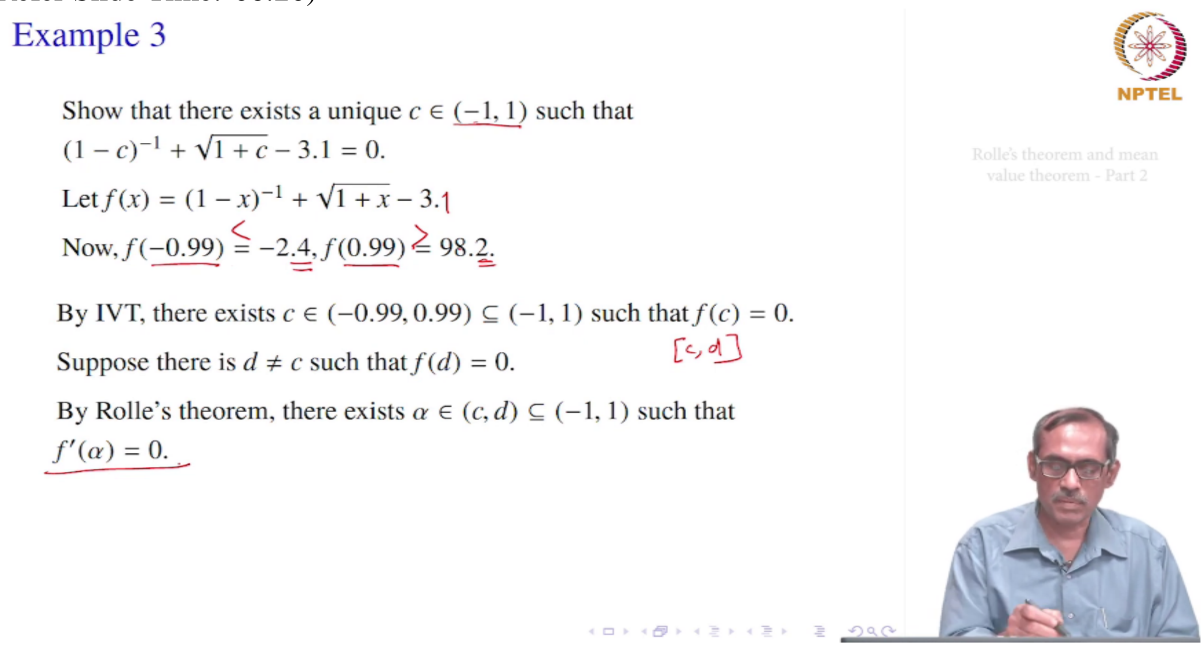

In fact, our computation may give 2.4; so, it is not greater than, it will be less than, since it is chopped off here. And here similarly it is chopped off; so, it is 98.2 plus something. You may get it to be greater than that; this is really less than or equal to this, and this is greater than or equal to this, because something is chopped here from the decimal points.

We see that $f(-0.99)$ is negative and $f(0.99)$ is positive. Therefore, by Intermediate Value Theorem, there exists a point c between this such that $f(c) = 0$. That gives us the existence of a root of this $f(x)$, or a zero of this $f(x)$. So, there exists a point c in between -0.99 to 0.99, which is again in between -1 to 1 such that that equation is satisfied for this c.

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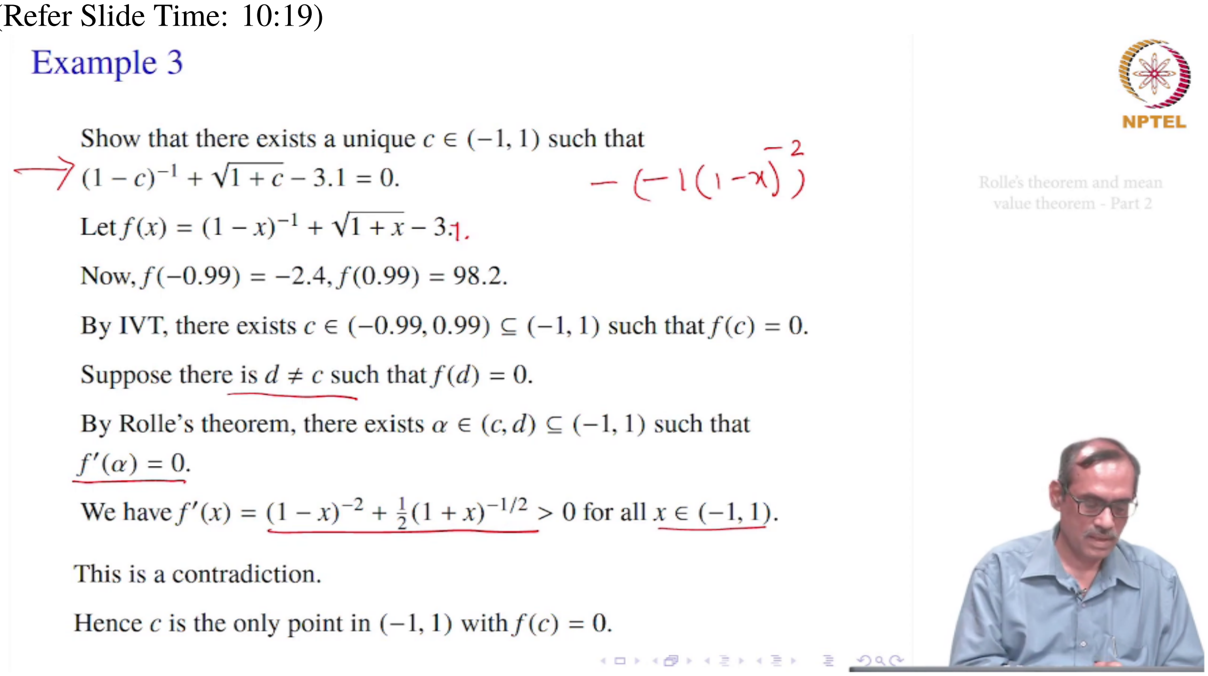

But we want to show uniqueness, that such a point is unique; that there is no other point between

-1 to 1 satisfying the equation. So, suppose that there exists another point, say d, where $f(d) = 0$. We already had $f(c) = 0$; now we also have $f(d) = 0$. Now, it is differentiable function, it is continuous on the closed interval $[c, d]$ and differentiable on the open interval (c, d). You can apply Rolle's theorem. It implies that there is a point α in between c and d, which again belongs to $(-1, 1)$ such that $f'(\alpha) = 0$. We expect that there should be something wrong.

Well, let us take $f'(x)$. Now, $f'(x) = (1-x)^{-2} + (1/2)(1+x)^{-1/2}$. How? When you differentiate it, $(1-x)^{-1}$ gives -1 times $(1-x)^{-2}$ times the derivative of $1-x$ with respect to x, that is, -1; and the second one $\sqrt{1+x}$ gives $(1/2)(1+x)^{-1/2}$. Now, when you take x between -1 to 1, this expression is always greater than 0. It is 2, here also 2 so it is always greater than 0.

Since f' is greater than 0, it contradicts our earlier derivation that $f'(\alpha) = 0$ for some α. That means, our assumption that there are two points such that $f(c) = 0$ and $f(d) = 0$ is wrong. That proves that there is a unique c such that this equation is satisfied with that particular c. That is it.

Let us take another example. Here we are asked to find a function whose derivative is $\sin x$, and whose graph passes through the point $(0, 2)$. That means we need a function $f(x)$ such that $f'(x) = \sin x$ and it passes through the point $(0, 2)$, that is, $f(0) = 2$.

So, what do we do? We apply our earlier result, which was a corollary of Mean Value theorem. It says that if two functions are having the same derivative, then they will differ by a constant. So first, we have to see what is a function whose derivative is $\sin x$. Of course, we know it is $\cos x$ or rather, $-\cos x$.

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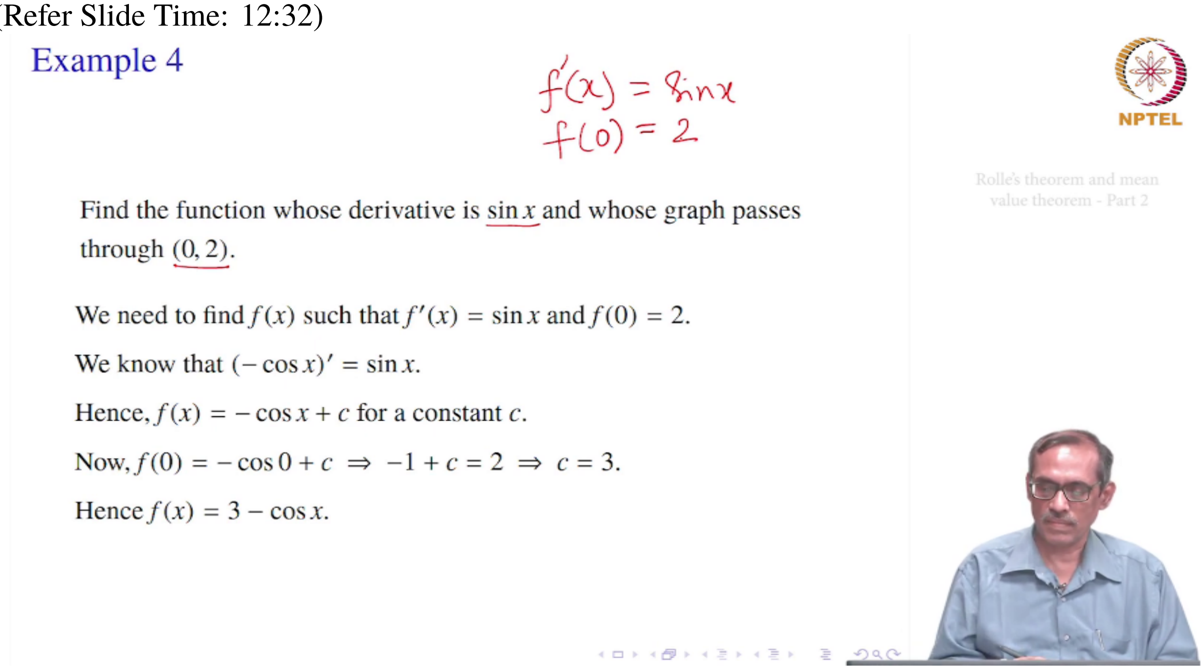

We know that $(-\cos x)' = \sin x$. We have the function $-\cos x$, and we have a function $f(x)$, whose derivatives are same, that is $\sin x$. Then, they must differ by a constant. That means, our $f(x)$ will be equal to $-\cos x$ plus some constant,. Write is as $f(x) = -\cos x + c$. Now we apply the second condition that $f(0) = 2$. By substituting $x = 0$, we have $2 = f(0) = -\cos 0 + c = -1 + c$. That gives $c = 3$. When $c = 3$, our function is $3 - \cos x$. That is what it is.

Sometimes, you may have to use this kind of argument using some corollary of Mean Value theorem. That is how we will be using our results.