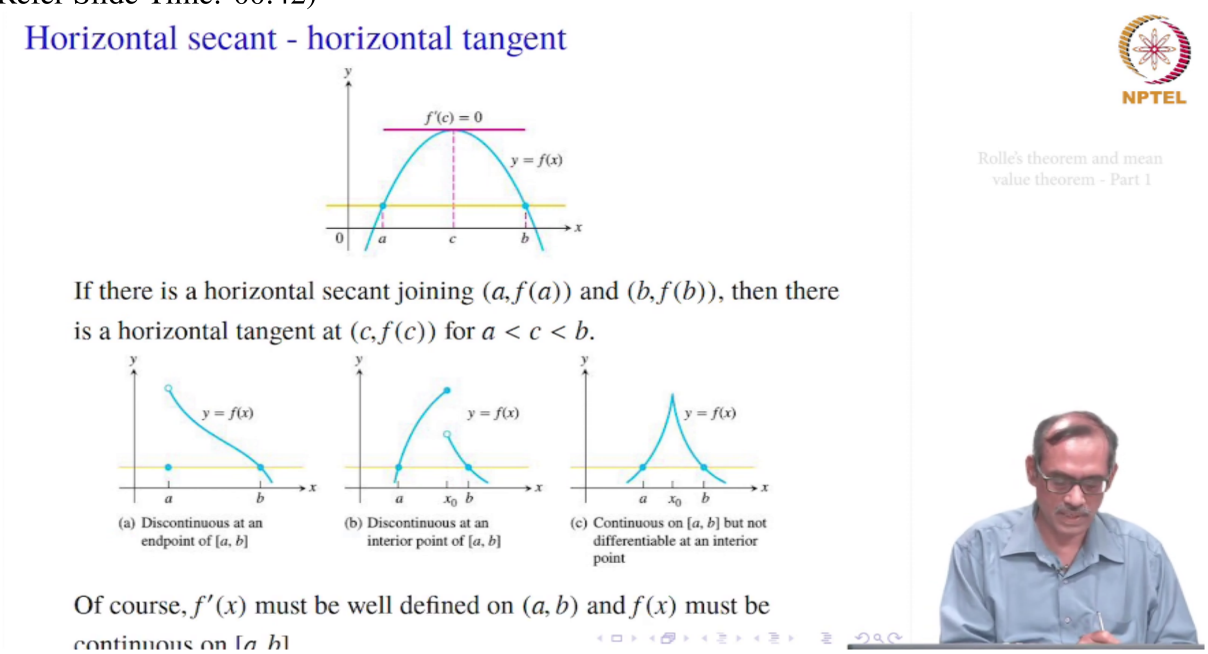
Basic Calculus - 1
Professor. Arindama Singh
Department of Mathematics
Indian Institute of Technology Madras
Lecture 18 - Part 1
Rolle's Theorem and Mean Value Theorem - Part 1

So, this is lecture 18 of Basic Calculus 1. As you recall, we had discussed about maximum and minimum values of a function defined over an interval. Today, we will be talking about Rolle's theorem and the Mean Value theorem, where we will be using those notions. These are very fundamental results for differential calculus. Let us see what it is.
(Refer Slide Time: 00:42)



Suppose you have the curve $y = f(x)$. For simplicity, we write this way; it will be easier to see what we want to say. You take one horizontal secant, that is in the yellow line. There is a horizontal secant joining the point $(a, f(a))$, which is here, and $(b, f(b))$. Once it is horizontal, it means $f(a) = f(b)$; the heights are same, that is why that blue line is horizontal. Then, we just hold that line and move it up. It looks that at some point, it will just touch the curve. That is, there is a horizontal tangent somewhere. We say it this way: there exists one point c between a and b such that it is horizontal. So, $f'(c)$ must be equal to 0, the slope of the tangent is 0. That is exactly the Rolle's theorem.

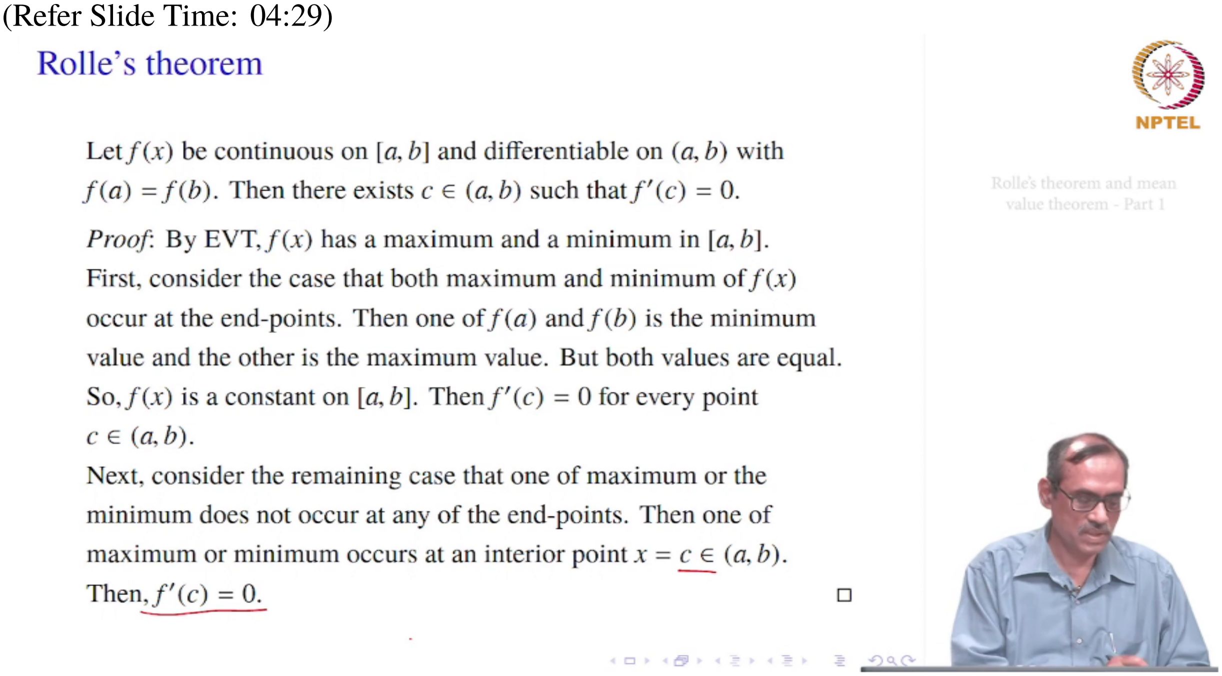
But we will see what is given and what is to be shown. Here, f is a function which is defined on the closed interval $[a, b]$; it is also defined at a and b. We assume that it is continuous over the whole closed interval $[a, b]$ and it is differentiable. You can say it is differentiable on the whole interval, closed interval $[a, b]$, but we require something less than that. We will say that it is differentiable in the open interval (a, b). That is at a and b, it may not be differentiable. Of course, a and b are the endpoints, where the function is defined, then the derivatives would be one sided.

But we even do not need it. What we need is that the function is continuous on the closed interval $[a, b]$ and it is differentiable in the open interval (a, b).

You see, suppose one of them is dropped, then you may not get this idea that there is this horizontal secant implies there is a horizontal tangent. For example, let us take the first one here, first picture below numbered (a). There is a discontinuity of the function at the endpoint a. In this case, you have the line joining $(a, f(a))$ and $(b, f(b))$. Though it is geometrically not a secant, but it satisfies $f(a) = f(b)$. You can see that it does not imply that there is a horizontal tangent. Nowhere there is a tangent to the curve which is horizontal. So, continuity on the whole interval $[a, b]$ is important. That is about the end points.

Similarly, if at an interior point, say at x_0, there is discontinuity of the function. Then also you see there is no horizontal tangent in the picture (b). We also assume that it is differentiable on the open interval (a, b). If there is a point say x_0, where it is not differentiable, then you see there was the possibility of a tangent being horizontal only there, but nowhere else there is a horizontal tangent. And it is not differentiable there, so, we cannot say $f'(x_0) = 0$. That is a like a cusp. So, the conditions as the hypotheses in the result are very important.

(Refer Slide Time: 04:29)



Let us formulate it and see how does it look. It says that $f(x)$ is a continuous function on the closed interval $[a, b]$, $f(x)$ is differentiable on the open interval (a, b), and $f(a) = f(b)$. These are the three conditions we are imposing. Under these conditions, we will see that there exists a point c in the open interval (a, b) where $f'(c) = 0$. That is the main idea of Rolle's theorem. Of course, we will give a proof.

The proof will use our notion of maximum and minimum. We have the Extreme Value Theorem, since it is continuous on the closed interval $[a, b]$. There is a point where the function achieves its maximum. There is also a point where the function achieves its minimum. So, we say that $f(x)$ has a maximum and a minimum in the closed interval $[a, b]$.

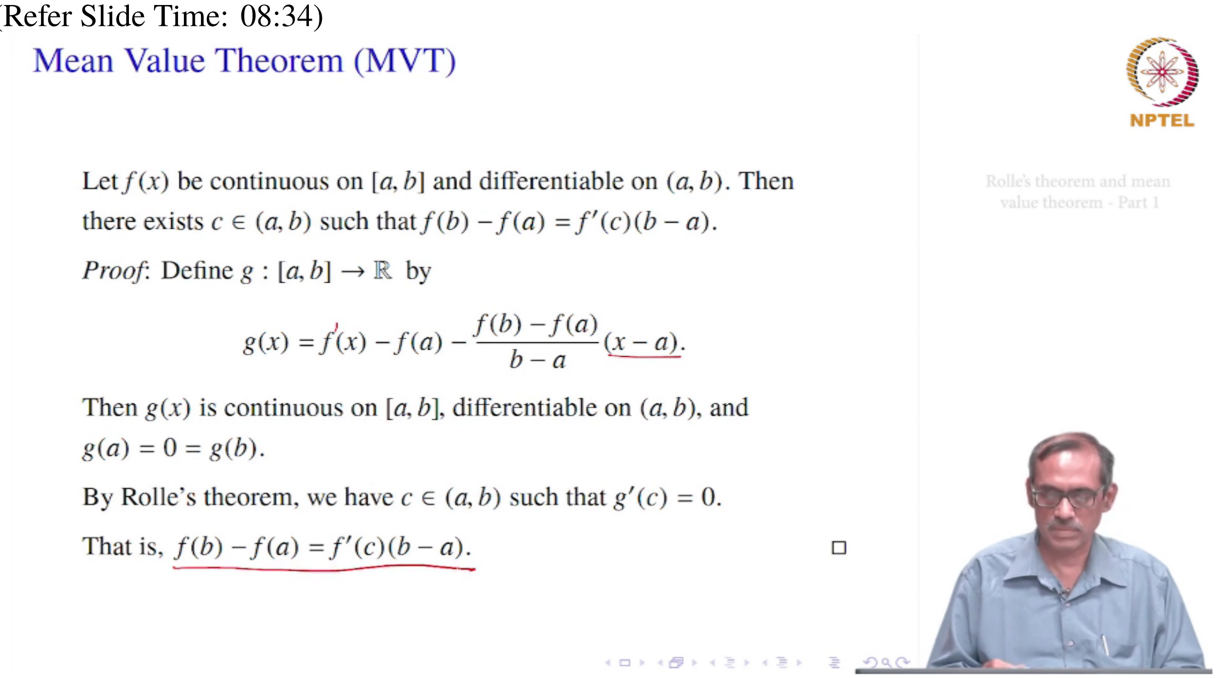
Suppose we consider the first case, where the maximum and the minimum occur at the endpoints. That is what we consider as our first case. So, the maximum occurs at an endpoint and also the minimum occurs at an endpoint. Let us consider this case first. In this case, one of them is a maximum, one of them is a minimum, but it is possible that both of them are maximum or minimum, that will be the easiest case, but we will show that, that is the case. Now, since one is maximum, one is minimum, and $f(a) = f(b)$, the maximum of the function is equal to the minimum of the function. That is why the function is a constant function. It is the same value (whatever is that maximum value or the minimum value), the value of the function throughout the interval. In that case $f(x)$ is a constant on the closed interval $[a, b]$. Once it is a constant, we know that its derivative is equal to 0 everywhere in the interval.

Of course, you see that we always assume implicitly that $a < b$; it is a non-degenerate interval. If it degenerates to a singleton then there is problem. There will not be an interior point, c may not belong to (a, b), and (a, b) will become empty. Our assumption is that $a < b$; it is a non-degenerate interval.

Now, $f(x)$ is a constant. So, $f'(x) = 0$ for every point x inside that the open interval (a, b). Now, you can choose any point and that will satisfy our conclusion of Rolle's theorem. Now, this is the case where maximum and minimum occur at the endpoints.

Suppose, it is not the case. That is the next case, where one of maximum or minimum does not occur at an endpoint. First, our case was both of them occur at the endpoints. Now, we say the other case which is not covered in that. That is, at least one of maximum or minimum does not occur at the endpoints.

(Refer Slide Time: 08:34)



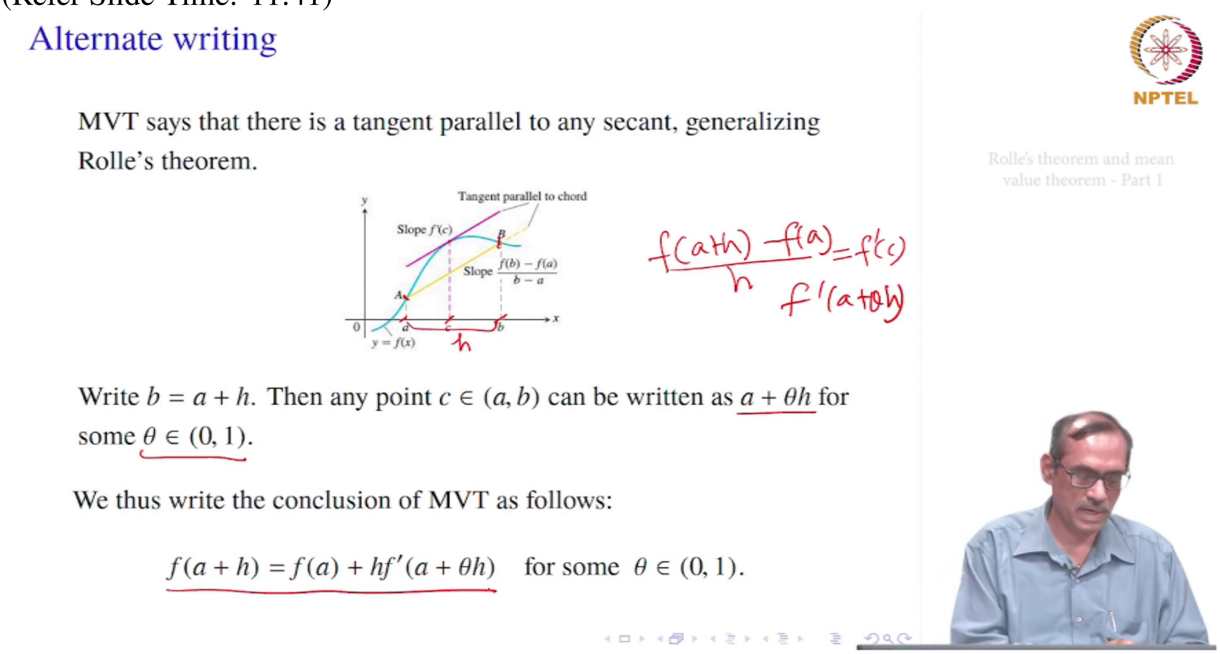
So, where does it occur? It occurs, because $f(x)$ is continuous on the closed interval $[a, b]$. So, there is an interior point c, where maximum or minimum whichever does not occur at an end point, occurs at this interior point c. That is, at c, we have a maximum or a minimum. And we know that it has to be a critical point. Since $f(x)$ is differentiable in the open interval (a, b), $f'(c)$ exists, and then it must be equal to 0. That is the conclusion of Rolle's theorem.

We will see many applications of this Rolle's theorem. The first application is a slight generalization of this idea, where we assume those two facts that $f(x)$ is continuous on the closed interval $[a, b]$ and it is differentiable on the open interval (a, b), but we do not assume that $f(a) = f(b)$. That is, there is some difference between the heights at $x = a$ and $x = b$. Then what it says is that there exists a point c in (a, b) such that $f'(c) = [f(b) - f(a)]/(b - a)$.

Soon we will see how does it look geometrically. But let us see how to prove this first. So, what do we do? We define a new function g on the closed interval $[a, b]$ to $\mathbb{R}$ by $g(x) = f(x) - f(a) - [(f(b) - f(a))/(b - a)](x - a)$. You se that this function is well defined. Because, we assumed that $a < b$, that is, $b - a$ is nonzero, then this is well defined. And since $f(x)$ is continuous on the whole of closed interval $[a, b]$, this function $x - a$ is also continuous. So, you see that the expression $g(x) = f(x) - f(a) - [(f(b) - f(a))/(b - a)](x - a)$ defines a continuous function on the closed interval $[a, b]$.

Also, we find that $g(a) = 0$. Why? Suppose I substitute $x = a$ here. This becomes 0 and this is now $f(a) - f(a)$, that is also 0. So, $g(a) = 0$. What about $g(b)$? It is $f(b) - f(a) - [f(b) - f(a)]/(b - a)$ times $(b - a)$ as we substitute $x = b$. Then, $b - a$ cancels, and we have $f(b) - f(a) - [f(b) - f(a)]$, which is also 0. Also, g is differentiable on the open interval (a, b) since f is differentiable and $(x - a)$ is differentiable. So, this is also differentiable. Therefore, we can apply Rolle's theorem on g.

(Refer Slide Time: 11:41)



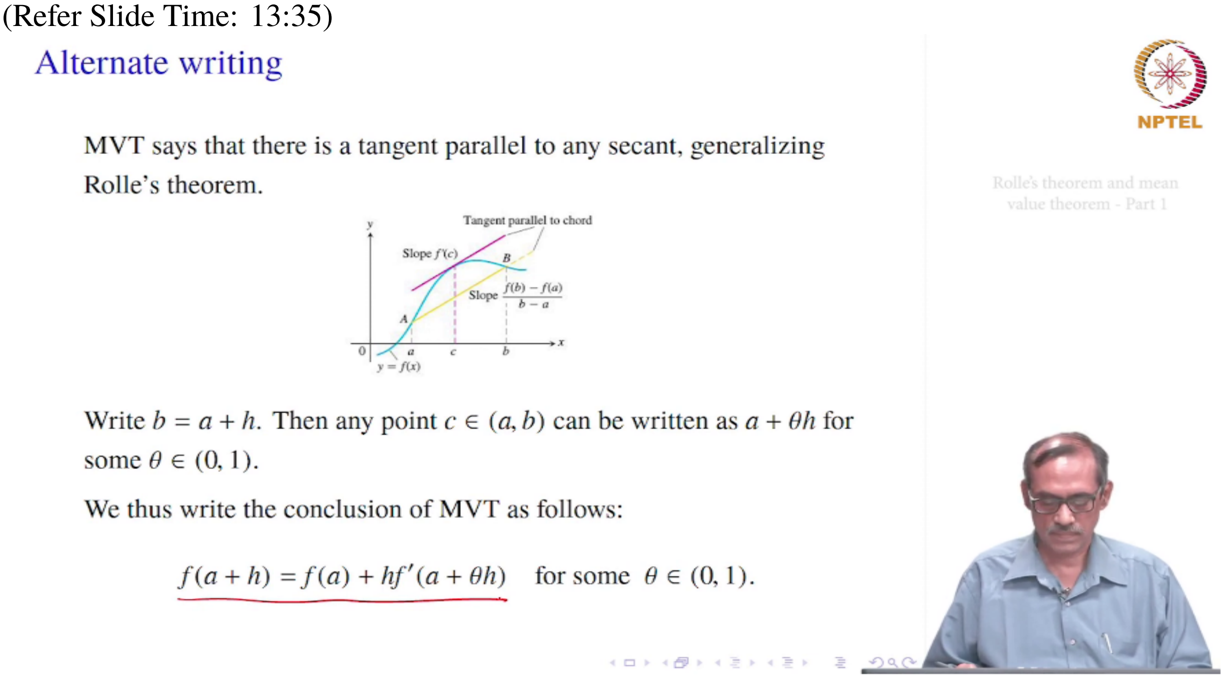
That means, there exists a point c, such that $g'(c) = 0$. But what is $g'(c)$? It is really $f'(x)$ minus the derivative of $f(a)$, which is 0 minus this expression times $(x - a)'$. This will be evaluated at c. Now, $g'(x) = f'(x) - [f(b) - f(a)]/(b - a)$ as the derivative of $x - a$ is 1. Then, $g'(c) = 0$ gives $f'(c) - [f(b) - f(a)]/(b - a) = 0$ That means $f(b) - f(a) = f'(c)(b - a)$. The proof is very simple, it is an obvious application of Rolle's theorem.

We will see how does it generalise Rolle's theorem. All that it says is this: Suppose you take the point A for $x = a$ and B for $x = b$. Look at the picture. We have the point A which is $(a, f(a))$ and B which is $(b, f(b))$. The slope of the secant is $[f(b) - f(a)]/(b - a)$. The Mean Value theorem says that, there exists a point c, where $f'(c)$ is equal to this slope. That means, there is a point inside the open interval (a, b) where the slope of the tangent is equal to the slope of the secant. That is how it is a generalization of Rolle's theorem. When $f(a) = f(b)$, you get directly Rolle's theorem from this. However, we have proved this using Rolle's theorem.

There is an alternate way of writing it, instead of b, which is to the right side of a, we will write $b = a + h$. That means $b - a$, this distance is h. Once you write $b = a + h$, with this distance as h, what do we get is, $f(b) - f(a)$ is $f(a + h) - f(a)$. Then, this divided by h is equal to $f'(c)$ for some point c. How do we write c? It is a point in between a and b. So, you can write $c = a + \theta h$, where θ is some number between 0 and 1. When it is equal to 1 it gives b, when it is 0, it gives a and in between points are expressed by $a + \theta h$. It says that $f'(a + \theta h) = [f(a + h) - f(a)]/h$.

Or we write this way: $f(a + h) = f(a) + hf'(a + \theta h)$ for some θ between 0 and 1. This is also another way of rewriting the Mean Value theorem.

(Refer Slide Time: 13:35)



We will see some applications of the Mean Value theorem. Our first application is a very important result, though very small reasoning is required to prove this. What we do is we start with a function f, which is defined on an interval I and it takes real values. Suppose it is differentiable on the interval, you may think of some open interval; if you take a closed interval, then you think of one sided limits there. If $f'(x) = 0$ for each x in the interval, then $f(x)$ is a constant function. We want to show that $f'(x) = 0$ for each x.

In fact, if you take $f(x)$ as a constant function, then its derivative will be equal to 0; we know that. But it says something about the converse. If it is an interval and on the whole of the interval, you get derivative equal to 0, then everywhere, at every point of the interval, $f(x)$ must be the same constant; $f(x)$ must be a constant function.

So, what do we do? We choose any two points. Suppose that $f'(x) = 0$. Let us choose any two points in the interval $[a, b]$, where $a < b$. Then f is differentiable on I to $\mathbb{R}$, where this closed interval $[a, b]$ is a subset of that I. We know that differentiability at a point implies continuity. That is, f is a continuous function on the closed interval $[a, b]$. Also its derivative exits everywhere inside the open interval (a, b); it is differentiable on the open interval (a, b). Since this happens, we apply Mean Value theorem to get $[f(b) - f(a)]/(b - a) = f'(c)$ for some c between a and b. Now, that $f'(c) = 0$; so, the right side becomes 0, and you get $f(b) = f(a)$. So, it says that if you choose any two points a and b inside the interval, then $f(b) = f(a)$. Therefore, f is a constant function. That is the simple proof; it is an application of Mean Value theorem.

But you can pose it in another way. Suppose, we have two functions f and g on an interval I; they are differentiable functions. And we know that $f'(x) = g'(x)$ for every point $x \in I$. Then you can think of $f(x) - g(x)$ as a new function. So, $f - g$ or, $f(x) - g(x)$ is defined from I to $\mathbb{R}$, and that $(f - g)' = 0$ on the interval I. Applying the previous result, we get: $f(x) - g(x)$ is a constant; it is a constant function, some number. Therefore, we conclude that $f(x) = g(x) + c$ for some constant c. This says that if the derivative of two functions are same inside an interval, at every point of the interval, then they will differ by a constant.

And there is another corollary, which is called as the Mean Value Inequality. It says that if $f(x)$ is a continuous function on $[a, b]$, differentiable on the open interval (a, b), and its derivative is bounded, say, $m \le f'(x) \le M$, for every x in that interval $[a, b]$, then $m(b - a) \le f(b) - f(a) \le M(b - a)$. [There is some mistake here. It is not $f(x)$, it is $f(b) - f(a)$ is less than or equal to M into $b - a$.]

Of course, it is done for $a,\ b$ only. Instead of $a,\ b$, suppose you choose any two other points, say, x_0 and x_1, where $x_0, x_1 \in [a, b]$. Then we would say $m(x_1 - x_0) \le f(x_1) - f(x_0) \le M(x_1 - x_0)$. It is also formulated this way. In fact, all that we have to do is think of the restriction of f to this closed interval $[x_0, x_1]$, and then apply this. This result is sometimes called the Mean Value Inequality. We will see some applications of these results to solving some problems.