Basic Calculus - 1

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Lecture 16 - Part 1

Differentiation Exercises - Part 1

This is lecture 16 of Basic Calculus 1. Recall that in the last lecture, we had given the rules of differentiation, and then applied that to some of the problems. Today's lecture will be simply problem based, we will solve some more problems relying on again those notions.

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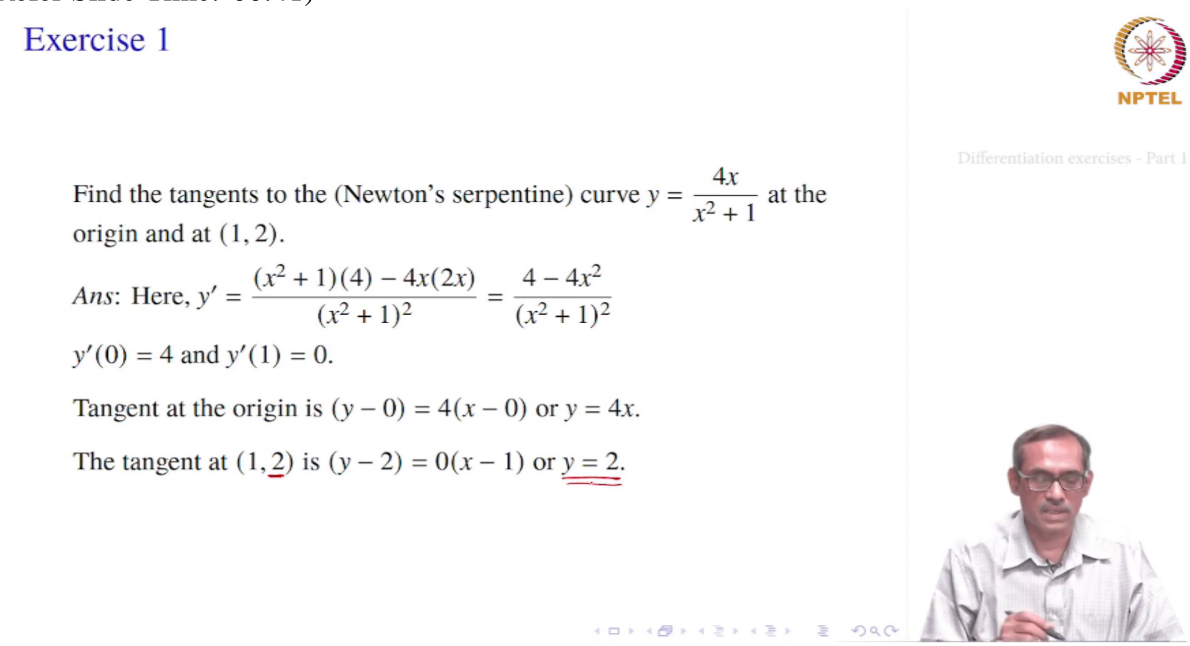

Let us go to the first problem. Exercise 1 asks to find the tangent to the curve $y = 4x/(x^2+1)$ at the origin, and also at $(1, 2)$. There are two points where we have to find the tangents. This particular curve is called Newton's serpentine due to its graph. Our function is $y = 4x/(x^2+1)$. We want tangents at the origin and at $(1, 2)$; at two points.

First, we differentiate and evaluate at 0 to get the slope. Similarly, at $x = 1$, we also need to get the slope. And then use the straight line formula that it passes through the origin and $(1, 2)$ having this and this slope.

For differentiation of $y = 4x/(x^2+1)$, see that it is in the form $f(x)/g(x)$. Its derivative will be $g(x)f'(x) - f(x)g'(x)$ divided by $[g(x)]^2$. Here, $g(x) = x^2+1$ and $f(x) = 4x$. So, in the numerator we have $(x^2+1) \times 4 - 4x(2x)$. In the denominator, we will have $[g(x)]^2$, which is $(x^2+1)^2$. Then we simplify. It is $(4 - 4x^2)/(x^2+1)^2$.

So, when we evaluate this at the origin, this x will be equal to 0. Once you substitute $x = 0$ there, in the numerator we get 4, denominator becomes 1, so, $y'(0) = 4$. Similarly, when you take $x = 1$, because we want at the point $(1, 2)$ also, it is $y'(1) = 0$. These are the slopes. When you say

y' equal to 0, that means, the tangent is parallel to the x-axis at $(1, 2)$. And at the origin, the slope is 4.

So, the equation of the tangent at the origin will be $y = 4x$, that is it. And the tangent at the point $(1, 2)$ is $y - y_1 = m(x - x_1)$, which is $y - 2 = 0(x - 1)$. You simplify. That gives $y = 2$; as it should be parallel to the x-axis. So, that is how we will be proceeding. This is an easier one where differentiation is easy.

We will take the next problem. In this problem, we are given that f is a function where $f(2) = 8$. We assume that it is defined everywhere, unless there is some problem. Next, there is another function, which is g. It is given that $g(2) = -3$. Also, we are given their derivatives, at 2 again; we are taking both at $x = 2$. So, $f'(2) = 1/3$ and $g'(2) = -3$. We are required to find the value of the derivative of $\sqrt{f^2 + g^2}$ at $x = 2$.

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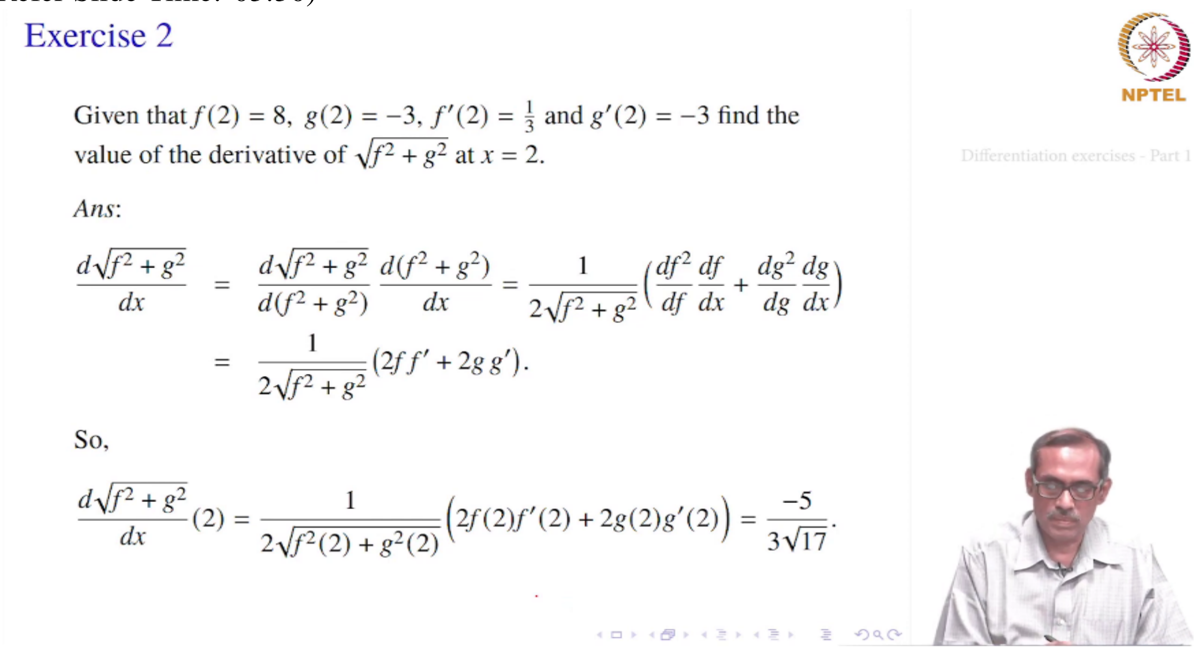

All that we have to do is differentiate this and express it in terms of f', g', f and g if possible, because these are given at 2 only. Then that should give us the answer. So, first we differentiate $\sqrt{f^2 + g^2}$. We have to use the composition rule. It is the square root of something. If you take that something as z, then it will be $d\sqrt{z}/dz$ and etc. So, we go on writing with our experience that as $d\sqrt{f^2 + g^2}/d(f^2 + g^2)$. And then it will be multiplied by $d(f^2 + g^2)/dx$. And all these things have to be evaluated at 2 only. So, let us find out.

This first one is $d\sqrt{z}/dz$, which is $1/(2\sqrt{z})$. Why? It is $z^{1/2}$. So, its derivative is $(1/2)z^{-1/2}$. This is $1/(2\sqrt{f^2 + g^2})$. Then, we differentiate $f^2 + g^2$ with respect to x. Again, we want the derivative of f^2 with respect to x. First, we differentiate with respect to f, and then f with respect to x. Similarly, for g^2. It will be the derivative of g^2 with respect to g and dg/dx. They will be added together with our plus rule. So, that is how it will look now.

Now, df^2/df would give us $2f$ and then df/dx is $f'(x)$ anyway. Similarly, dg^2/dg is $2gg'$. We do not know how to evaluate this at any point, but we are given the values at 2. Our data says $f(2)$,

$g(2)$, $f'(2)$ and $g'(2)$. And of course, we want the derivative of $\sqrt{f^2+g^2}$ at 2. So, we evaluate them both the sides at 2. Then we would get the answer at 2 as $1/2\sqrt{f^2(2)+g^2(2)}$ multiplied with the pother factor. We substitute the values and simplify to get the answer and that comes out as $-5/3\sqrt{7}$. You should verify this. That is how we will be proceeding by using the chain rule.

Let us go to next problem. Here, we are asked to find dy/dx given that y is equal to a function of x, which is $[1+\tan^4(x/12)]^3$. Again as you see, it is a composition of many functions. We have to differentiate the inside first like z^3 with respect to z, then differentiate z with respect to x. Here, z will be $1+\tan^4(x/12)$. So, we proceed slowly.

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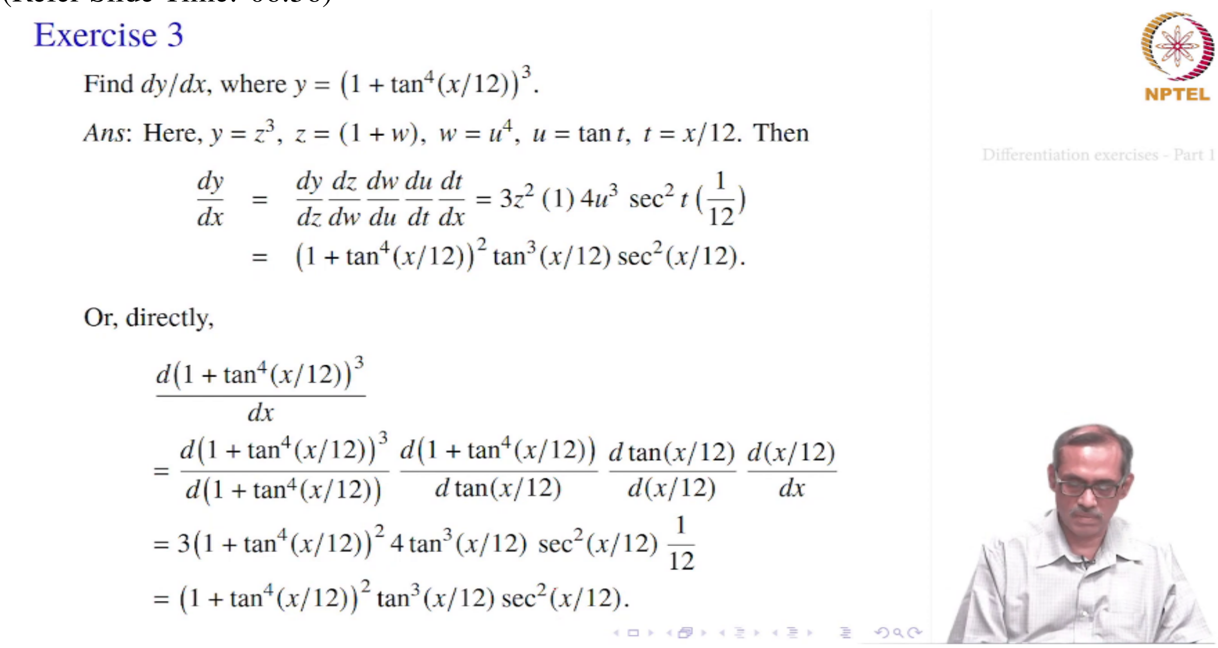

Suppose, we write it as z^3 with $z=\tan^4(x/12)$. We proceed the other way. Let us say $t=x/12$ and $u=\tan t$. That means, $u=\tan(x/12)$. Let $w=u^4$; that means $w=\tan^4(x/12)$. Next, take $z=1+w$; that means z is the inside expression $1+\tan^4(x/12)$. And, then $y=z^3$. That is how it will look now. If you read from the left side, then, $y=z^3$, $z=1+w$, $w=u^4$, $u=\tan t$ and $t=x/12$. So, our formula will be

$$\frac{dy}{dx}=\frac{dy}{dz}\,\frac{dz}{dw}\,\frac{dw}{du}\,\frac{du}{dt}\,\frac{dt}{dx}.$$

Again, it is an application of the composition rule. If we proceed slowly, we see that $dy/dz = d(z^3)/dz = 3z^2$, $dz/dw = d(1+w)/dw = 1$, $dw/du = d(u^4)/du = 4u^3$, $du/dt = d(\tan t)/dt = \sec^2 t$ and $dt/dx = d(x/12)/dx = 1/12$. Then, you substitute their values to get the answer. As z is $1+\tan^4(x/12)$. It will be 3, it will be 3 and this 4 cancels with 12. So, we have canceled it, what remains is $\left(1+\tan^4(x/12)\right)^2\tan^3(x/12)\sec^2(x/12)$.

Or you could have done it directly, with a slight experience, instead of substituting all these. We just write d of the total equal to d of that cubed divided by that one which is z. Then dz/dw would look like d of $1+\tan^4(x/12)$ divided by our $d\tan(x/12)$. Then, $d\tan(x/12)$ over $d(x/12)$ and $d(x/12)/dx$. Then immediately you get it. It is tan fourth, so it gives three times tan square, and this gives us 1 plus tan fourth. It is like 1 plus z fourth by z. That gives $4\tan^3 z$, or, $4\tan^3(x/12)$, and then $\tan(x/12)$ divided by $x/12$, that will be $\sec^2(x/12)$, next $x/12$ by dx is $1/12$ as earlier. We get the same answer.

So, this is how we will be proceeding. Sometimes the direct method will be easier to follow. But if there is a confusion, it is better to substitute and see slowly how it goes.

We go to the next problem, the fourth one. We are required to find the derivative of the function $y = f(x)$ where the function $f(x)$ is not given as an explicit expression, but x and y are given parametrically. There are some problems under this.

We take the first problem, where the parametric equation for x and y are given as $x = a\cos t$, $y = a\sin t$ for t varying between 0 to π. What is this really, $x = a\cos t$, $y = a\sin t$? It is a circular arc, where a is the radius. It is really the upper semicircle. Because t varies from 0 to π, it is the upper semicircle $x^2 + y^2 = a^2$ or, we write it as $y = \sqrt{a^2 - x^2}$. So, we have now two ways open. We can directly take $y = \sqrt{a^2 - x^2}$ and differentiate this. Or, we may go to parametric differentiation, find dx/dt, dy/dt and then take suitable multiplication or division to find dy/dx.

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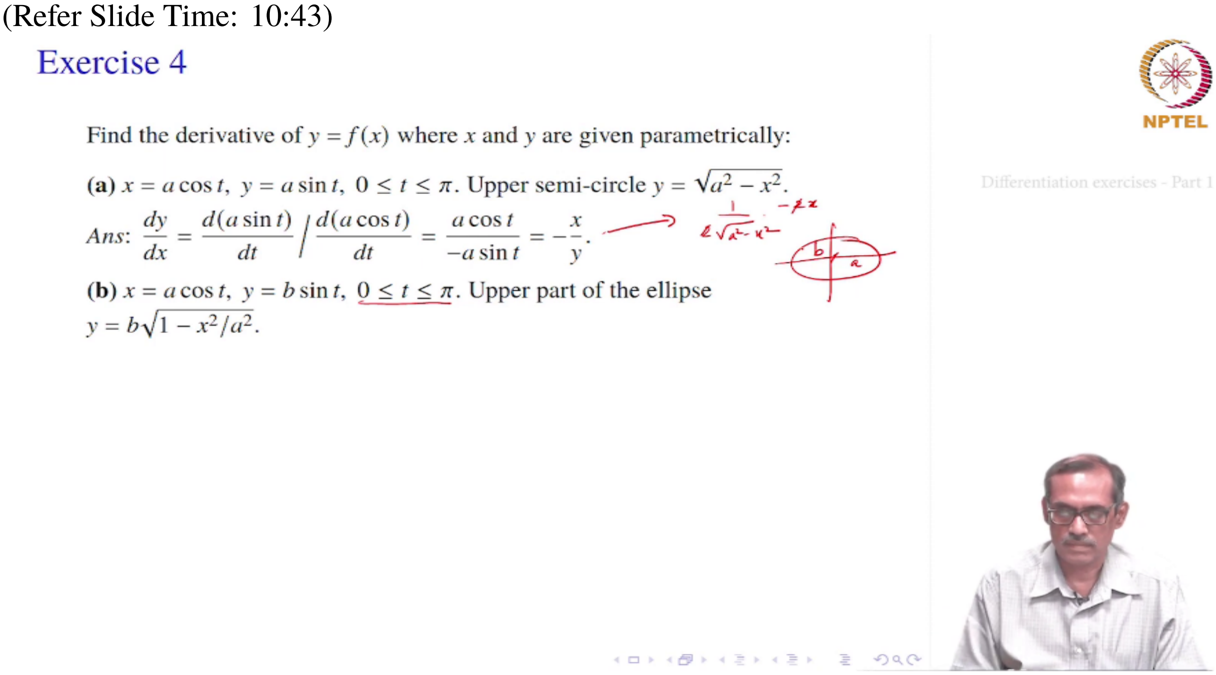

So, let us do it the parametric way. Now dy/dx is dy/dt divided by dx/dt. The first one gives $d(a\cos t)/dt$, which is $-a\sin t$; this divided by $d(a\sin t)/dt$, which is $a\cos t$. So, a cancels, we have a minus sign. Now, without canceling a, we can directly use $a\cos t = y$ and $a\sin t = x$ so that it is just $-x/y$.

Or, if you do directly, you can get the same answer from $y = \sqrt{a^2 - x^2}$. Here, dy/dx will be the square root with respect to that factor first. That will give you $1/2\sqrt{a^2 - x^2}$ times the derivative of $a^2 - x^2$ with respect to x. That would give $-2x$. Then, this 2 cancels and you get $-x/\sqrt{a^2 - x^2}$, which is same as $-x/y$.

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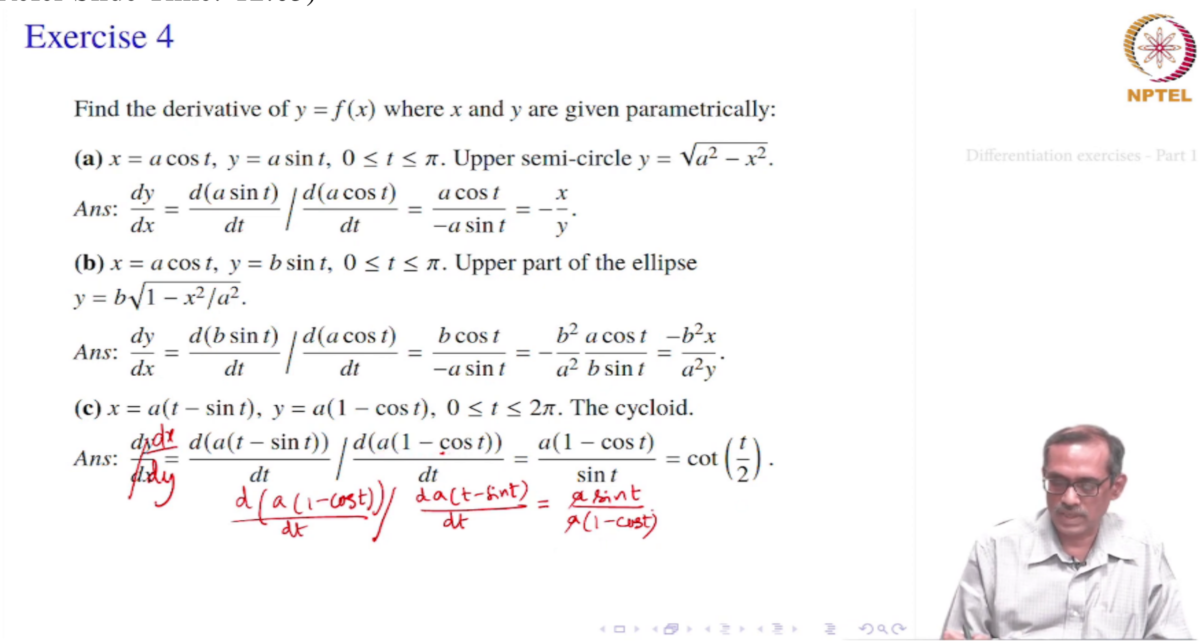

We go to second problem under this fourth one. Here, x and y are given parametrically and we want to find dy/dx. The equations are $x = a\cos t$, $y = b\sin t$, where t varies from 0 to π. This is an ellipse, the upper part of the ellipse, with a as this semi-major axis and b as the semi-minor axis. That is how it looks like. Again, you have two options, whether to do parametrically or directly.

Let us do one of them, say we will use parametric differentiation. Here dy/dx is dy/dt divided by dx/dt as earlier. Now, $y = b\sin t$. Its derivative with respect to t is $b\cos t$. And, $x = a\cos t$; so, its derivative is $-a\sin t$. So, $dy/dx = -b\cos t/(a\sin t)$. We need to express in terms of x and y if possible. So, you multiply ab on the numerator and denominator. That gives $-b^2/a^2$ into $a\cos t$ divided by $b\sin t$. Now, $a\cos t$ gives you x and $b\sin t$ gives you y. So, our answer is $-b^2x$ divided by a^2y. Of course, you can work with $y = f(x)$ directly as you did earlier for the case (a). You should get the same answer.

You have the third problem under this, where our parametric equations are $x = a(t - \sin t)$, $y = a(1 - \cos t)$ for $0 \le t \le 2\pi$. This is called a cycloid. Wwe proceed again similarly, and compute dy/dx, which is dy/dt divided by dx/dt. There is some problem here. It should be dy/dx not dx/dy. We are computing dy/dx. So, this should be $d(a(1 - \cos t))/dt$ divided by $d(a(t - \sin t))/dt$. That would give $a\sin t$ divided by $a(1 - \cos t)$. That is how it looks.

It is easier to keep as it is. Well, a just cancels, and you get $\sin t$ divided by $1 - \cos t$. 1 minus cosine t. What we calculated here was giving dx/dy not dy/dx. If you compute dx/dy, that will be equal to this; and the top one, whichever is given. That is how we will be proceeding. And then this is $\sin t$ divided by $1 - \cos t$.

Let us go to next one. The fourth one says that x is given as $a\cos^3 t$ and $y = a\sin^3 t$. Again, t varies from 0 to 2π. This is called an asteroid. If you eliminate t from here, you would get the equation in this form: $x^{2/3} + y^{2/3} = a^{2/3}$. To verify you substitute. That gives you, as $x = a\cos^3 t$, $[a\cos^3 t]^{2/3} = a^{2/3}\cos^2 t$; and as $y = a\sin t$, $y^{2/3} = a^{2/3}\sin^2 t$. So that would simplify to $a^{2/3}$.

That is how you eliminate t.

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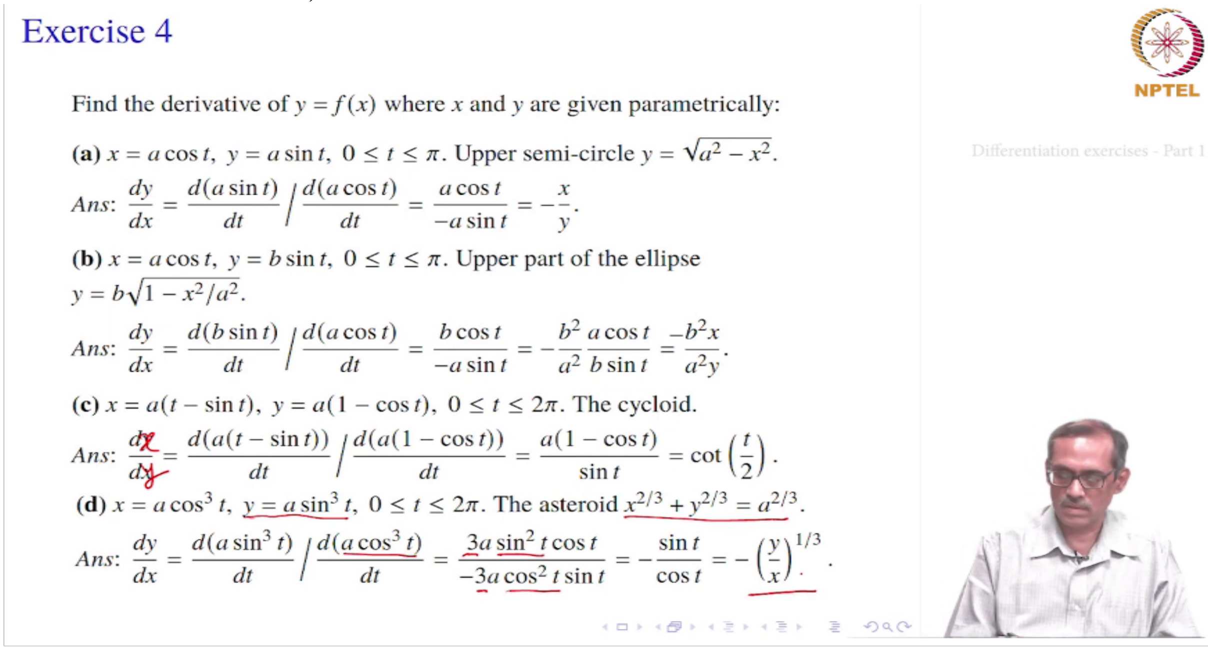

You want to find out dy/dx. Again, that is dy/dt divided by dx/dt. Now, dy/dt is $d(a\sin^3 t)/dt$. Use composition formula. It is equal to $ad(\sin^3 t)/d(\sin t)$ multiplied with $d\sin t/dt$. That gives $a \times 3\sin^2 t \times \cos t$. Now, dx/dt is equal to $d(a\cos^3 t)/dt$. It similarly gives $ad(\cos^3 t)/d(\cos t)$ multiplied by $d\cos t/dt$. This is $a \times 3\cos^2 t \times (-\sin t)$. When you divide, $3a$ cancels and $\sin t\cos t$ cancels. So, you get $-\sin t/\cos t$, which you can express also in terms of y/x. Now, this is a nice one to verify directly. If you use implicit differentiation for this equation directly, you should also get the same answer.

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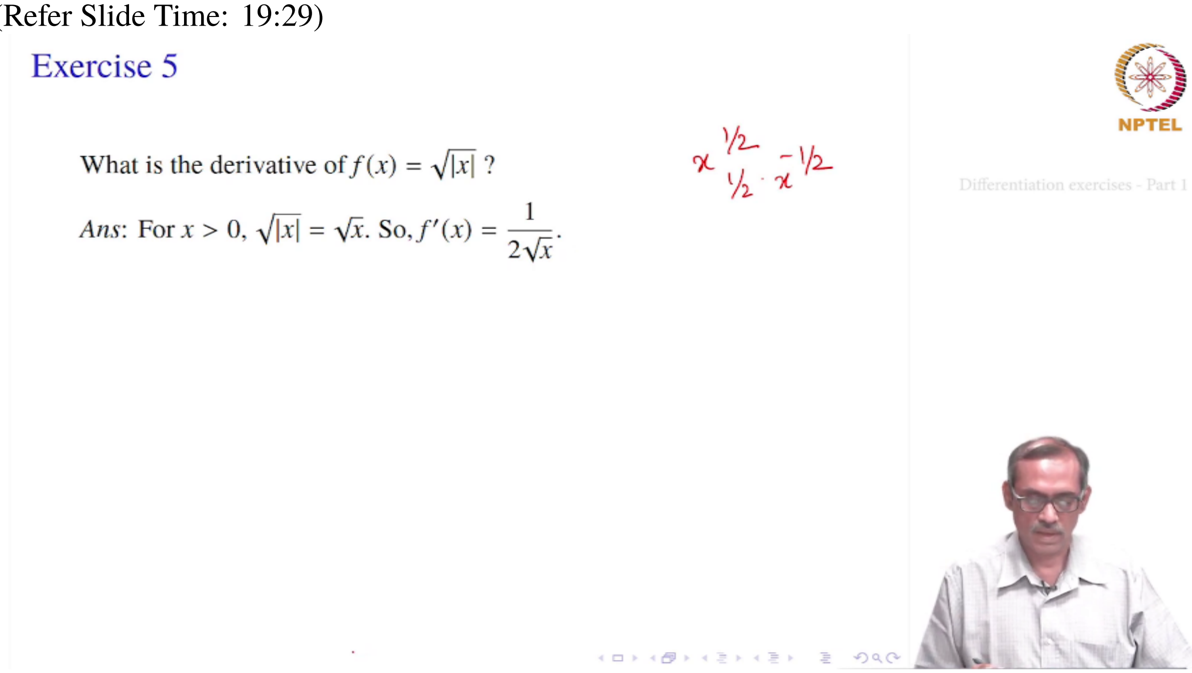

So, we go to next problem. Here, we are asked to find the derivative of $\sqrt{|x|}$. As we know, if you take $|x|$, then its derivative we can find this way: when $x < 0$, it is $-x$. So, its derivative is -1 at every point $x < 0$. When $x > 0$, it is x. So, its derivative is 1 for $x > 0$. And, it is not differentiable at $x = 0$. But we are asked to find the derivative of $f(x) = \sqrt{|x|}$. Since mod is non negative, its square root is well defined for any x. So, its domain is the whole of real numbers. Then, as our experience suggest, we must consider three cases separately: when $x < 0$, when $x = 0$ and when $x > 0$.

Let us take the simplest one $x > 0$. Then, $\sqrt{|x|} = \sqrt{x}$ because $|x| = x$ for $x > 0$. And we find its derivative directly. So, $f'(x) = 1/[2\sqrt{x}]$ as $x^{1/2}$ gives $1/2$ times $x^{-1/2}$. So, for $x > 0$, this is the derivative.

Now, what happens at $x = 0$? What will be $f'(0)$, the derivative at $x = 0$? These formulas would not give us anything. We go back to our definition. By definition, it is the limit of $[f(0+h) - f(0)]/h = f(h)/h$. Now, what is $f(h)$? Here, h can be positive or negative. When h is positive, $f(h) = \sqrt{h}$. There is no 1 here; it is square root of h. But we are computing this divided by h. So, when you divide by h, that gives $1/\sqrt{h}$. That is fine.

Next, when $h < 0$, $f(h) = \sqrt{-h}$. In $\sqrt{-h}/h$, write this h as $-(-h)$. Now $-h$ is positive. Then, $\sqrt{-h}$ cancels; you would get $\sqrt{-h}$ here. There is a negative sign, we take it to the top, and write $-1/\sqrt{-h}$. Is that correct? For $h < 0$, this is how this ratio is evaluated. To find the derivative, we have to really compute the limit of this as h goes to 0; that is, when $h > 0$ and when $h < 0$. So, you have to take the right hand and left side limits.

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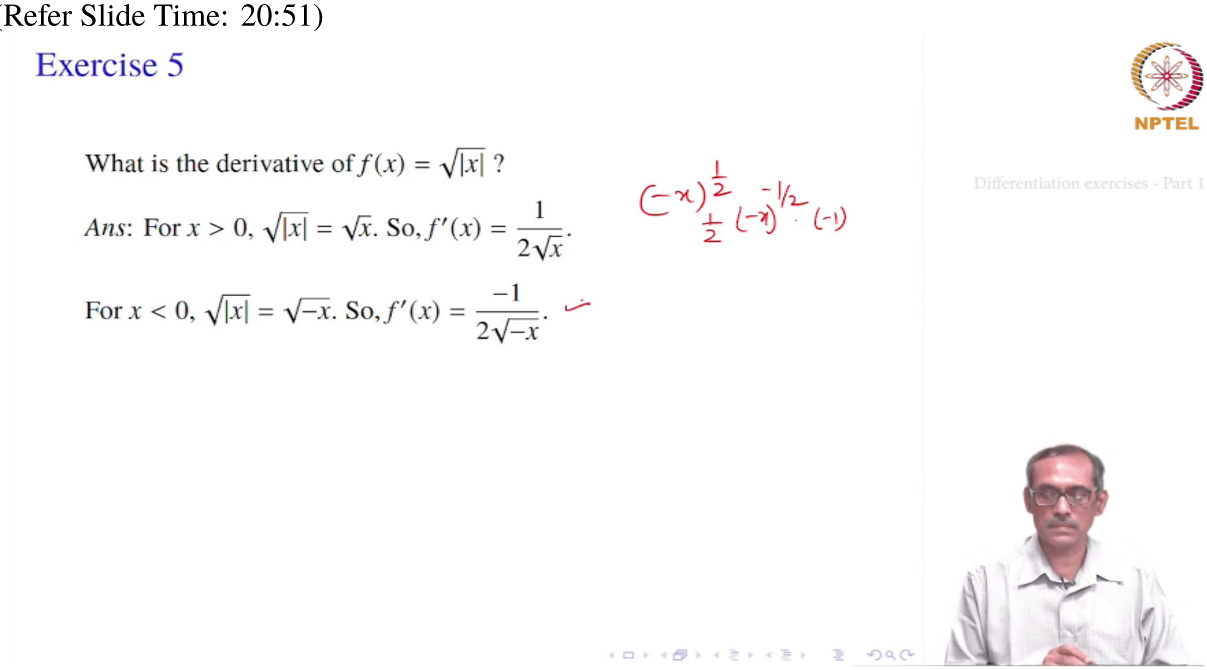

Now, when $x < 0$, $|x|$ becomes $-x$. So, $\sqrt{|x|}$ is $\sqrt{-x}$ for $x < 0$. For $x < 0$, $-x$ is non negative, so its square root is meaningful. Now, how do we proceed for its differentiation? We differentiate with respect to x not $-x$. So, it will be the differentiation of $(-x)^{1/2}$. It will be $1/2$ and $(-x)^{-1/2}$. We are not going to write $(-x)^{-1/2}$ in terms of x; here, $-x$ is positive. So, it will kept as it is. And then, $d(-x)/dx$ gives -1. So, you get the answer as $-1/2\sqrt{-x}$; that is what it is.

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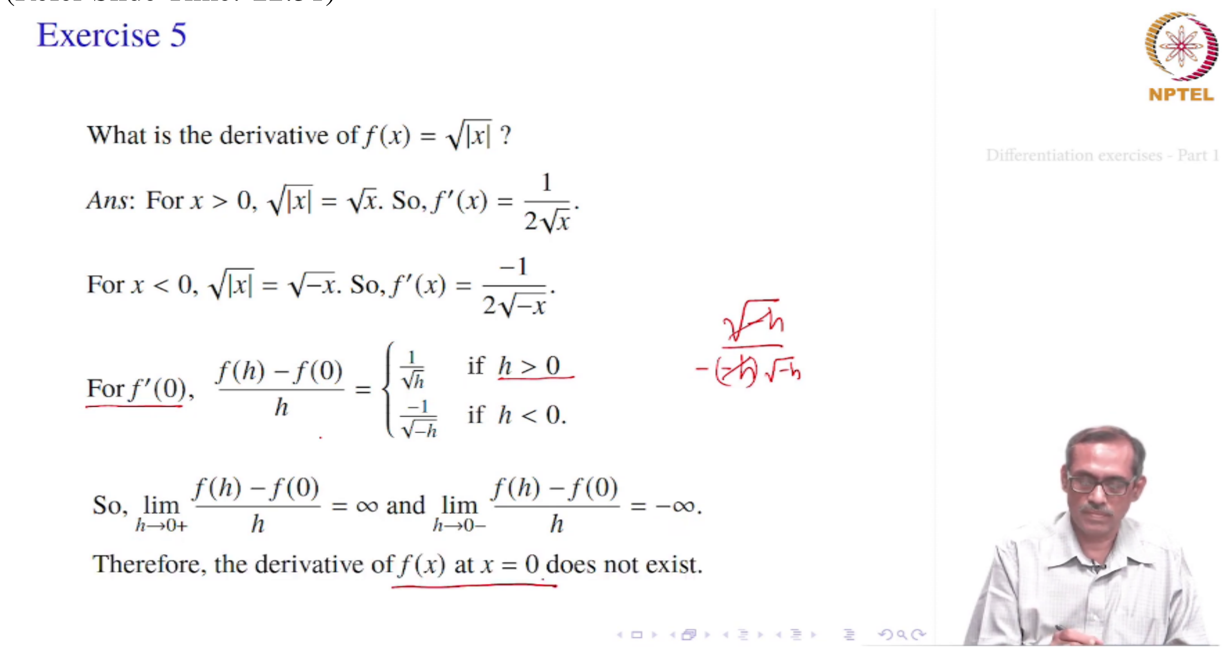

And what will be the limit? When $h > 0$, that is $h \to 0+$, it is $1/\sqrt{h}$. That becomes ∞ as its limit. When $h < 0$, taht is, $h \to 0-$, it is $-1/\sqrt{-h}$; it gives you $-\infty$ in the limit, as there is a negative sign. This the how the limit looks like. Therefore, $f'(0)$ does not exist. That is, the function $f(x) = \sqrt{|x|}$ is not differentiable at $x = 0$. Even we cannot redefine it anyway we like, because they are blowing up. They become ∞ on one side and $-\infty$ on the other side. So, we say that the derivative of $f(x)$ at $x = 0$ does not exist.