Basic Calculus - 1
Professor. Arindama Singh
Department of Mathematics
Indian Institute of Technology Madras
Lecture 16 - Part 2
Differentiation Exercises - Part 2

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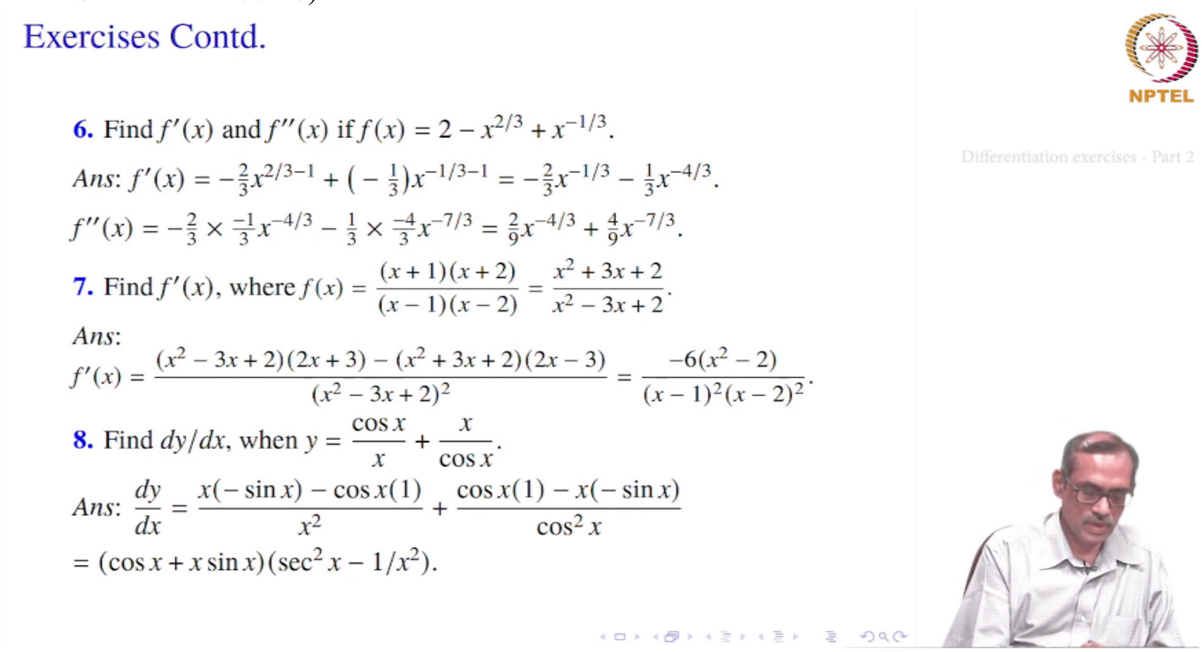

Let us try the next problem. Here, $f(x)$ is given as $2 - x^{2/3} + x^{-1/3}$. We want to find $f'(x)$, and also its derivative $f''(x)$. We should proceed slowly. For $f'(x)$, the derivative of 2 is 0, minus sign remains as minus sign, then $x^{2/3}$ gives $2/3$ into $x^{2/3-1}$; next, $x^{-1/3}$ gives $-1/3$ into $x^{-1/3-1}$. We simplify. It is $-(2/3)x^{-1/3} - (1/3)x^{-4/3}$. That is what our $f'(x)$. Once more we have to differentiate. We proceed from this place. It is $-2/3$, it remains as a constant, and $(-1/3)$ into $x^{-1/3-1}$, which is $-4/3$. And the next factor is $-1/3$ into $-4/3$ into $x^{-4/3-1}$, which is $-7/3$. That is how it looks now. And simplification gives $(2/9)x^{-4/3} + (4/9)x^{-7/3}$.

We go to next problem. Here again $f(x)$ is given as a rational function. On the top there is a polynomial, on the bottom also there is a polynomial. It is given as $(x+1)(x+2)/(x-1)(x-2)$. Sometimes it is easier to differentiate when it is in factored form. But we go to the expanded form that may be easier here.

So, let us write it again. It is $x^2 + 3x + 2$ divided by $x^2 - 3x + 2$. Now, this is a function in the form f/g. The denominator is g and the numerator is f. We remember the formula. It will be g, that is, the denominator times differentiation of the numerator minus and so on. So, let us find it.

That is how it looks: $gf' - fg'$ divided by g^2. Here, f' is $2x + 3$ and g' is $2x - 3$. Our formula gives $(x^2 - 3x + 2)(2x + 3) - (x^2 + 3x - +2)(2x - 3)$ divided by $(x^2 - 3x + 2)^2$. After simplification it turns out to: $-6(x^2 - 2)$ divided by $(x - 1)^2(x - 2)^2$. That is what $f'(x)$ is. These are simple applications of whatever rules of differentiation we have learned.

We go to next problem. Here again, we want to find out dy/dx, given that $y = (\cos x)/x + x/\cos x$. Of course, it is not differentiable at $x = 0$, because this is given, and also at $x = \pi/2$. For all odd multiples of $\pi/2$, there is a problem. So, except that wherever it is differentiable, we need to find dy/dx. That is the way we interpret the question, that wherever this function is defined, what about dy/dx at those points?

We just use our rules. So, dy/dx is the derivative of the first term plus the derivative of the second term. And for the first term the denominator is x, so its derivative is x times the derivative of $\cos x$, that gives $-x \sin x$, then minus the derivative of x times $\cos x$, and then divided by x^2. And we tackle the second one similarly. It is $\cos x$ into the derivative of x which is 1 and minus x into the derivative of $\cos X$ which is $-\sin x$, and then this divided by $\cos^2 x$. We simplify to get $(\cos x + x \sin x)$ times $(\sec^2 x - 1/x^2)$. You should verify it later; whether this is correct or not.

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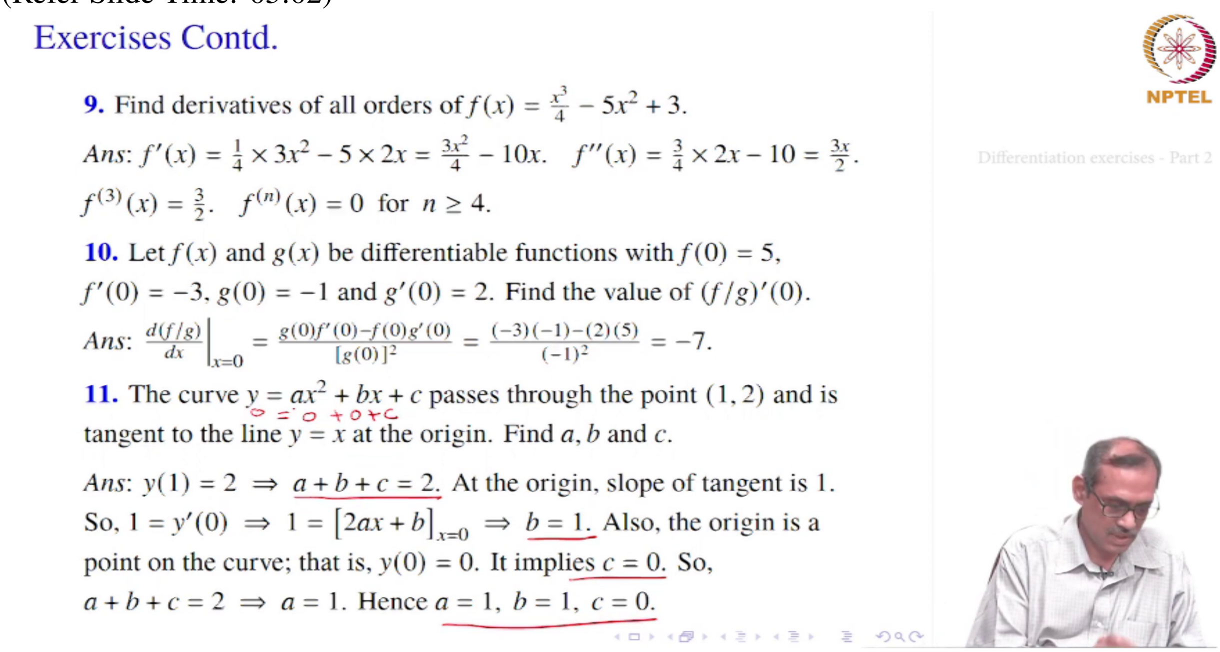

We go to next problem. Here, you want to find the derivatives of all orders of $f(x) = x^3/4 - 5x + 3$. On the face of it, it looks that we have to get $f^{(n)}(x)$ for each natural number n. But let us see. Because it is a polynomial, each time you differentiate, one power of x is reduced; x^n gives nx^{n-1}. That reduces the degree of the polynomial. Then after some finite number of differentiations, the polynomials will reduce to 0. That is how we see it now. So, we can find up to third derivative, the next derivative will be 0.

With that expectation, let us compute its derivative. Now, $f'(x)$ is the derivative of $x^3/4$ minus the derivative of $5x$ plus th derivative of 3. So, $x^3/4$ gives $1/4$ times, x^3 gives $3x^3$; then -5 remains, x^2 gives $2x$. So, that is $3x^2/4 - 10x$. Once more we differentiate. You get $3/4$, that constant, then x^2 $2x$; then $-10x$ gives -10. That simplifies to $3x/2$. When you go to the third derivative of $f(x)$, it is the derivative of $3x/2$; that is just $3/2$. And then when you take the fourth derivative, that will be constant, which is 0. Fifth derivative will be 0, the differentiation of 0 is 0. Everything else will be 0 afterwards; $f^{(n)}(x) = 0$ for $n \geq 4$. That is what we expected from the polynomial.

We go to next problem. Suppose we have two differentiable functions: $f(x)$ is differentiable and $g(x)$ is also differentiable. It is known that $f(0) = 5$. Similarly, we know that $g(0) = -1$. Also, you know that the derivatives at 0, that is $f'(0) = -3$ and $g'(0) = 2$. We want to compute the value of $(f/g)'(0)$. All that we have to do is use the differentiation of the ratio formula, the division. And then see what happens.

Well, for get about 0. Now, $d(f/g) = [gf' - fg']/g^2$. When we evaluate at $x = 0$, everything will be evaluated at 0. So, it is $[g(0)f'(0) - f(0)g'(0)]/[g(0)]^2$. You substitute the values $g(0) = -1$, $f'(0) = -3$, $f(0) = 5$, $g'(0) = 2$. That gives this factor, which simplifies to -7. That is fairly easy.

We go to next problem. Here, we have a curve. Of course, it is a parabola given by $y = ax^2 + bx + c$. It is known that this curve, which is given by this function, $y = ax^2 + bx + c$ passes through the point $(1, 2)$. That means though you have written a, b in symbols, they have certain values. For those values, we know that the curve passes through $(1, 2)$. And we know that the curve is tangent to the line $y = x$ at the origin. That means the origin is also a point on the curve. We want to find out a, b and c. That is what being asked. So, for which curve, this will happen?

It should pass through $(0, 0)$, through $(a, 2)$ and its tangent at $x = 0$ has equation $y = x$. First thing is, let us find out the values. It is given that $(1, 2)$ is a point on the curve. So, y is 2 at $x = 1$, or $y(1) = 2$. Once we substitute $x = 1$, the left side gives 2, and the right side gives $a + b + c$. So, $a + b + c = 2$. That is how it gives one equation.

Next, we see that at the origin, slope of tangent is 1. How do you say that? Because $y = x$ is a tangent to the curve; and its slope is 1. That means the derivative of this function evaluated at $x = 0$ should give us 1. That is, $y'(0) = 1$. Now differentiate y; you get $2ax$ from ax^2, and the next one gives b. So, $2ax + b$ at $x = 0$ must be equal to 1. Once you substitute $x = 0$ that gives b; so, $b = 1$.

Now, you get $b = 1$, $a + b + c = 2$, and of course, the curve passes through the origin because at the origin there is a tangent. So, when you substitute 0 here, you get $0 = 0 + 0 + c$. That gives $c = 0$. We know $c = 0$, $b = 1$ and $a + b + c = 2$. So, a must be 1. Therefore, we conclude that $a = 1$, $b = 1$ and $c = 0$.

We discuss next problem. Here, a curve is given by the equation $y = \tan x$. We are taking a segment of that curve, a portion of the curve which is traced as x varies from $-\pi/2$ to $\pi/2$, but not equal to $-\pi/2$ or $\pi/2$. Notice that $\tan x$ also is not defined there. That means we are taking one branch of $\tan x$. It goes from $-\pi/2$ to $\pi/2$ this way as we are taking only one branch of that. It repeats again, but we are not taking that.

So, this is our curve. We want to find all points on this curve, where the tangent is parallel to the line $y = 2x$. What does that mean? Let us take a point say $(c, y(c))$, where the tangent to this curve will be parallel to the line $y = 2x$. Once it is parallel, its slope is known. Its slope is 2. Recall that if it is in the form $y = mx$, its slope is m. Now, at that point, the slope of the tangent is 2.

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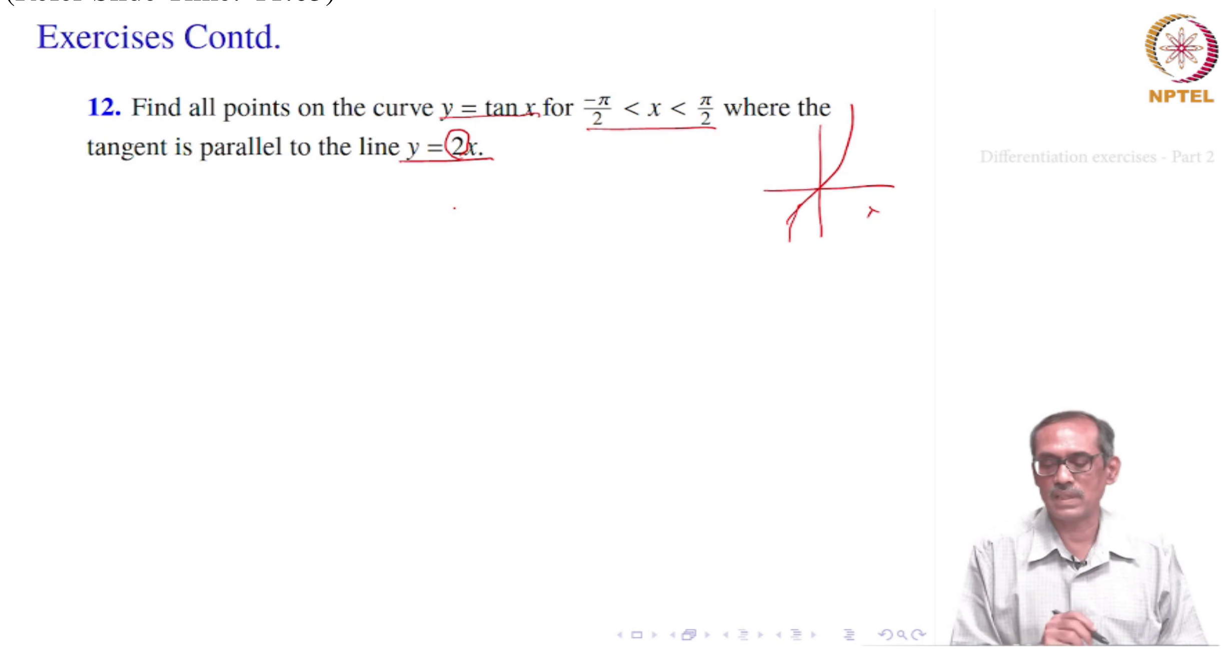

First of all, at that point (a, b), it will satisfy $b = \tan a$ where a is in between $-\pi/2$ to $\pi/2$. Second thing is, the slope, that is, the derivative of this evaluated at that point (a, b) would give us 2. These are the two equations we get. We should get the equations first.

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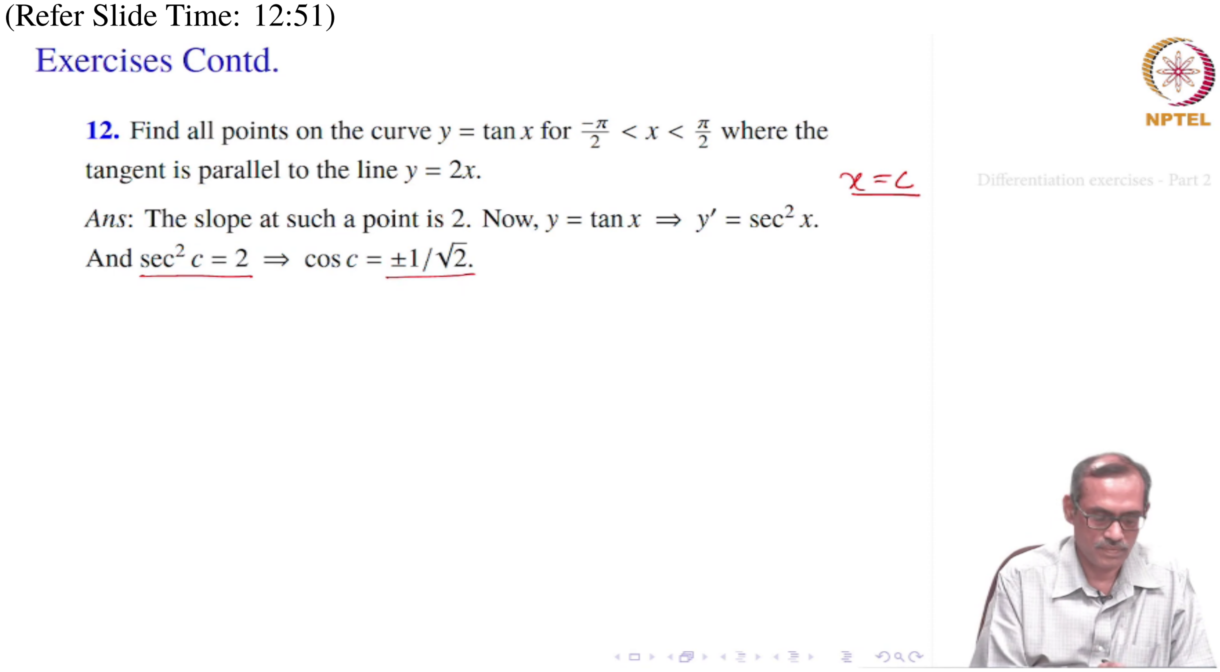

The slope at such a point is 2. Now, $y = \tan x$ gives us $y' = \sec^2 x$. If that point is c, say, $x = c$, then at that point, $\sec^2 c$ must be equal to 2. Or, $1/\cos^2 c = 2$, or, $\cos c = \pm 1/\sqrt{2}$. Since $c \in (-\pi/2, \pi/2)$, c must be $\pm\pi/4$; that is what we say. So, from the slope itself, we get some hint that c must be $\pm\pi/4$. And that is it.

So, we got all the points; both of them are there. The points are $(\pi/4, \tan(\pi/4))$ and $(-\pi/4, \tan(-\pi/4))$. As $\tan(\pi/4) = 1$ and $\tan(-\pi/4) = -1$, we get the points as $(\pi/4, 1)$ and

$(-\pi/4, -1)$. These are the two possible points where the tangent to the curve $y = \tan x$ is having slope equal to 2.

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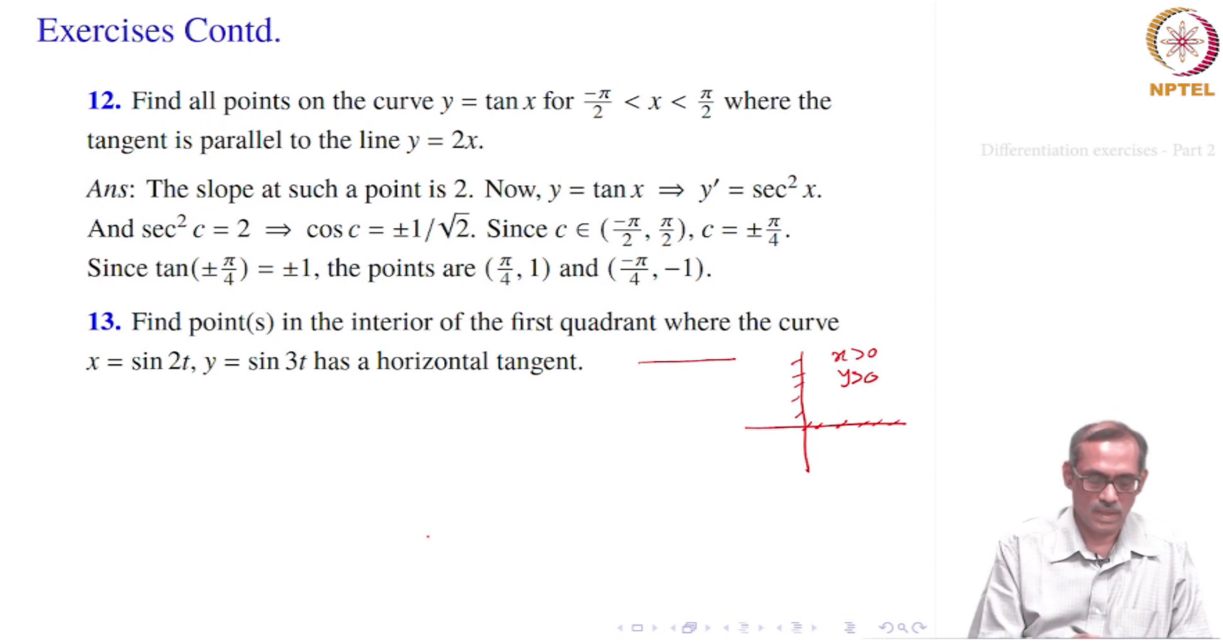

We will go to the next problem. Here we are asked to find points in the interior of the first quadrant. What does it mean? This is the first quadrant. Its interior means these lines are excluded. So, it says that $x > 0$, $y > 0$. In the interior of the first quadrant the curve $x = \sin(2t)$, $y = \sin(3t)$, which is given parametrically, has a horizontal tangent. So, such a tangent should be something like $y = 0$. That should be the tangent, where the slope is 0.

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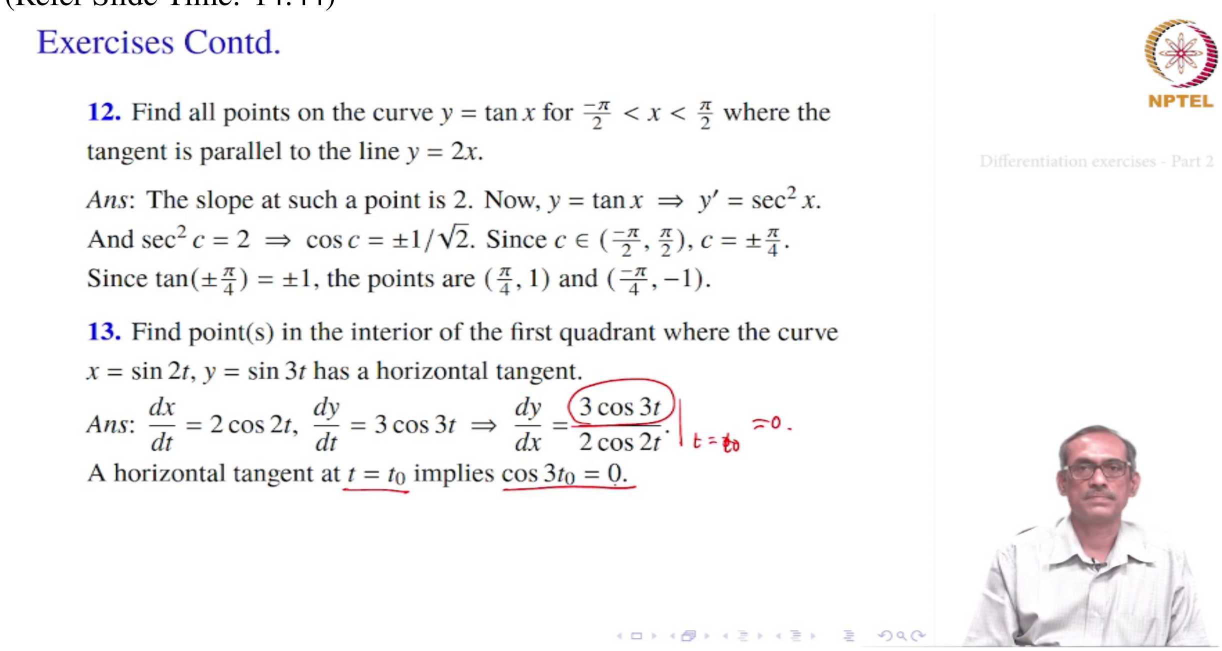

We will find the slope. Since it is given parametrically, we have $dx/dt = 2\cos(2t)$, $dy/dt = 3\cos(3t)$. So, this gives $dy/dx = 3\cos(3t)/(2\cos(2t))$. This is this is the slope at the point t. At the point t means the point on the curve is $(x(t), y(t))$. A point t means its coordinates are $x(t)$ and $y(t)$. A horizontal tangent at $t = t_0$ means this derivative at $t = t_0$ must be 0. Since the derivative is a fraction, it is 0 means its numerator must be 0. So, you get $\cos(3t_0) = 0$.

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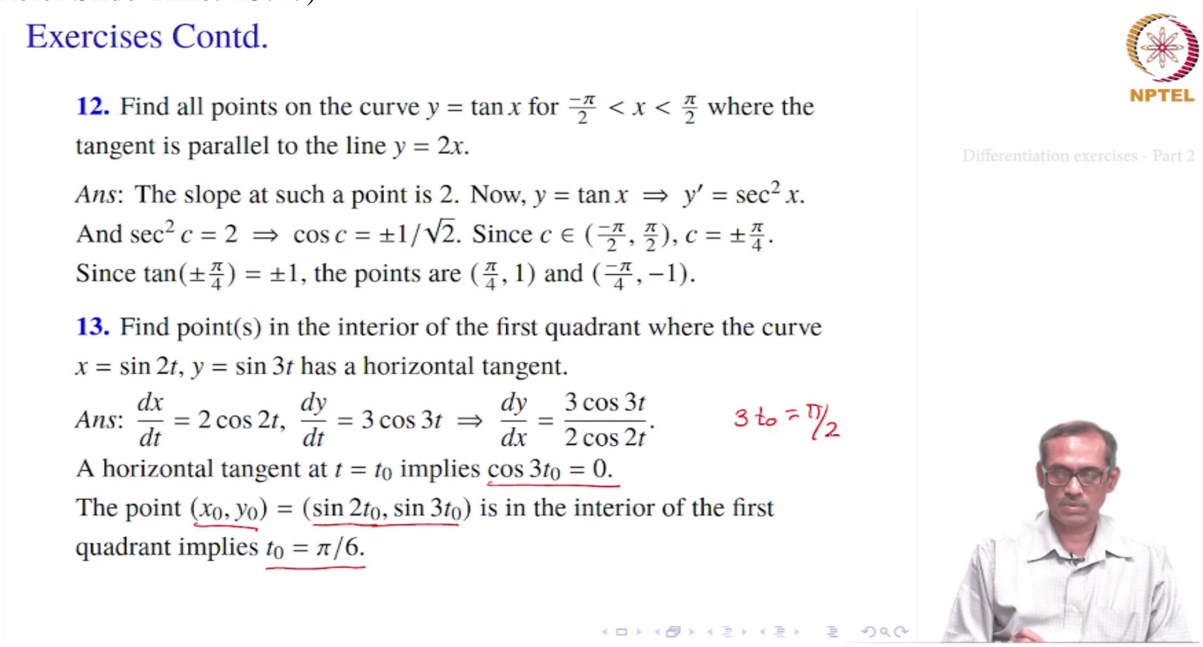

So, if the point is (x_0, y_0), that is, if $(\sin(2t_0), \sin(3t_0))$ is the point, then it must be in the interior of the first quadrant. That means $\sin(2t_0)$ and $\sin(3t_0)$ should be positive. And the slope condu=ition implies that $\cos(3t_0) = 0$. Now, where cos becomes 0 in the first quadrant? It is at $\pi/2$. There is no other solution. So, $3t_0 = \pi/2$ implies $t_0 = \pi/6$. See that at $t_0 = \pi/6$, both $\sin(2t_0) = \sin(\pi/3)$ and $\sin(3t_0) = \sin(\pi/2)$ are positive.

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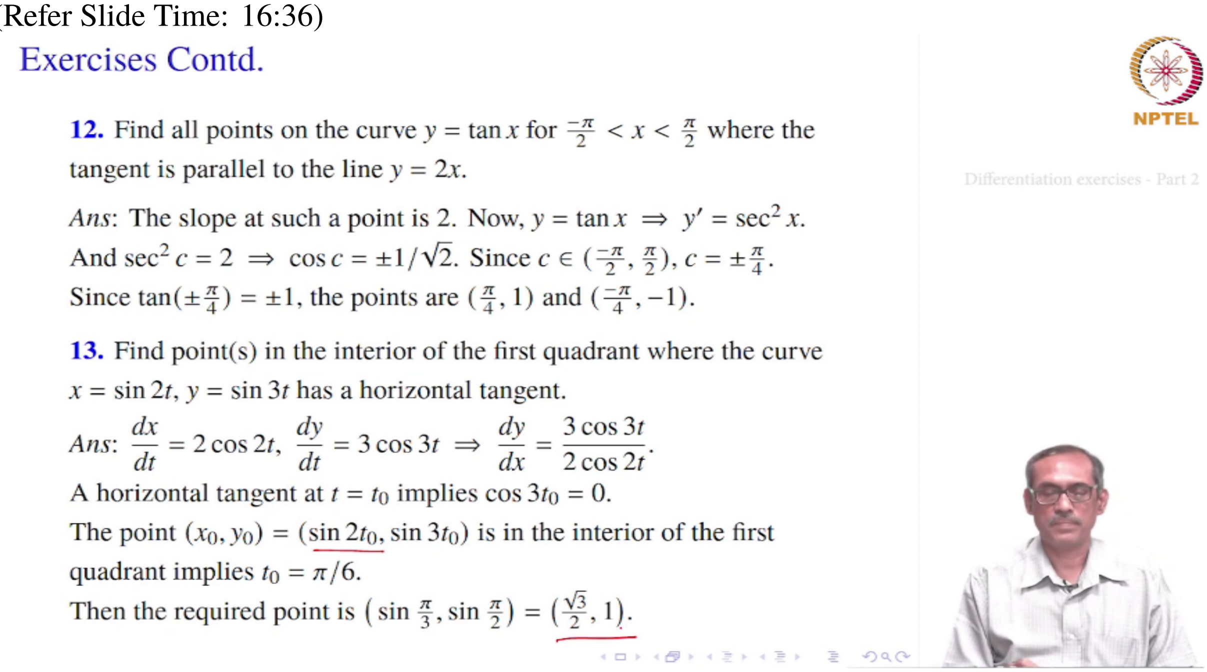

So, the point is $(\sin(2t_0), \sin(3t_0))$, or, $(\sin(\pi/3), \sin(\pi/2))$, which is $(\sqrt{3}/2, 1)$.

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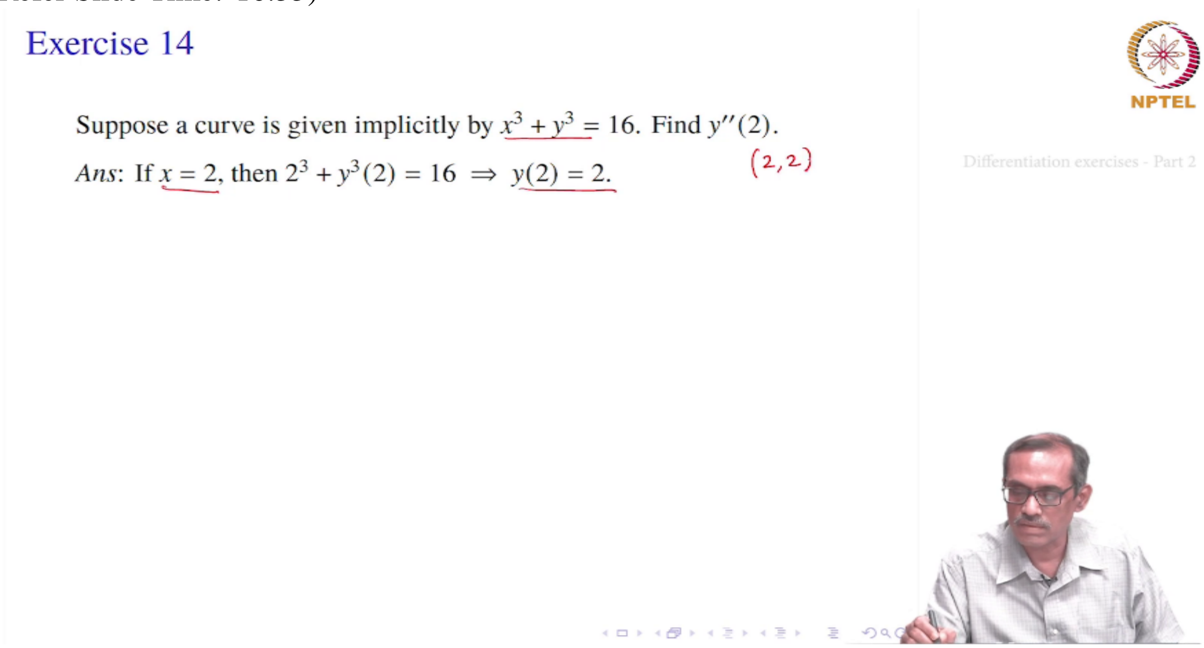

We may solve one more problem. Suppose a curve is given implicitly by $x^3 + y^3 = 16$. We want to find $y''(2)$. We need implicit differentiation here. Then we find out $f''(x)$ and substitute $x = 2$ to get the result.

First thing is, the point corresponding to $x = 2$ must be on the curve. That is if $x = 2$, then what is y? If $x = 2$, then then the equation says $2^3 + [y(2)]^3 = 16$. It gives $[y(2)]^3 = 16 - 8 = 8$. So, $y(2) = 2$. That means, $(2, 2)$ is a point on the curve.

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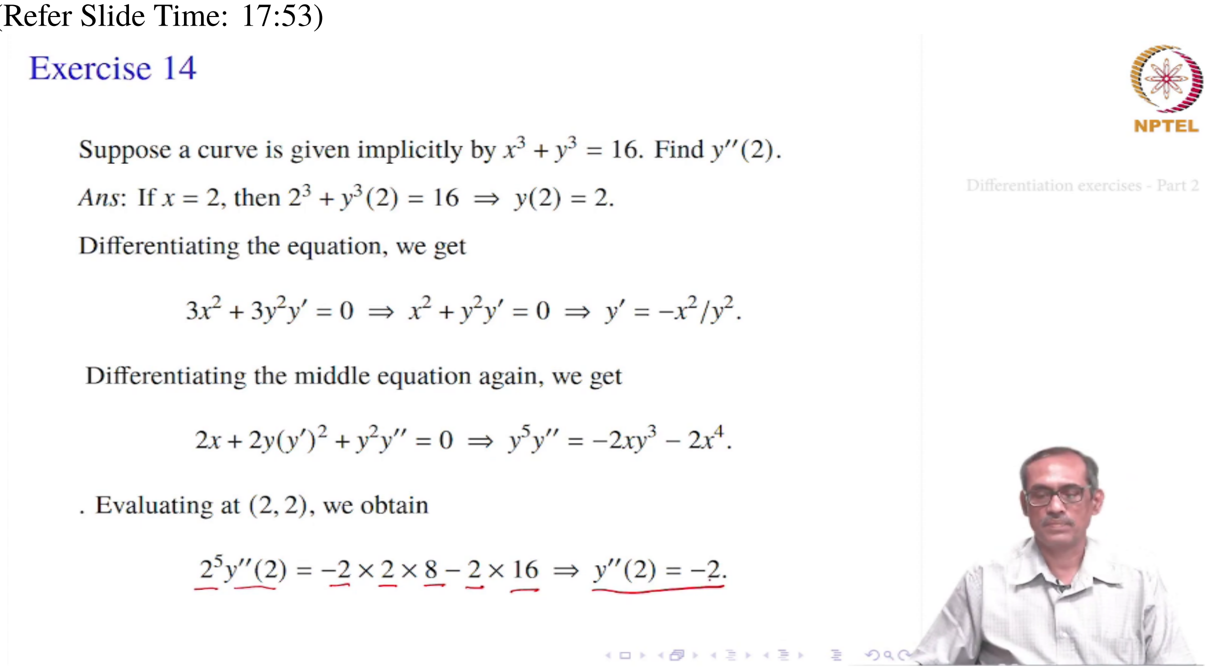

Now let us differentiate the equation. We have: x^3 gives $3x^2$ and y^3 gives y^3 with respect to y and then y with respect to x. We are differentiating with respect to x. That is, $3x^2 + 3y^2y'$. And on the right side of course constant, so that gives 0. That gives $y' = -x^2/y^2$. And we want to find at

2, of course, but it is not $y'(2)$; it is $y''(2)$. So, you need to find one more differentiation.

Instead of differentiating $y' = -x^2/y^2$, where we have to use the ratio formula or the division formula, we consider this equation itself. That might be easy. We have $x^2+y^2y' = 0$. Differentiation of x^2 gives $2x$; for y^2y', we use the product formula. It is the differentiation of y^2 multiplied by y' plus y^2 into the derivative of y'. The differentiation with respect to x of y^2 gives $2yy'$. (It is y^2 with respect to y into dy/dx.) That gives $2yy'$ into y' plus yy''. That is equal to 0. If you simplify, you get y'' times y^5 is equal to $-2xy^3 - 2x^4$.

We are interested in $y''(2)$. So, evaluate everything at 2. It gives $[y(2)]^5y''(2) = -2 \times 2 \times [y(2)]^3 - 2 \times 2^4$. As $y(2) = 2$, that gives $y''(2) = -2$. That is how we will be solving these problems. Let us stop here.