Basic Calculus - 1
Professor. Arindama Singh
Department of Mathematics
Indian Institute of Technology Madras
Lecture 17 - Part 2
Maxima and Minima - Part 2

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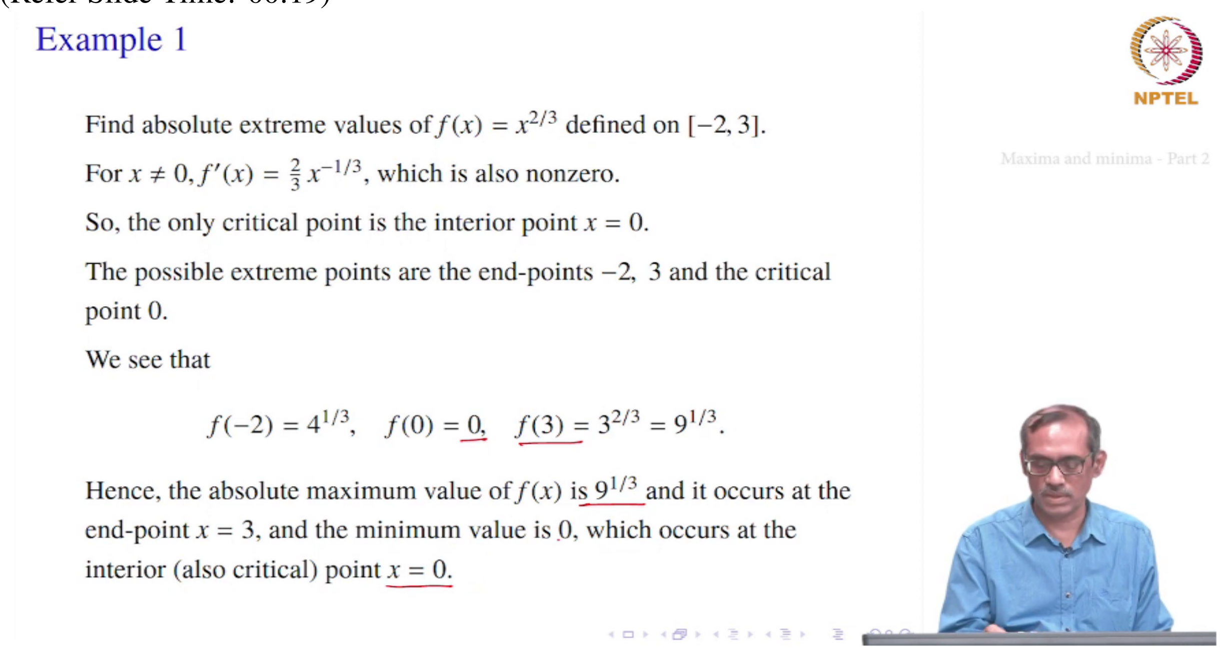

Let us see, how to go about it this. Consider an example. Find the absolute extreme values of $f(x) = x^{2/3}$. Where is our function defined? It is only on $-2, 3]$. Though this function can be defined elsewhere, our search is limited, the function is defined on this interval only, on $[-2, 3]$. And how does it act? It acts as $f(x) = x^{2/3}$.

Now, how do we get its maximum, or minimum values? First, the end points -2 and 3 are possible points. Then, we will go for the critical points. Those critical points and along with these end points are possible values where its extreme values can occur. So, let us find the critical points. Of course, if $x = 0$, then $f(x)$ can be 0.

To find the critical points we need the derivative of $f(x)$. If you take the derivative, it is $f'(x) = (2/3)x^{-1/3}$. This is not defined at 0. So, at 0, we do not define it. $f(x)$ is not differentiable at 0. But if x is nonzero, then its derivative is $(2/3)x^{-1/3}$. We substitute. If $x \neq 0$, then $f'(x) \neq 0$. That means the only possible critical point is 0, where $f(x)$ is not differentiable. And there is no other point.

There are points where $f(x)$ is differentiable, but those points are not critical points, because f' does not vanish there. So, no other critical points; 0 is the only critical point.

Now you have three things. The possible extreme points are the end points -2 and 3, and the only critical point 0. Then, all that we have to do is compare the values and find out which are the extremum, that is, maximum or minimum. We have $f(-2) = (-2)^{2/3} = 4^{1/3}$ from here, $f(0) = 0$, and $f(3) = 3^{2/3} = 9^{1/3}$. Out of these, we find that the maximum value is $9^{1/3}$ and the minimum value is 0. Here, the minimum is achieved at the critical point 0, and the maximum is achieved at 3. So, the point of absolute maximum is 3, and the point of absolute minimum is 0, and the absolute maximum value is $9^{1/3}$, the absolute minimum value is 0. That is what we conclude from this.

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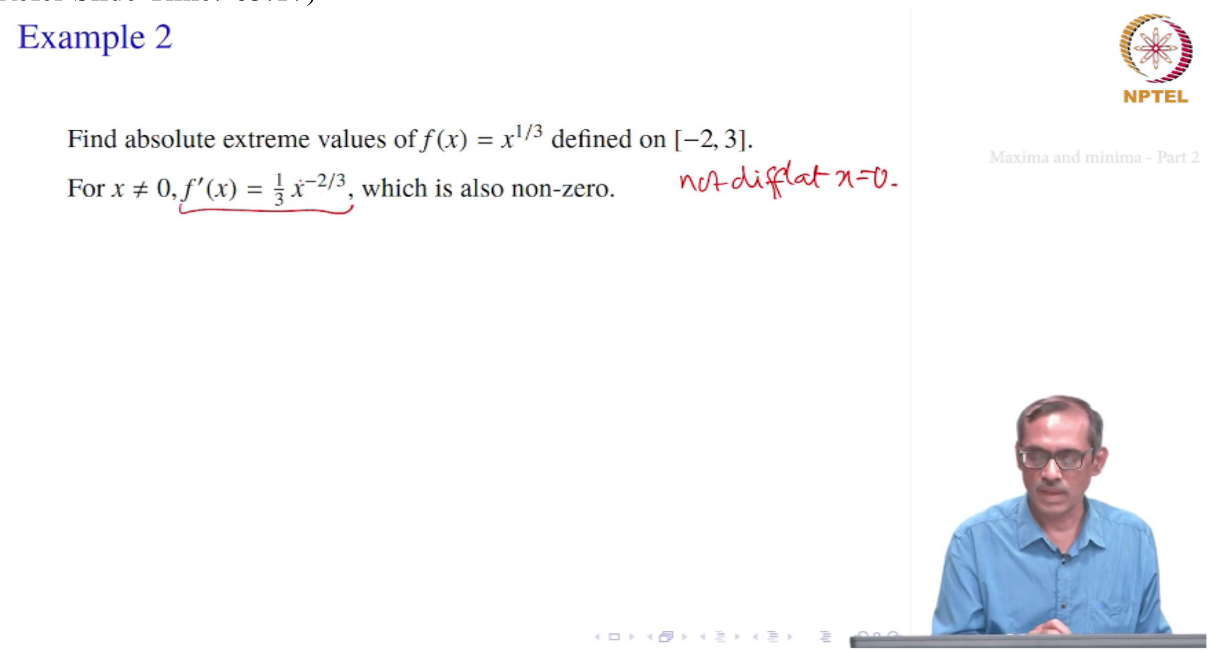

Let us take another similar problem. Suppose you have $f(x) = x^{1/3}$. Again, it is defined on the same closed interval $[-2, 3]$. Let us find out the derivative and see the critical points. It is not differentiable at $x = 0$. And, if $x \neq 0$, then it is differentiable with the derivative as $(1/3)x^{-2/3}$. But for nonzero x, $f'(x)$ is never equal to. So, nonzero points are not critical points.

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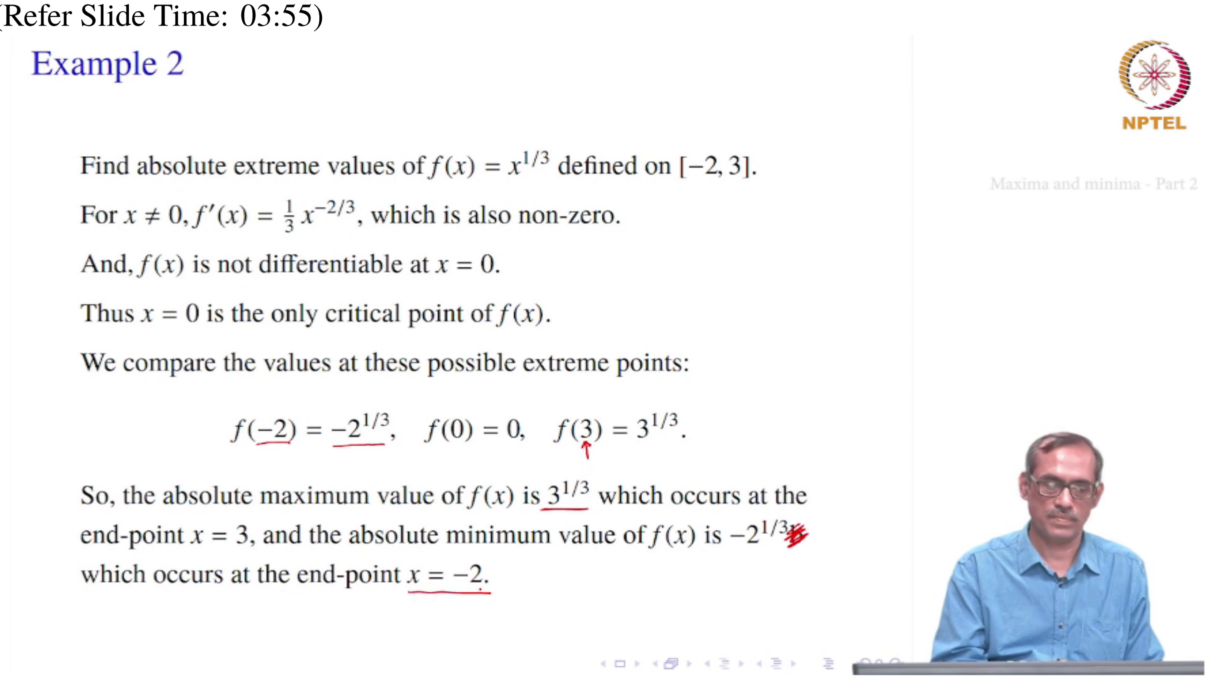

So, the only critical point again is 0. And we have the endpoints, which are -2 and 3. You compare the values: $f(-2) = -2^{1/3}$, $f(0) = 0$, and $f(3) = 3^{1/3}$. Again, you get the maximum value as $3^{1/3}$, which occurs at 3. So, we say that the maximum value of $f(x)$ is $3^{1/3}$ and the point of absolute maximum is 3. Now, $f(0) = 0$ is not the minimum, because there is a negative value. So, -2 is a point of absolute minimum; and the absolute minimum value is $-2^{1/3}$.

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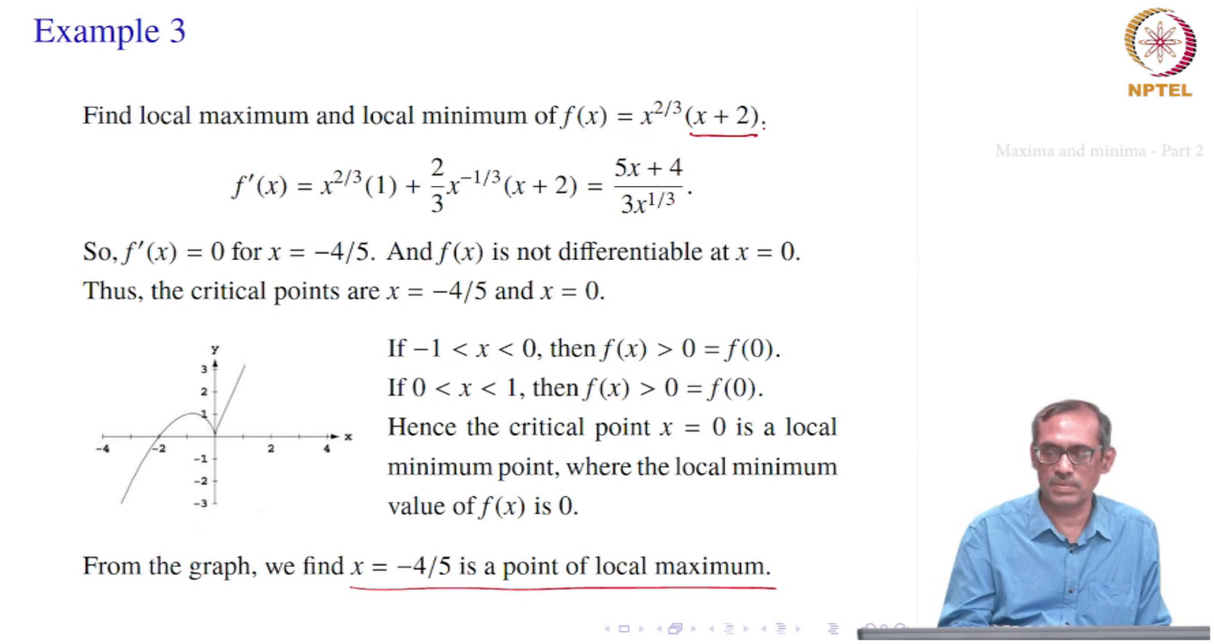

We will take some more examples. Let us take the function $f(x) = x^{2/3}(x+2)$. What is the domain? It is defined everywhere, of course. We have to find its local maximum and local minimum. That means they will occur only at critical points, as there is no endpoint here; its domain is the whole of $\mathbb{R}$, all points are interior points.

Now, we find the derivative. Here, $f(x) = x^{2/3}(x+2)$. It is fg form, product of two functions. So, first we take f times derivative of g, that is fg', which with f as $x^{2/3}$ and g as $x+2$, gives $x^{2/3} \times 1 = x^{2/3}$. Next, the derivative of $x^{2/3}$ times $(x+2)$. It gives $(2/3)x^{-1/3}(x+2)$. You simplify their sum. That gives $(5x+4)/(3x^{1/3})$. To find the critical points, we should see where this becomes 0. It is 0 at the point $x = -4/5$, from this numerator. And where else? The critical points can be those, where $f(x)$ is not differentiable. The function $f(x) = x^{1/3}$ is not differentiable at $x = 0$. Now, you have two critical points. One is, where it is differentiable and derivative 0, another where it is not differentiable. These are $-4/5$ and 0.

Out of these two points, we should find out or decide about the maximum and minimum. How do we go about it? If you go for absolute maximum or absolute minimum, you just substitute $x = -4/5$ and $x = 0$ to get $f(-4/5)$ and $f(0)$. And whichever is larger that will be maximum, whichever is smaller that will be absolute minimum.

But we want local maximum and local minimum. That means, in some neighborhood of $-4/5$ and in some neighborhood of 0, we have to compare the values. Let us take the point 0 first, that will be easier. Let us take some neighborhood of 0; it may be, say, $(-1, 1)$. You could have taken $(-1/2, 1/2)$ also; but bigger might give a trouble; smaller is better to find out extreme points.

In this neighborhood for $-1 < x < 0$, $f(x)$ is negative or positive?. As x is negative, this becomes negative; you get this to be positive now, it is $2 - 1$. So, you get $f(x) > 0$, which is $f(0)$. Is it okay? You are taking the left-hand neighborhood, where $f(x) > 0 = f(0)$. That is what you see from the graph also. Now, you take right neighborhood. There also, you find that $f(x) > 0 = f(0)$. So, at 0 there is a minimum. The critical point $x = 0$ is a local minimum point, where the local minimum value of $f(x)$ is 0.

Now, this kind of analysis will be difficult for $x = -4/5$. Because the function is complicated, but still it can be done by computing with the inequalities. However, from the graph, you can see directly that $x = -4/5$ is a point of local maximum. You have to really do this kind of thing. Take a neighborhood around $-4/5$. Limit the radius of the neighborhood to 1 so that it will be easier to compute this one. And it will carry through.

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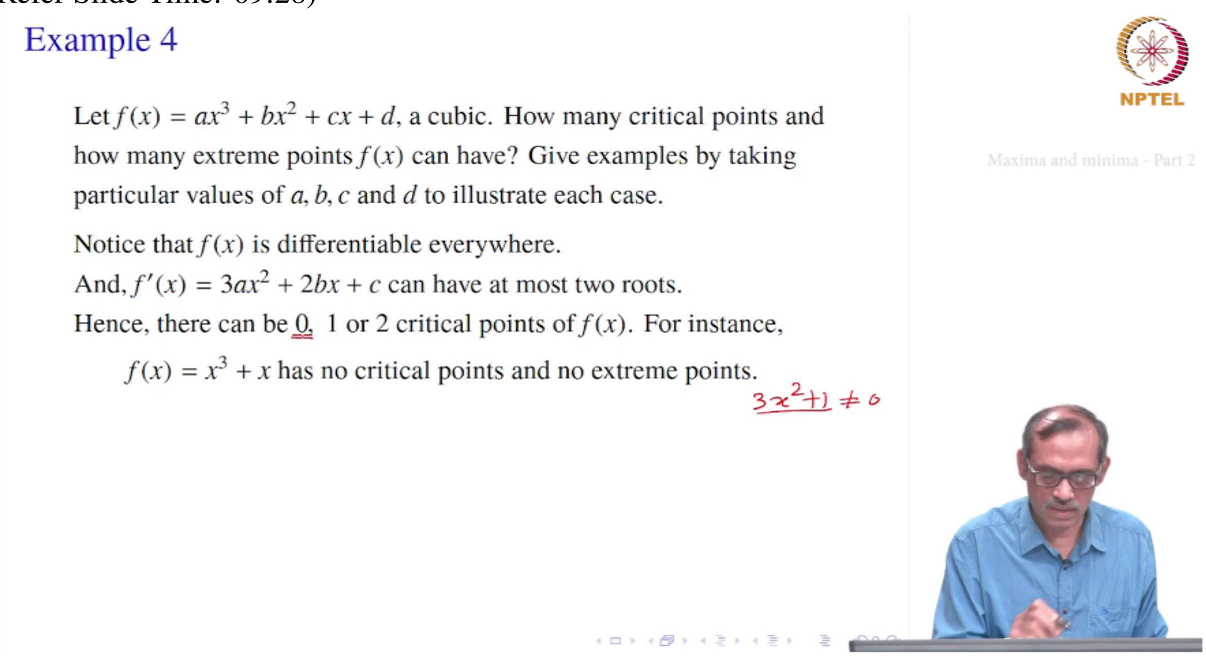

Let us take another example; say, $f(x) = ax^3 + bx^2 + cx + d$. It is a polynomial of degree 3, a cubic. We are not going to find the critical points or extreme points, it is asking something else. It asks: how many critical points and how many extreme points $f(x)$ can have? How do we proceed?

To find the critical point, see that this is a polynomial; so it is differentiable everywhere; and it is defined over the whole of $\mathbb{R}$. Then, every point is an interior point. Now, f' will be a quadratic; it is $f'(x) = 3ax^2 + 2bx + c$. That can vanish at most at two points. If there are real roots, there will be exactly two; if not, there will be none; the two roots will be complex. It will be depending on a, b and c.

That is what we are going to do. Now, $f'(x) = 3ax^2 + 2bx + c$ can have at most two real roots. Since there are maximum of two real roots, it can have say, 0 or 1 or 2 critical points, maximum of two critical points; that is what it says. Let us see how to proceed.

Suppose I take the polynomial $x^3 + x$. My a is 1, b is 0, d is 0, and c is 1. Now, does it have any critical point? We get the derivative as $3x^2 + 1$. This is never equal to 0 for any real number.

So, it does not have a critical point. Also, look at the graph of $y = x^3 + x$. Here, x^3 is growing and x will be added to it. You will see that there is no extreme point, neither negative nor positive. It grows to infinity and also it decreases to minus infinity. It will neither have maximum points nor minimum points; and it has no critical points. That serves an example for the 0 case.
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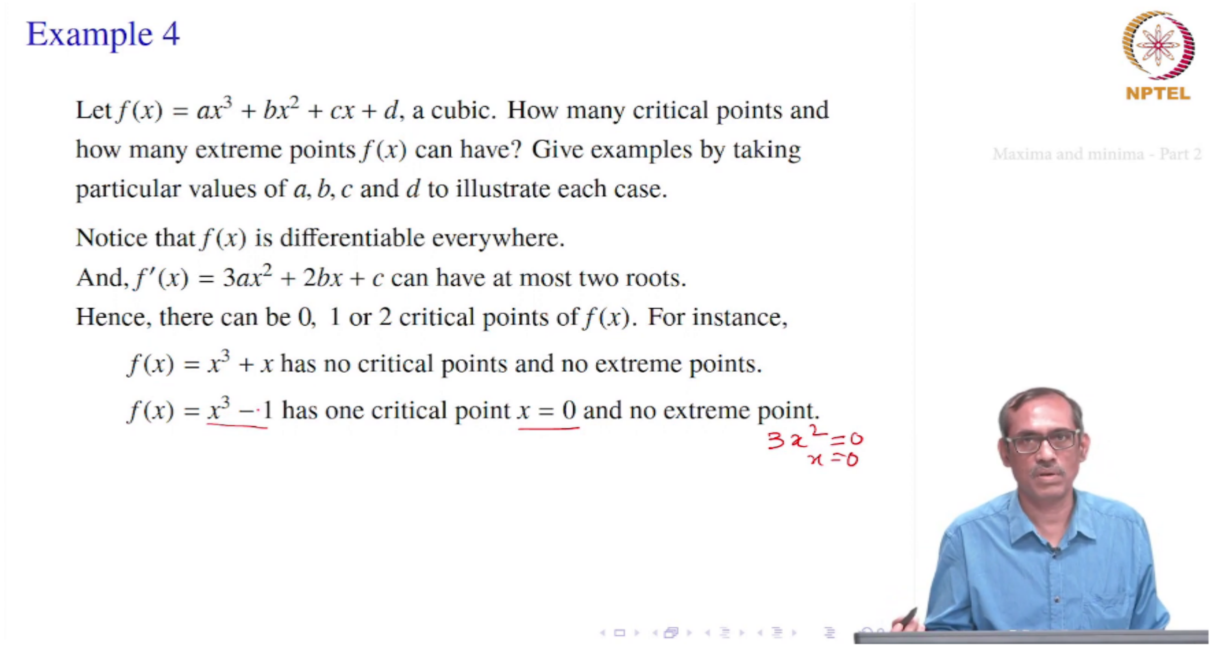

Now let us take $f(x) = x^3 - 1$. Its derivative is $f'(x) = 3x^2$. You want critical points. So, equate that to 0. You get $x = 0$, the only critical point. But it is $x^3 - 1$; it is growing with -1 shifted. So, it has no extreme points. But it has one critical point. It serves the 1 case.
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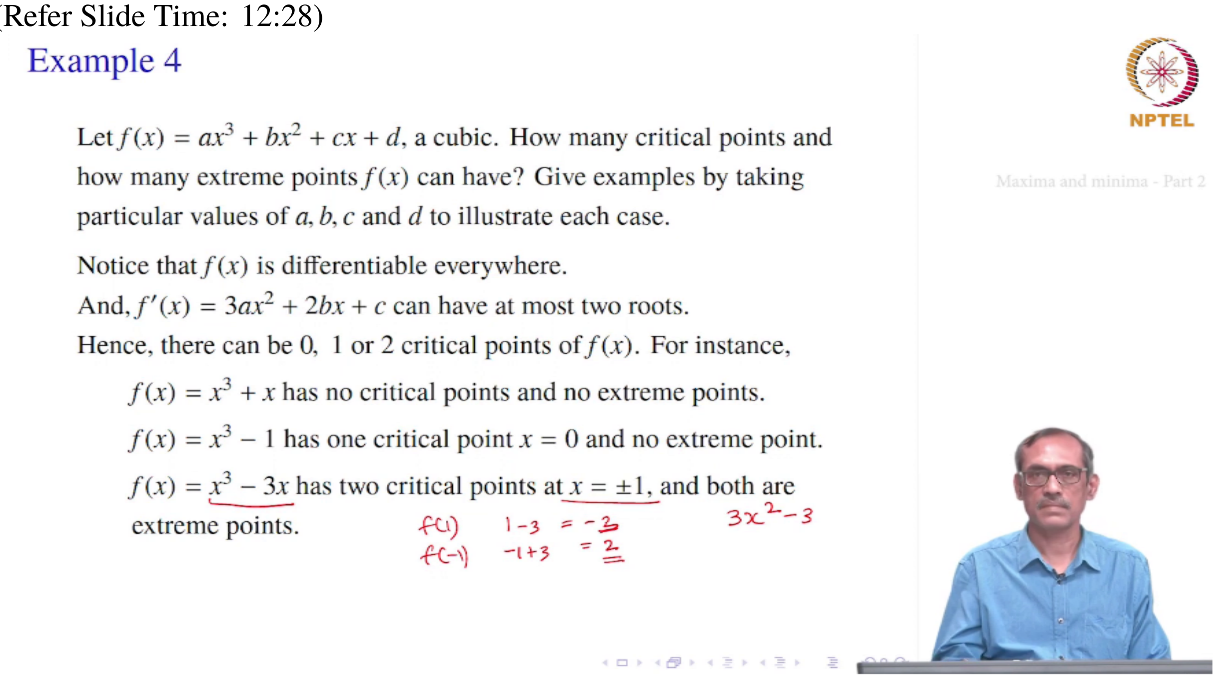

Let us take another example: $f(x) = x^3 - 3x$. For the critical points, you differentiate. It gives $f'(x) = 3x^2 - 3$. If you equate that to 0, you will get x $x = \pm 1$. These are the critical points. There are two critical points.

In the first case, there are 0 critical points; next, 1 critical point; next, 2 critical points. Here, both the critical points are also extreme points. That is clear from $x^3 - 3x$. Both are really extreme points. You have to find some neighborhoods and then complete the argument. But that is what you see intuitively. It will have one extreme point $+1$. If you substitute, it gives $1 - 3 = -2$. That is, $f(1) = -2$. And $f(-1) = -1 + 3 = 2$. There is a maximum value at -1 and a minimum value at 1. So, all sorts of things could happen for a cubic. That is what we see from these examples.

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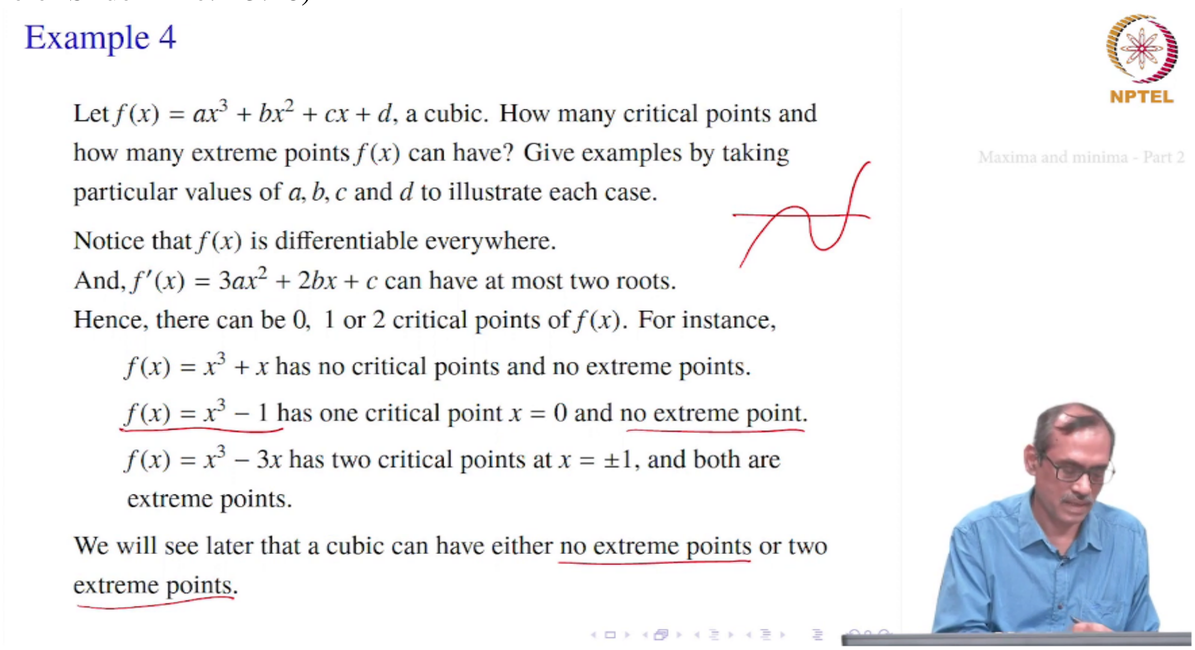

As you see, a cubic can have either no extreme points, or as we have shown in an example, it will have two extreme points. We will see later why 'only one' does not happen? It either has no extreme points (here you see two examples), no extreme points; or two extreme points (the last one). For $f(x) = x^3 - 1$, there is one critical point, no extreme points. We will see latter, why such a thing is happening. Why for a cubic, it is not possible that there is only one extreme point. Of course, you can se it from its graph. In general, the graph of a cubic would be something like this; and that should give the answer. But we will see analytically why this happens.

Let us take another example. It is a different kind of thing; it is given that $f(x)$ is an even function. Also, $f(x)$ has a local maximum at $x = c$. We need to find out whether there is a local maximum or a local minimum or some such thing happens at $x = -c$. Because it is even, it may have something to do with c and $-c$. So, what do we do?

If you take $f(c)$, it is equal to $f(-c)$ because it is even. What can you say about $x = -c$? If it is absolute maximum then of course, you could have told directly there is the absolute maximum at $x = -c$, because it is the same value $f(-c) = f(c)$. But it is only asking for local maximum at $x = -c$. Since given is a local maximum at $x = c$, we have a neighborhood of c, say, $(c - \delta, c + \delta)$ where if x belongs to that neighborhood, then $f(x) \leq f(c)$. That is what it says.

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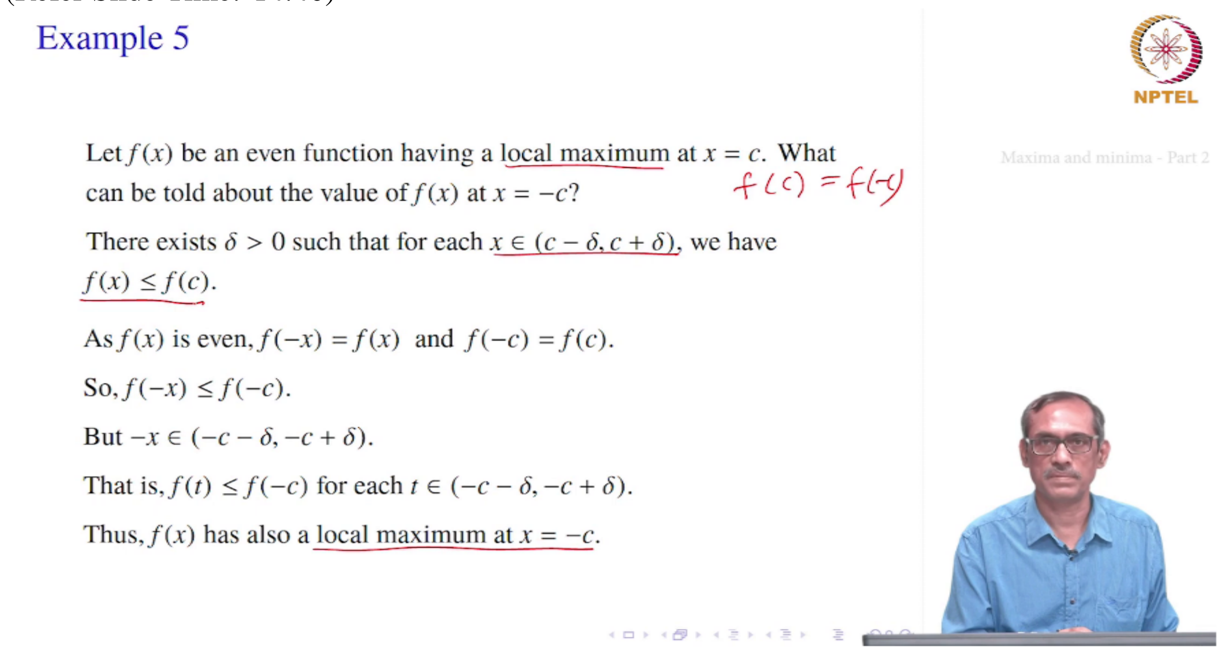

Now, what will happen in a neighborhood of $-c$? Well, since f is even, $f(-x) = f(x)$ and $f(-c) = f(c)$. We conclude that $f(-x) \le f(-c)$; but for which x? Whenever $x \in (c - \delta, c + \delta)$. Then, in that case, $-x$ belongs to what? If $x \in (c - \delta, c + \delta)$, then $-x$ belongs to minus of this thing. But the inequalities will be reversed, and thus, $-x \in (-c - \delta, -c + \delta)$. Since $-x$ belongs to this, we have a neighborhood of $-c$, where if this is say t, then t belongs to that neighborhood, and then $f(t) \le f(-c)$. So, $f(-c)$ is a local maximum value. Where does it occur? It occurs at $x = -c$. So, you would say that there is a local maximum of f or f has a local maximum at $x = -c$ also. Is that fine?

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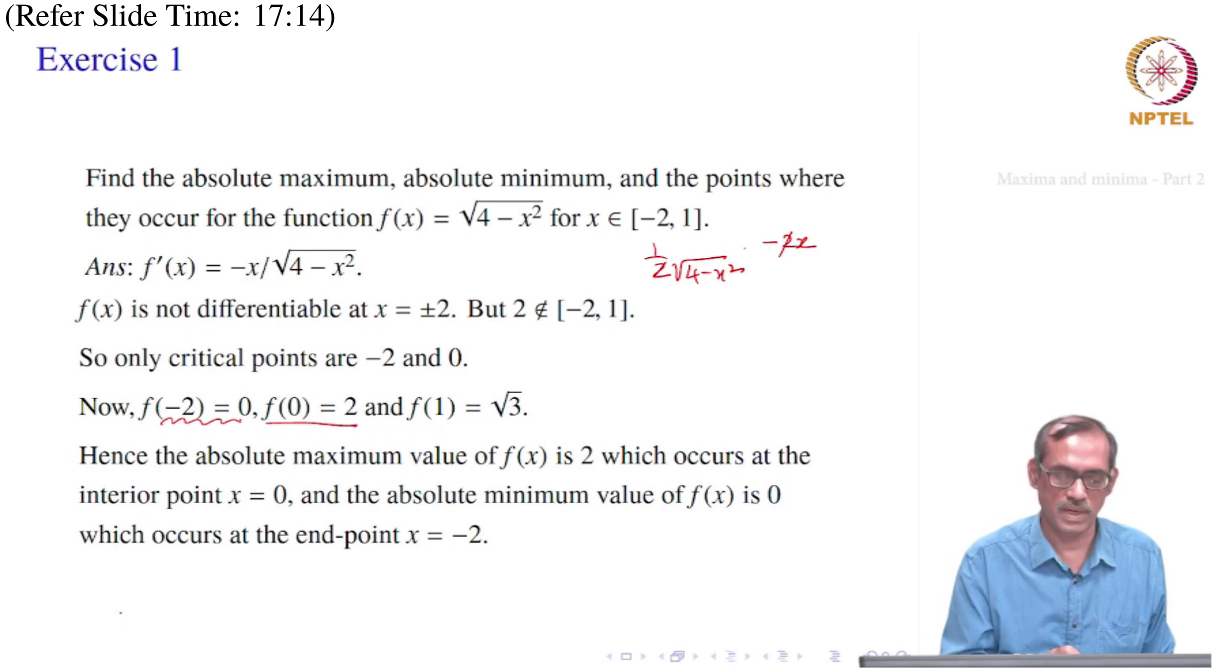

Let us solve some more problems and see how these ideas are exploited. Here, we are required to find the absolute maximum, absolute minimum and the points where these absolute maximum, absolute minimum occur; for the function $f(x) = \sqrt{4 - x^2}$ where $x \in [-2, 1]$. All that we have to do is: find the critical points and and the end points and then compare the values at these points.

Differentiate. For $(4 - x^2)^{1/2}$, it should be half the derivative with respect to $4 - x^2$ and then derivative of $4 - x^2$ with respect to x. That gives $(1/2)$ times $(4 - x^2)^{-1/2}$ times $-2x$. The 2 there cancels and we get $-x/\sqrt{4 - x^2}$.

And this is equal to 0, when x equal to 0. Therefore, the only critical point is 0. Also, 0 is an interior point. Thus, you have now three points: $-2,\ 0,\ 1$. And, where is it not differentiable? They are also critical points. So, we have to see, where it is not differentiable. It is not differentiable at -2, and 2. Now, 2 is not in the domain; and -2 is an endpoint, which we are going to consider anyway.

So, we just evaluate f at $-2,\ 0$ and 1. We find that $f(-2) = 0$, $f(0) = 2$ and $f(1) = \sqrt{3}$. So, it has a maximum value 2 at $x = 0$, and it has a minimum value 0 at $x = -2$. It is fairly straightforward.

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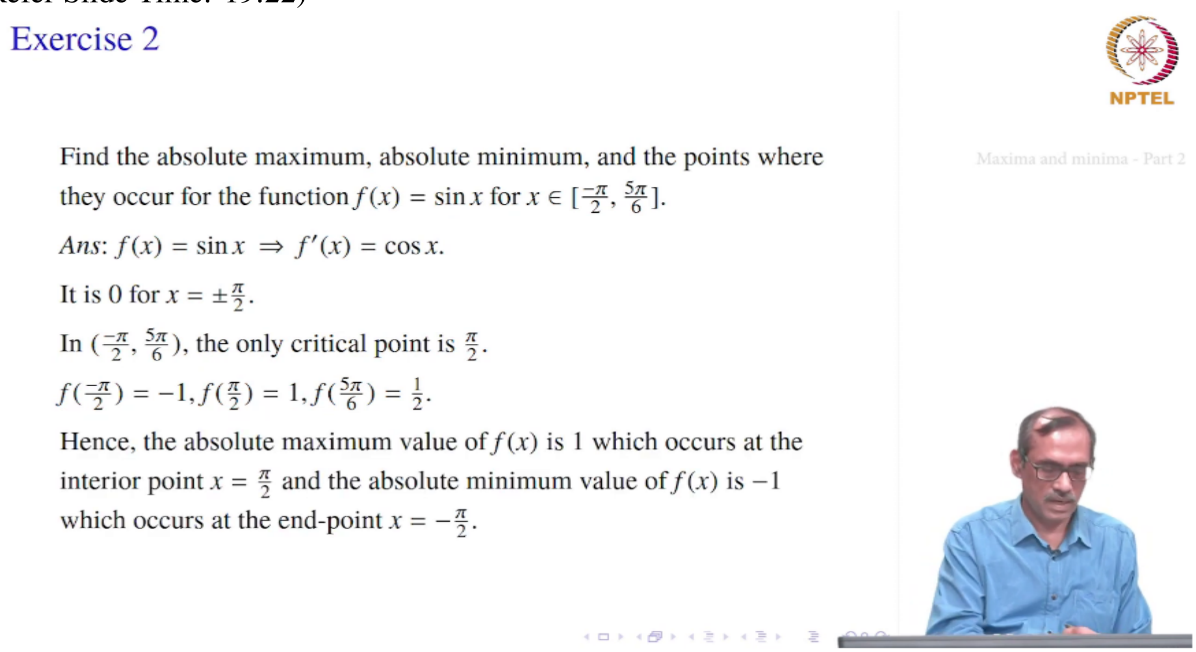

We take one more exercise. Find the absolute maximum, absolute minimum and the points where they occur for the function $f(x) = \sin x$, where x belongs to the closed interval $[-\pi/2, 5\pi/6]$. Again, we differentiate. We get $f'(x) = \cos x$; and we equate that to 0 to get the critical points. Of course, it is differentiable everywhere; so we do not have to worry about other types of critical points. Now, $f'(x) = \cos x$ is equal to 0 when x is an odd multiple of $\pi/2$; that is, at $-\pi/2$ and also at $\pi/2$. But $-\pi/2$ is an endpoint. So the only interior point is $\pi/2$. That is, the only critical point is $x = \pi/2$.

Then we have two more points, the endpoints. We have to compare the values. Now, $\sin(-\pi/2) = -1$, $\sin(5\pi/6) = 1/2$ and $\sin(\pi/2) = 1$. Out of these, the maximum is here at $\pi/2$, and the minimum is at $-\pi/2$. That is what it says.

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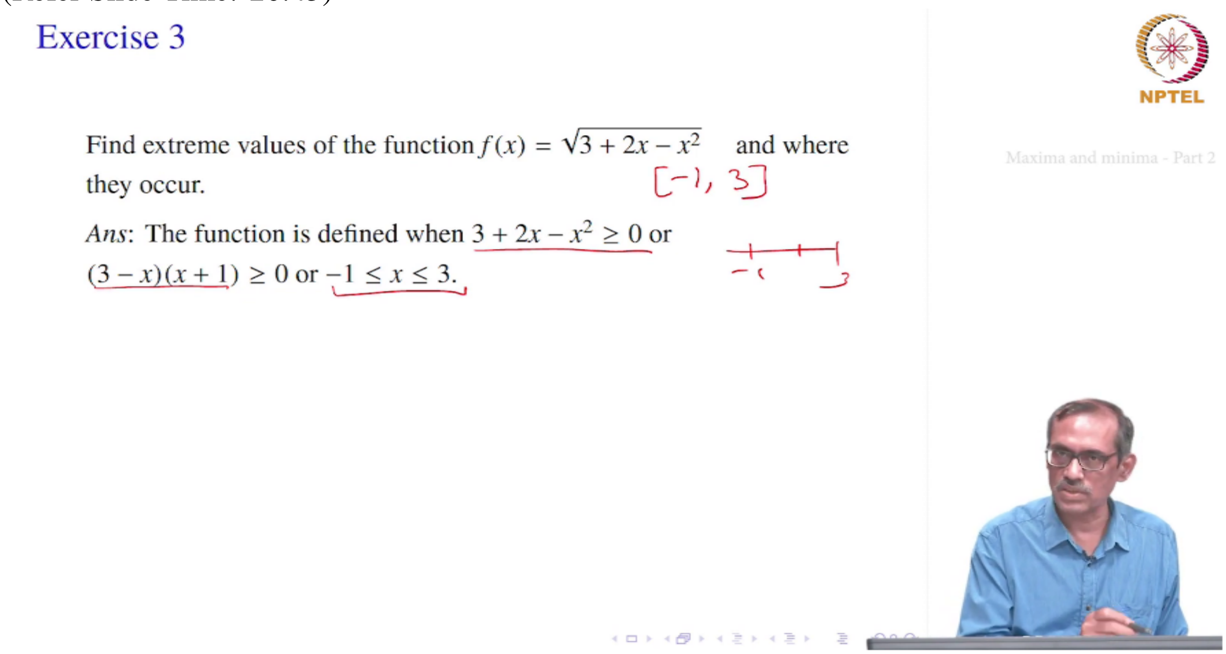

Let us go to next problem. Find the extreme values of the function $f(x) = \sqrt{3 + 2x - x^2}$; and also, find the extreme points of the same. All these things we have to find. Again, we proceed the same way. You see that the domain is not given. What is the domain? Since the square root is there, $3 + 2x - x^2$ must be greater than equal to 0. If you factorize that, you get $(3 - x)(x + 1)$. It is greater than or equal to 0 means x must lie between -1 to 3. Why? The expression is $-(x - 3)(x - (-1))$. If x is inside $[-1, 3]$, then that $x - (-1) \geq 0$ and $x - 3 \leq 0$ so that the expression is greater than or equal to 0. So, that gives $x \in [-1, 3]$. That is our domain now; it is $[-1, 3]$. We consider the interior points, the critical points, the endpoints, and then compare. That is what we do.

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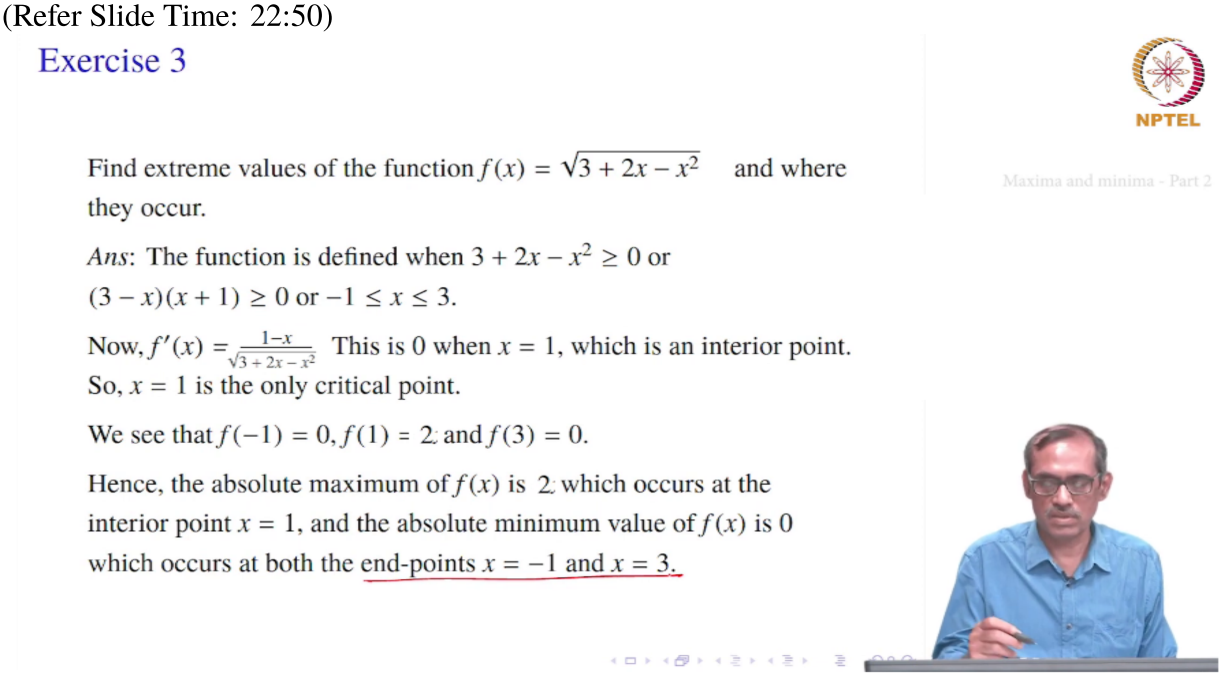

When you differentiate, you get $f'(x) = (1 - x)/\sqrt{3 + 2x - x^2}$. And this is 0, when $x = 1$. But

$x = 1$ does not belong to our domain. Is that fine? No. It is -1 to 3. So, $x = 1$ belongs to the domain. And that is an interior point, of course.

That point $x = 1$ is the only critical point. Then, we compare the values of f at all these points. Now, $f(-1) = 0$, $f(1) = 2$ and $f(3) = 0$. We find that the maximum value of 2 occurs at $x = 1$, and the minimum value of 0 is now occurring at two points: at -1 and at 3. The minimum value of the function is 0. And the points of absolute minima are -1 and 3. Also, these are the end points. Let us stop here.