Basic Calculus - 1

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Lecture 15 - Part 2

Rules of Differentiation - Part 2

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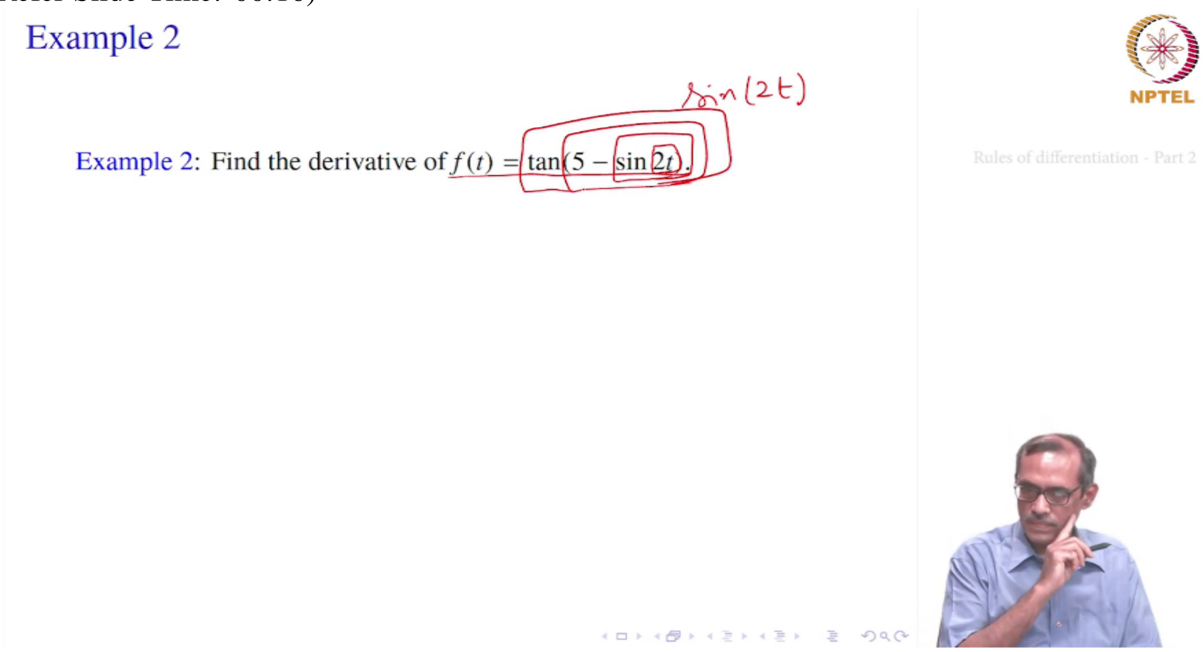

Let us apply this. We want to find the derivative of this function $f(t)$. Instead of x it is written as t. Here, $f(t) = \tan(5 - \sin 2t)$. See that $\sin 2t$ means really $\sin(2t)$; we write it as an abbreviation: $\sin 2t$. Here, there is a series of compositions. First, t to $2t$; next $2t$ to $\sin 2t$; next $5 - \sin 2t$; next tan of that. There is a series of compositions here. We have to proceed slowly.

So, let us say $s = 2t$ and $x(s) = 5 - \sin s$. We are directly taking $5 - \sin s$ instead of another composition. Let $f(x) = \tan x$; this is your x now. So, $f(x) = \tan x$. We have three compositions. Our interest is in finding what happens to df/dt. So, $df/dt = (df/dx)(dx/dt)$ as f is a function of x and x is a function of t. But $dx/dt = (dx/ds)(ds/dt)$ as x is a function of s and s is a function of t. That is how the chain rule is applied in succession. That is, $df/dt = (df/dx)(dx/ds)(ds/dt)$. Now, $df/dx = \sec^2 x$ as $f(x) = \tan x$; $dx/ds = -\cos s$ as 5 gives 0 minus from $\sin s$, you get $\cos s$, so it is $-\cos s$. And then $ds/dt = 2$; it is the derivative of $2t$ giving 2. So, you simply get $-2\cos s \sec^2 x$. Then, you replace to get in terms of t. That is, $-2\cos t$, s is $2t$ and in $\sec^2 x$, x is $5 - \sin 2t$. So, we write $5 - \sin 2t$. The answer is $-2\cos 2t \sec^2(5 - \sin 2t)$. That is how, we will be using the chain rule.

Now there is a shortcut. We do not have to always write $s = 2t$, x equal to this and f equal to $\tan x$. We proceed directly this way:

$$\frac{d\tan(5 - sin2t)}{dt} = \frac{d\tan(5 - \sin 2t)}{d(5 - \sin 2t)} \frac{d(5 - \sin 2t)}{d(2t)} \frac{d(2t)}{dt}.$$

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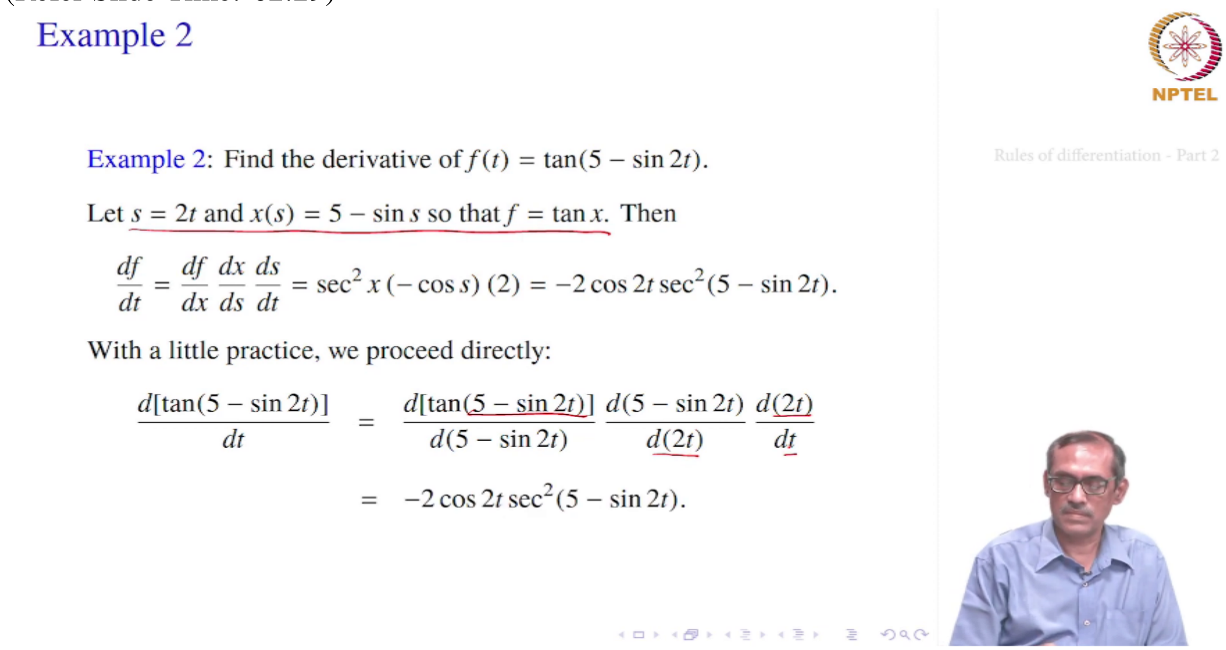

We can write it directly and then proceed immediately to get the answer. This is how we will be doing in a shorter way.

Let us take another example. We are required to find the derivative of f, where $f(x)$ is given as $(5x^3 - x^4)^7$. Of course, if you expand it by binomial theorem, you can do it, but there is no need, the chain rule will come of help.

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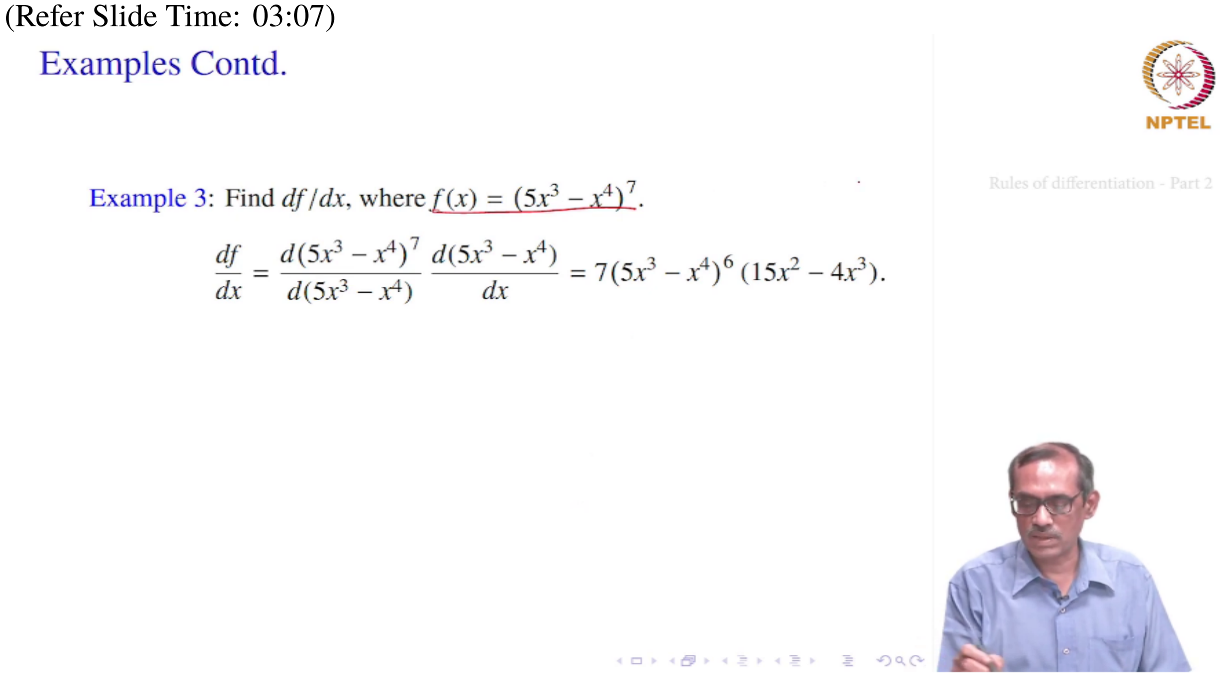

We use the chain rule directly. Now, df/dx has f as $(5x^3 - x^4)^7 5$. Here, we think of $5x^3 - x^4$ as one variable. So, it is d of this variable to the power 7, by d of that, and then multiplied with d of $5x^3 - x^4$ by dx. The last one can be differentiated because we know differentiation of polynomials. The first one gives dy^7/dy, where y is $5x^3 - x^4$; it is $7y^6$, that is, $7(5x^3 - x^4)^6$. The next one gives

$5 \times 3x^2 - 4x^3$. That is what it is. So, it is pretty straightforward.

We will take one more example. We are required to find the derivative of the function $f(x)$, which is given implicitly by an equation. What is the equation? It is $x^2 + \sin(xy) + y^2 = 0$. We want to find dy/dx, which is $f'(x)$. We differentiate the whole thing implicitly. Of course, using chain rule whenever required because y is there.

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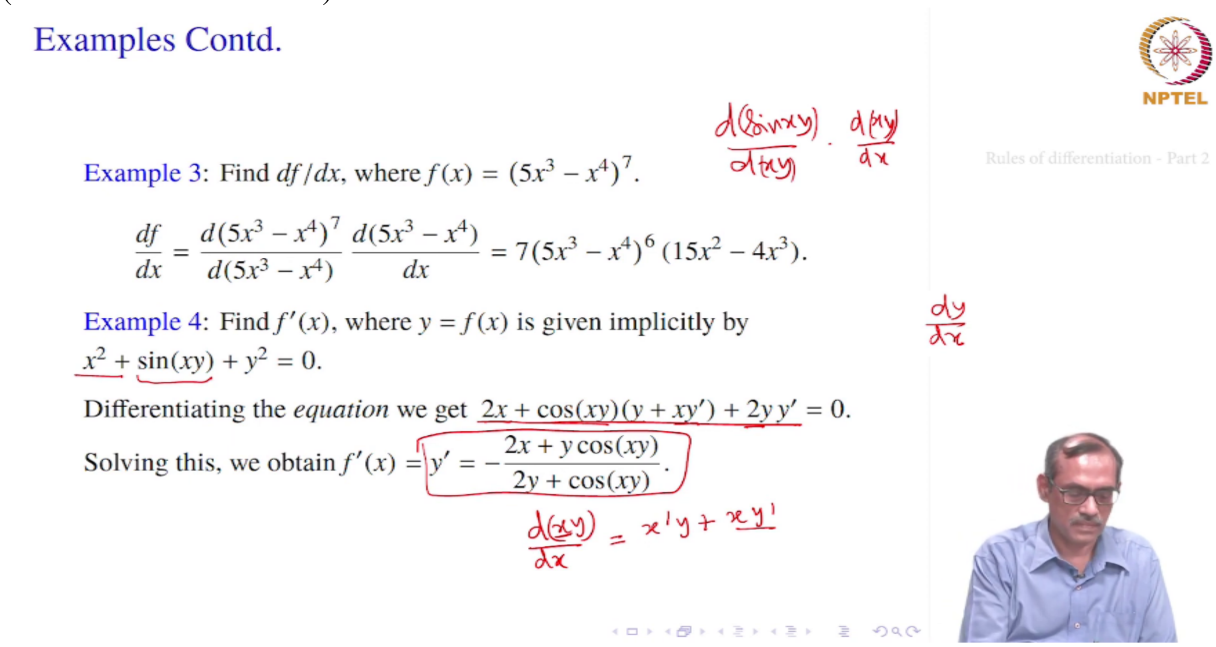

When we differentiate the equation, x^2 gives $2x$, $\sin(xy)$ gives $d\sin(xy)/d(xy) \times d(xy)/dx$. Of course, you have to put a bracket to disambiguate. That gives us $\cos(xy) \times d(xy)/dx$. Now, what is $d(xy)/dx$? Let us find out. It is the the multiplication formula: $d(fg)/dx = f'g + fg'$. Here, f' is x', then y plus x times y'. As $x' = 1$, we get $d(xy)/dx = y + xy'$. That is how you get for the first one, for the differentiation of $\sin(xy)$.

Next, we have y^2. So, dy^2/dx will use the chain rule again. It is dy^2/dx is dy^2/dy into dy/dx, which is equal to $2yy'$. That is how we get the equation. We thus arrive at the equation

$$2x + \cos(xy)(y + xy') + 2yy' = 0.$$

This is okay. This equation involves the derivatives. We solve it for y'. So, we keep y' on one side and all the other expressions on the other side. We get y' is equal to this expression: $-[2x + y\cos(xy)]/[2y + \cos(xy)]$. That is how we can differentiate implicitly.

Let us take another example. We are required to find the tangent and the normal to the curve $y = f(x)$. It is a function; its graph will be the curve. We will write that as the curve $y = f(x)$. It is given implicitly by this equation. We are interested in the tangent and the normal at $x = 2$.

Suppose that it is differentiable; and that the derivative at 2 exists, so that the slope of the tangent can be found out, which is the derivative evaluated at $x = 2$. Once you know the slope of the tangent, you can find the slope of the normal. It is just the perpendicular direction. Then we would find the equation of the tangent and normal using our formula: $(y - y_1) = m(x - x_1)$ if it passes through the point (x_1, y_1). So, you have to find out $y(2)$ also. What is that point when $x = 2$? What is $y(2)$? We have to compute all these things, and then come back to find the tangent and the normal. That is our plan.

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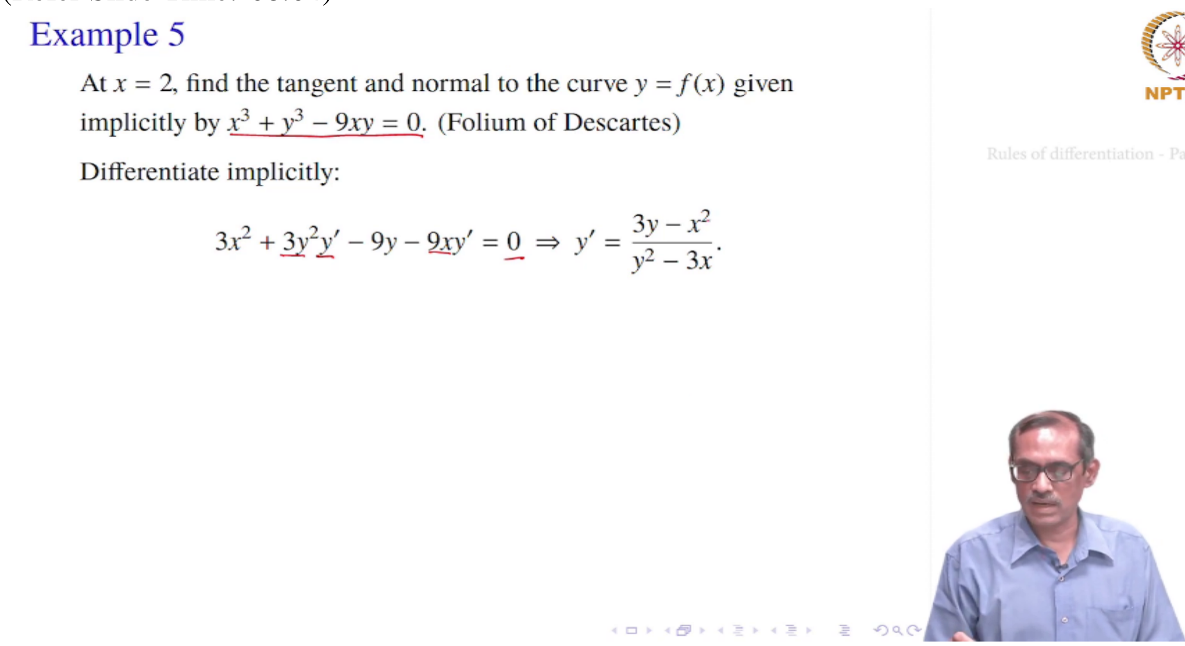

First, we differentiate. Our equation is $x^3 + y^3 - 9xy = 0$. We differentiate the left side and the right side. The right side gives you 0; the left side gives: from x^3, we get $3x^2$; for y^3, we have dy^3/dy into dy/dx, it is $3y^2y'$; then we go to the other one; from $-9xy$, the -9 comes out, it is derivative of xy, which we know to be $y + xy'$, using multiplication formula. So, the left side is $3x^2 + 3y^2y' - 9y - 9xy'$ and the right side is 0.

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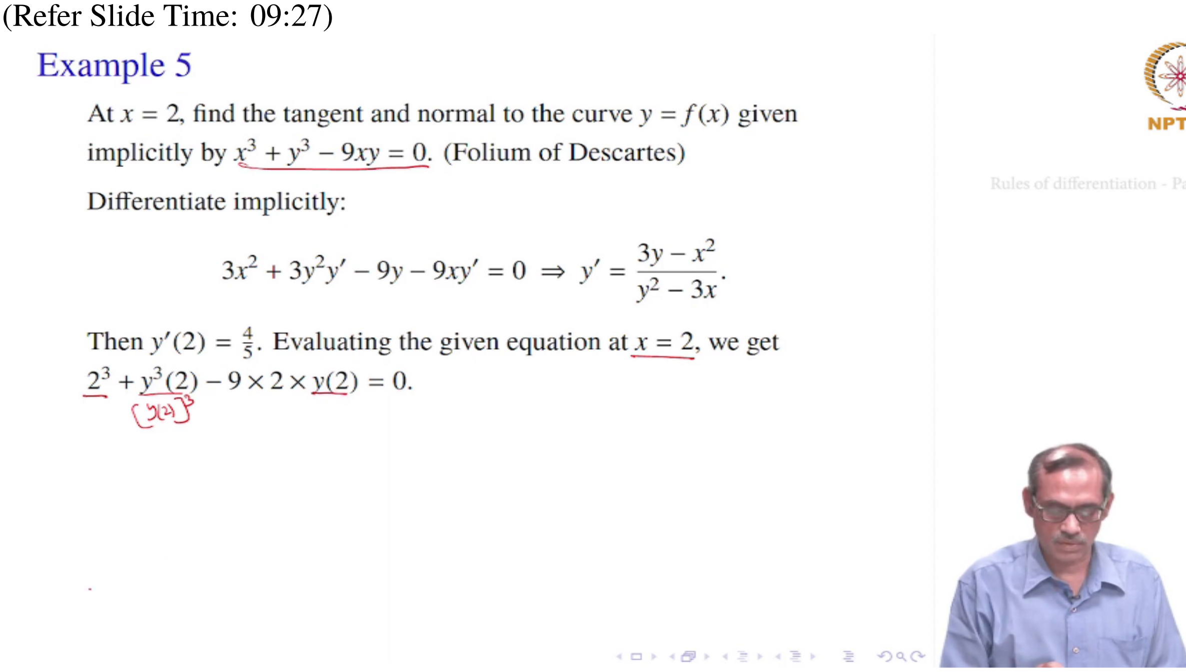

We solve it keeping y' on one side and taking all other factors to the other side and then divide by the coefficient of y'. After simplification, we get $y' = [3y - x^2]/[y^2 - 3x]$. That is our y'. You need $y'(2)$ and also $y(2)$, which can be obtained from the equation itself.

Since y' is this, we would like to find out what is its value at 2. You need first to find the value of y at 2. Once you replace $x = 2$ here in the equation $x^3 + y^3 - 9xy = 0$, we would get $2^3 + [y(2)]^3 - 9 \times 2 \times y(2) = 0$. This gives us, if you simplify this, that $y(2) = 4$.

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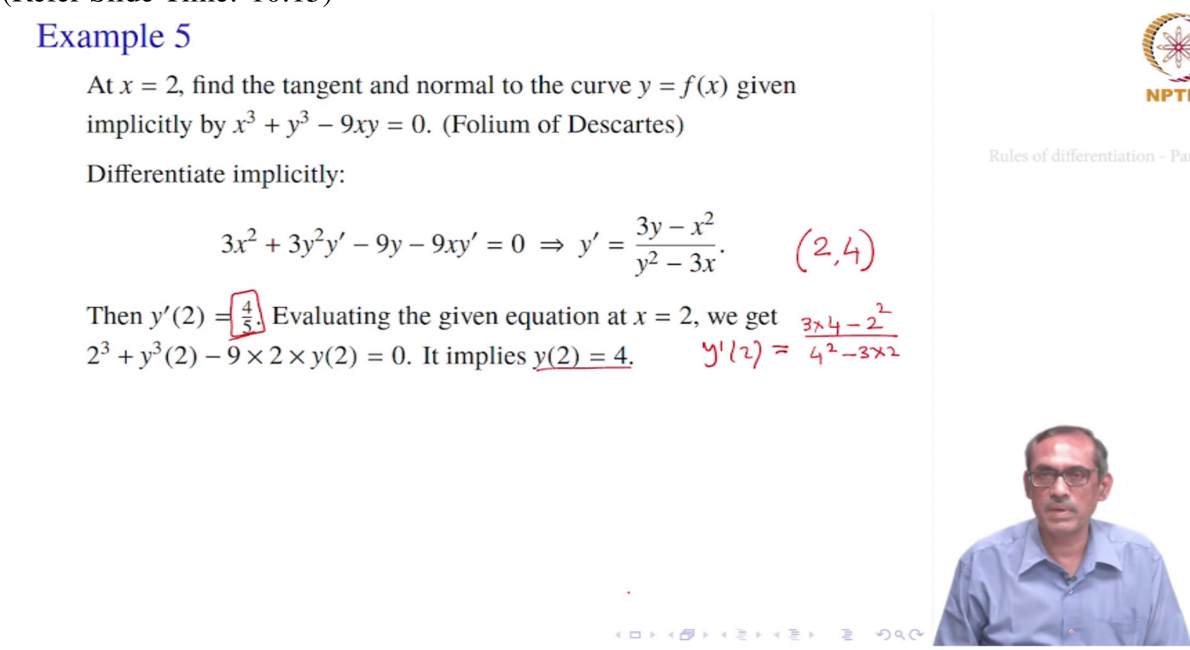

That means we have the point in the plane as $(2, 4)$. We are interested in finding the tangent and the normal at this point. We know that $y(2) = 4$. We use this here to compute $y'(2)$. So, $y'(2) = [3y(2) - 2^2]/[[y(2)]^2 - 3 \times 2]$. This gives you $y'(2) = 4/5$. Now, we have got $y(2)$ and also $y'(2)$.

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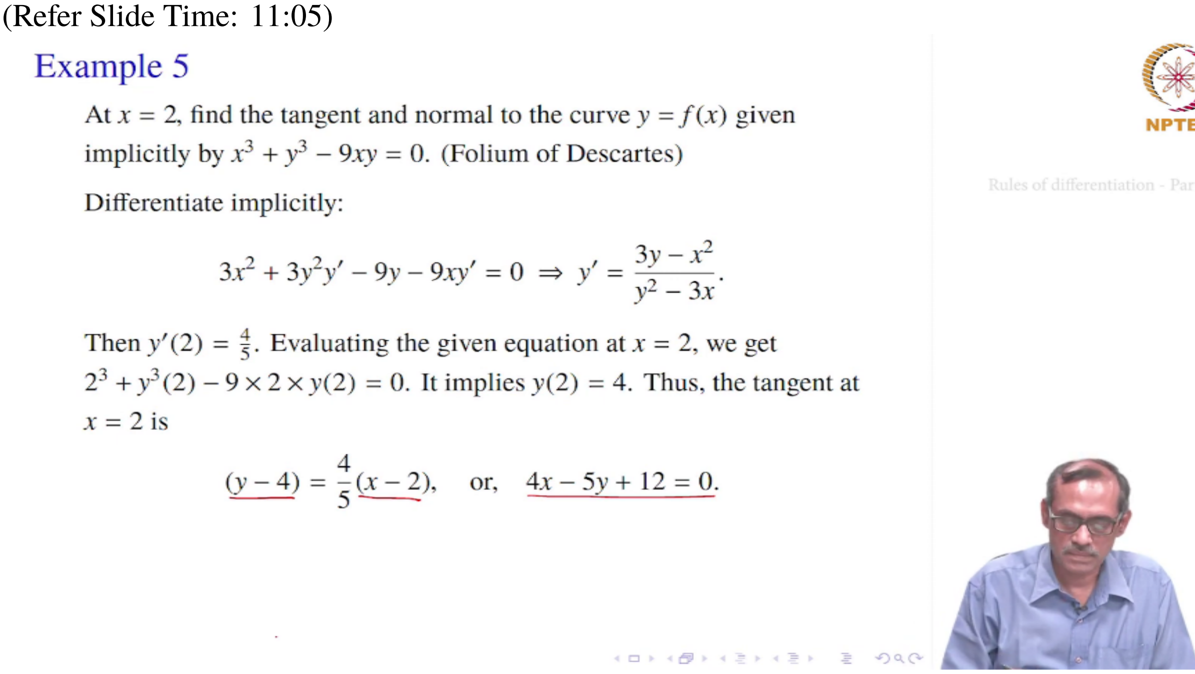

Then, at $x = 2$, the equation of the tangent will be $y - y_1$ equal to the slope, which is $y'(2) = 4/5$ times $x - x_1$. Simplifying we would get the equation of the tangent as $4x - 5y + 12 = 0$.
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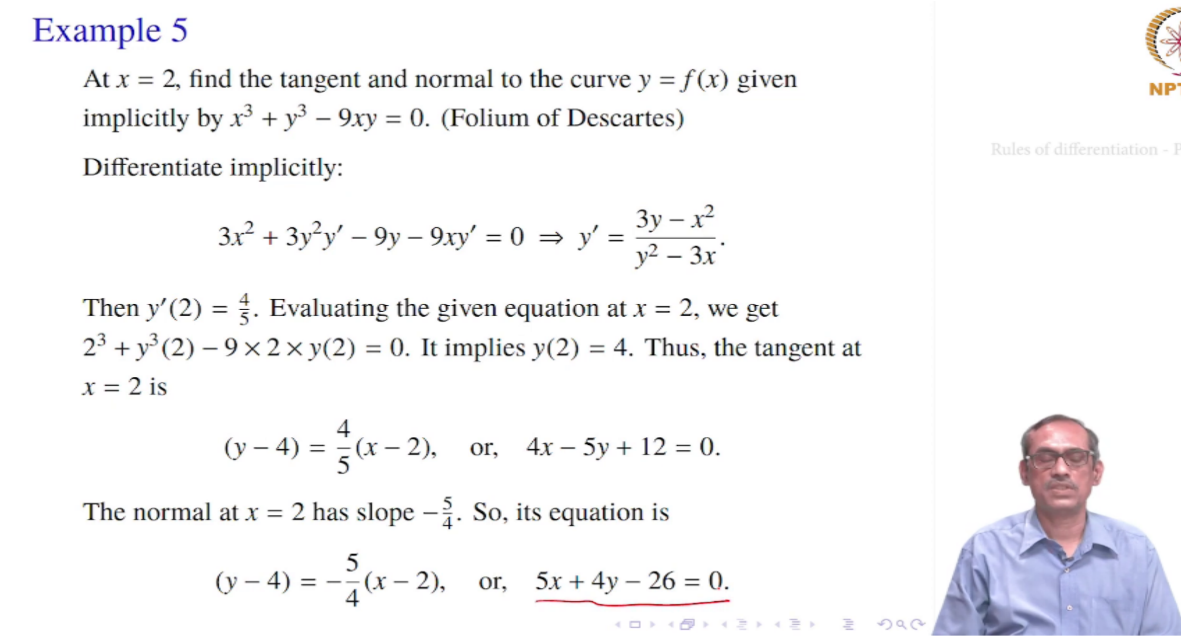

For the equation of the normal, we need the slope. If two straight lines are perpendicular their slopes when multiplied give us -1. So, if the slope of the tangent at that point $(2, 4)$ is $4/5$, then the slope of the normal will be $-5/4$. That is what it says, that the slope of the normal at $x = 2$ is $-5/4$. We apply the same formula for getting the normal. It will be $y - 4 = (-5/4)(x - 2)$. That gives $5x + 4y - 26 = 0$. So, it is an application of the idea of the derivative to compute x' and y' for curves given implicitly.
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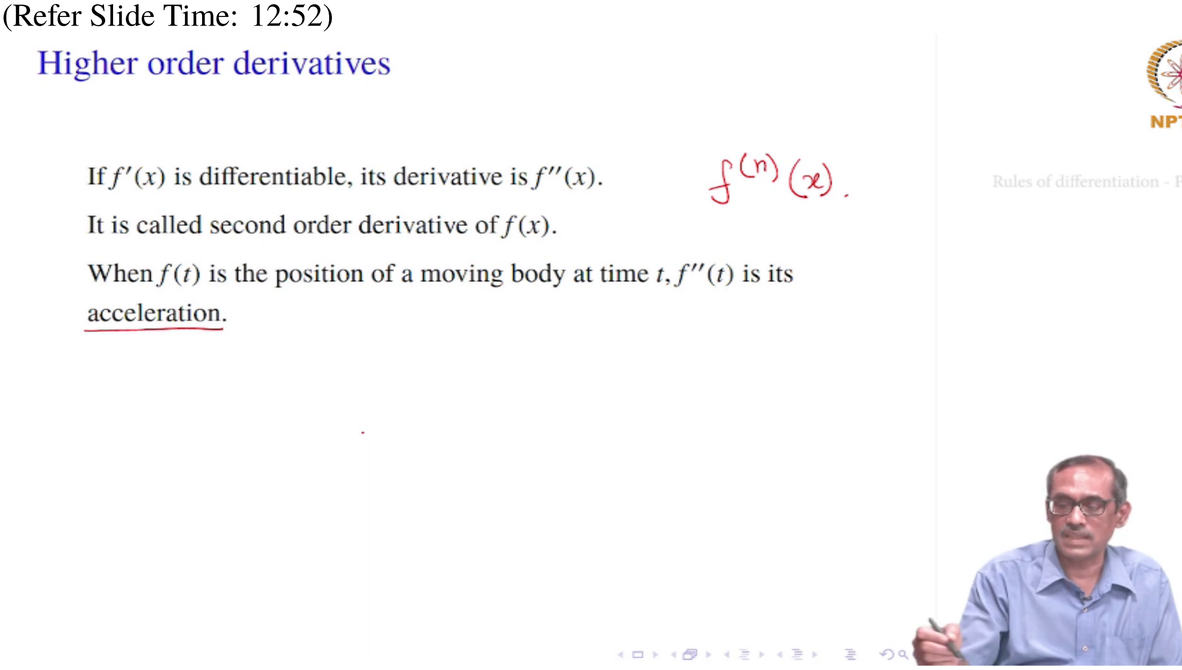

See all the while we have taken only one derivative such as $f'(x)$. You can really take more derivatives. If $f'(x)$ exists at every point of the domain of f, then it is a function also defined on the same domain. And if that function is again differentiable, then its derivative we would write as $f''(x)$. We can compute this from the derivative. This is called the second derivative or second order derivative of $f(x)$.

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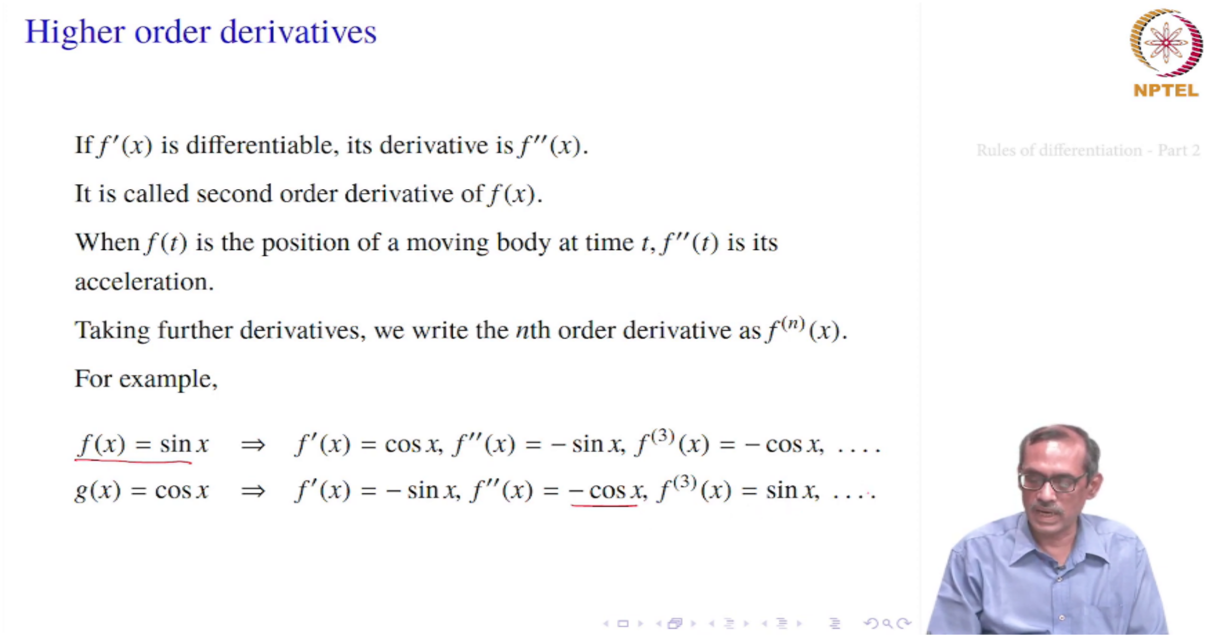

Similarly, we have third order derivatives, fourth order derivatives and so on. In general, the nth order derivative is written as $f^{(n)}(x)$. What does the second derivative mean physically? Suppose, t is time and $f(t)$ is the displacement of a body. So, $f'(t)$ gives the speed at a point and $f''(t)$ will be its differentiation; it is really acceleration. Acceleration takes care of direction also, but in the absence of any such, we are thinking f this as acceleration. It is really the absolute value of that, it is the length of that vector without direction. In this sense, $f''(t)$ can be thought of as acceleration.

If we take further derivative, we may go to $f^{(n)}(x)$. As an example, consider $f(x) = \sin x$. We would get $f'(x) = \cos x$. That is also differentiable. We take $f''(x)$; it is $-\sin x$. That is also differentiable. And its derivative is the third order derivative $f'''(x) = f^{(3)}(x) = -\cos x$. Again, that is also differentiable; and it gives the fourth order derivative $f^{(4)}(x) = \sin x$. After that the pattern repeats; we get $\sin x$ from where we have started. So, after every four derivatives, the pattern will repeat.

Similarly, for $\cos x$ we would get $f;(x) = -\sin x$, $f''(x) = -\cos x$, and $f^{(3)}(x) = \sin x$. After this we get $f^{(4)}(x) = \cos x$ again, and the pattern would repeat thereafter.

We will take another example to see how it proceeds. Suppose we are given the function $y = f(x)$, but parametrically, that is, $x = x(t)$ is a function of t, and $y = y(t)$ is also a function of t. So, $x(t)$ is given as $t - t^2$ and $y(t)$ is given as $t - t^3$. We are interested in getting the second derivative of y with respect to x. This is another way of writing the second derivative; either you write $f''(x)$ or you can write $d^2 f/dx^2$. Similarly, the nth derivative will be written as $d^n f/dx^n$. Do not confuse this with x to the power n; it is just a notation.

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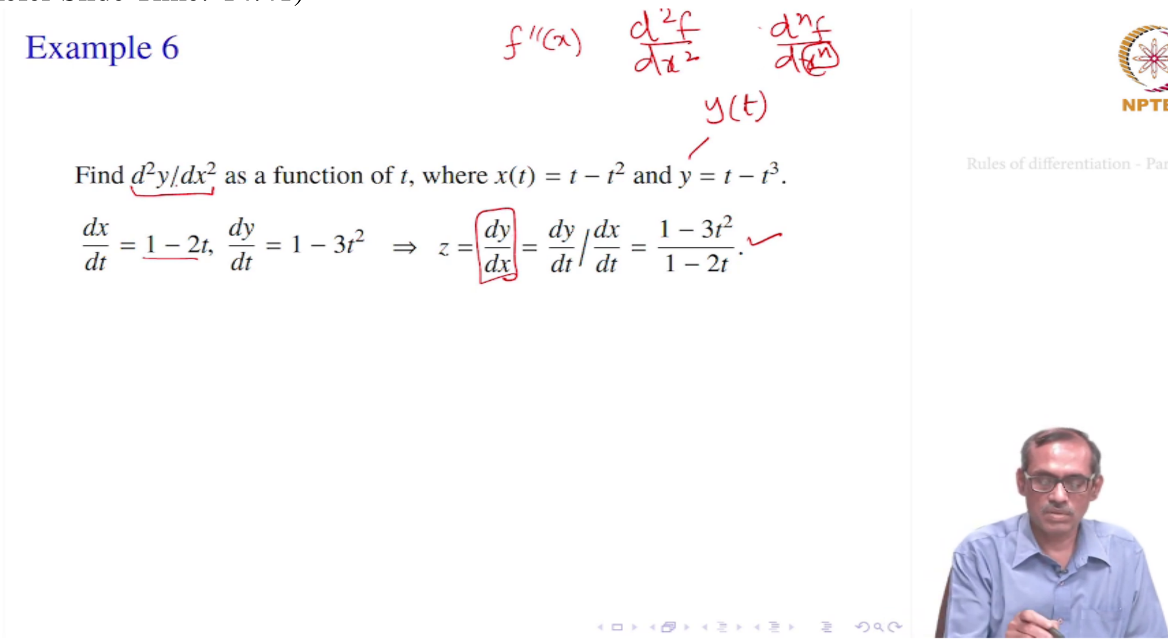

Now, coming to second derivative of this, what do we do? We compute the first derivative first. It is a different function $x(t)$ is given as $t - t^2$. Its derivative is $1 - 2t$. That is, $dx/dt = 1 - 2t$ and $dy/dt = 1 - 3t^2$. That gives $dy/dx =$. How? It is dy/dt multiplied by $(dx/dt)^{-1}$. After simplification, that gives $dy/dx = (1 - 3t^2)/(1 - 2t)$. There is no more simplification.

So, dy/dx is now given in terms of t; it is $(1 - 3t^2)/(1 - 2t)$. Sometimes it becomes possible to simply and express this as a function of x. Then, directly you can take d^2y/dx^2. But here it is not. So, you have to go for the chain rule and find the derivatives again for t. That is why we write $z = dy/dx$; this notation will be helpful.

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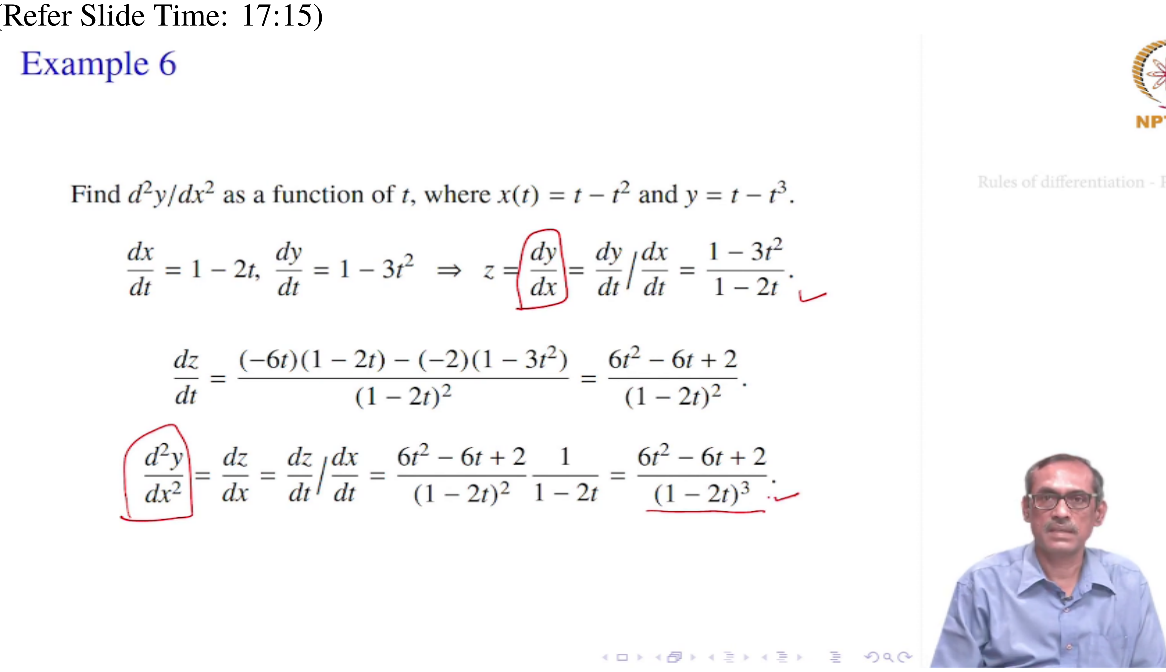

Next, you find dz/dt, that is the derivative of dy/dx with respect to t. We apply $f(t)/g(t)$-

formula for the derivative. That is, the derivative of $f(t)/g(t)$ is $f'(t)g(t) - g'(t)f(t)$ divided by $[g(t)]^2$. We have here $f(t) = 1 - 3t^2$ and $g(t) = 1 - 2t$. So, dz/dt is equal to $(-6t)(1 - 2t) - (-2)(1 - 3t^2)$ divided by $(1 - 2t)^2$. If you simplify, that gives $(6t^2 - 6t + 2)/(1 - 2t)^2$.

Now, this is dz/dt. But we want the second derivative of y with respect to x, which is d^2y/dx^2. Since dy/dx is z, we need $dz/dx = (dz/dt)(dx/dt)^{-1}$. That is dz/dt divided by dx/dt. We know $dz/dt = (6t^2 - 6t + 2)/(1 - 2t)^2$ and $dx/dt = 1 - 2t$. So, we can get dz/dt in terms of t now. That gives $d^2y/dx^2 = dz/dt = (6t^2 - 6t + 2)/(1 - 2t)^3$.

In fact, we can leave it there because that is what we want to find in terms of t. Sometimes it is possible to express it in terms of x also, but not always. We should be satisfied with that once it is expressed in terms of t. So, we have z, which is dy/dx in terms of t and also its second derivative d^2y/dx^2 in terms of t again. That is how we will be proceeding with functions given parametrically. So, we stop here.