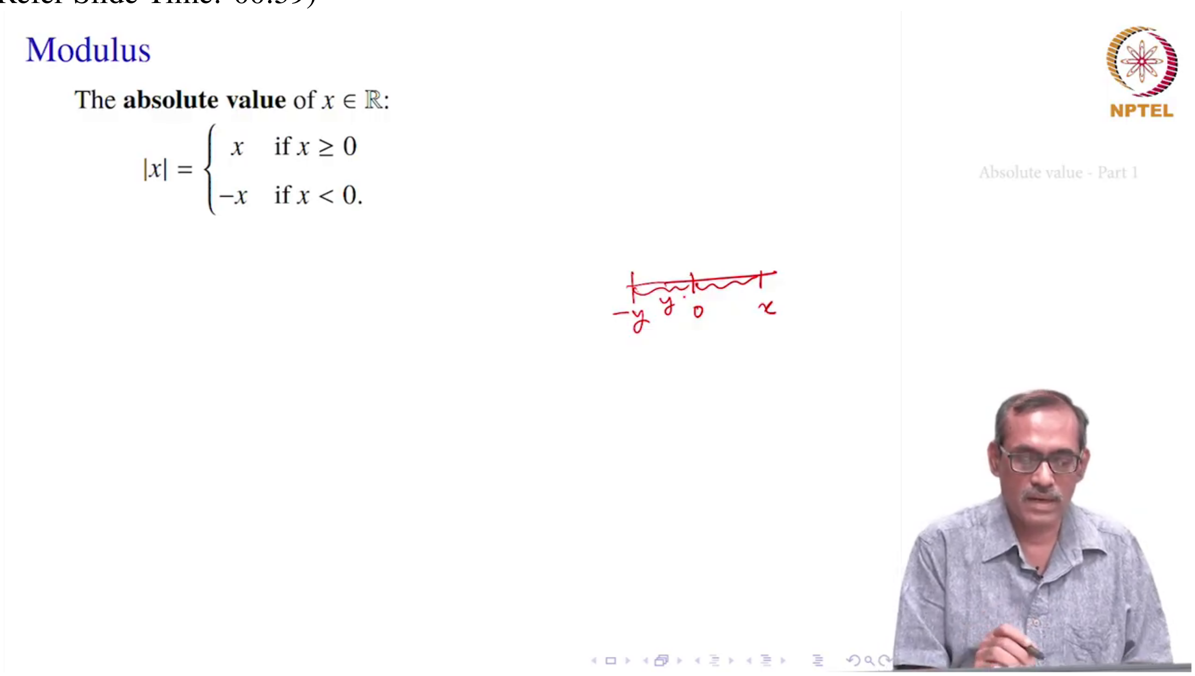
Basic Calculus - 1
Professor. Arindama Singh
Department of Mathematics
Indian Institute of Technology Madras
Lecture 3
Absolute Value - Part 1

So, this is lecture 2 of Basic Calculus 1. In the last lecture, we had discussed about the real line, the intervals and neighborhoods, then interior points and the left and right endpoints of intervals. In this lecture we will be discussing about the absolute value. Every real number has an absolute value. That is what we will be discussing today. We will also discuss its use in finding out how the neighborhoods look like in terms of this absolute value, and so on.

(Refer Slide Time: 00:59)

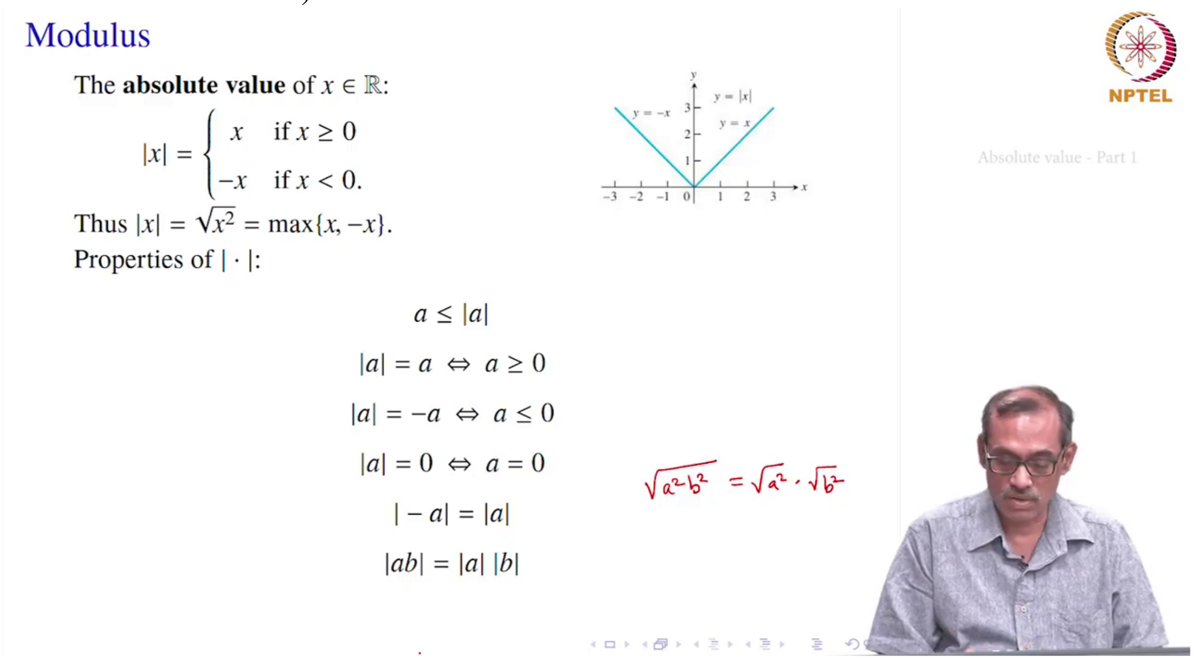


Let us start with the definition of absolute value which is also called the modulus. Suppose x is a real number. Then its absolute value is defined in two cases. If the real number $x \geq 0$, we will tell that the absolute value of x is equal to x, or $|x| = x$, and if x is negative, it is less than 0, then we will say that the absolute value of x is equal to $-x$.

What does it really say? Suppose $x = 1$, then its absolute value will be equal to 1 and if $x = -1$, then its absolute value is equal to $-(-1) = 1$ again. So, this absolute value really finds out the distance of the real number x from 0 on the real line. Suppose you take the real line. You have 0 here. You take x here which is positive. Then the absolute value of x will be equal to x, which is equal to this distance itself. And if x is negative, say here, it is $-y$, then its absolute value will be equal to y. Here, if we say $-y = -3$, then $y = 3$ is posiytive. And then the avsolute value of $-y$ is the distance from this point to 0, which is again equal to y.

We can, of course write in different ways. For example, we can write $|x| = \sqrt{x^2}$. Whenever we have this symbol square root, it is always non-negative square root of that. For example the square root of 9 can be 3 or -3. But this square root symbol says you have to take the non-negative value. If it is 0, you have to take 0 anyway. If it is positive then the result should be positive.

(Refer Slide Time: 03:25)

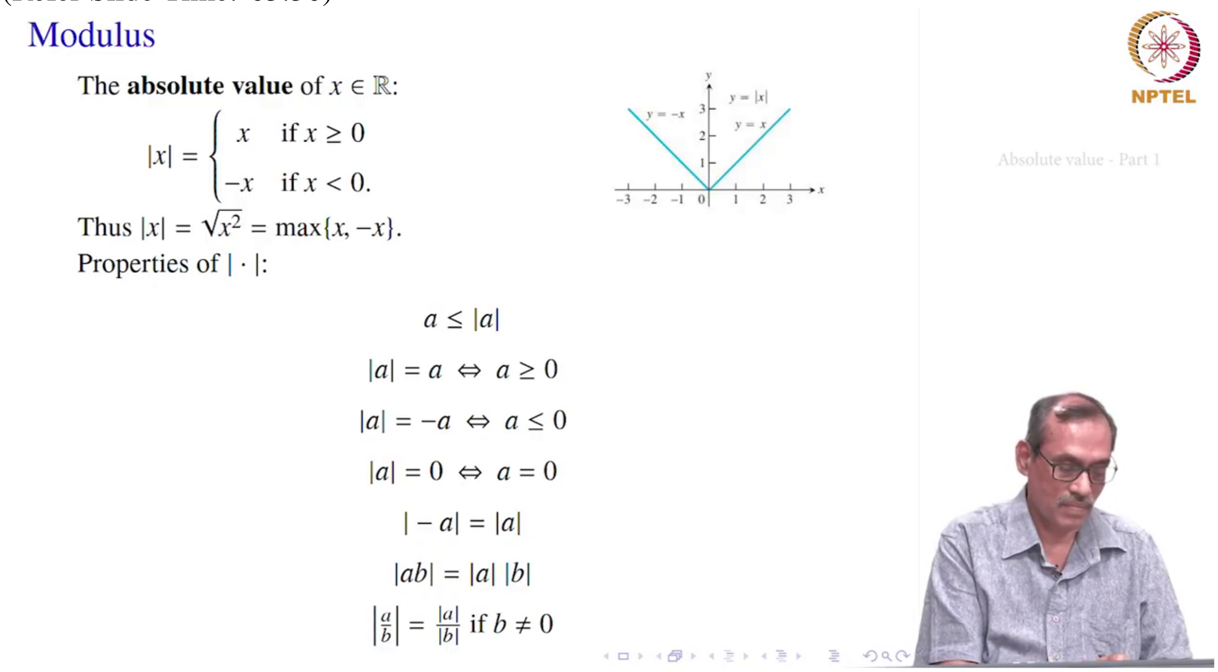


So, we can write mod x equal to the non-negative square root of x^2, which is also equal to maximum of both the values of x and $-x$. For instance, suppose $x = 1$. Then you get the maximum of 1 and minus 1 as 1; and if $x = -1$, then you get maximum of minus 1 and 1, which is 1. That is what the modulus means. These are two different ways of writing mod x.

We can really graph it. Suppose we take x to be positive. Then $|x|$ will be positive. Say, we write $|x|$ on the y-axis and on the x-axis, write x. When x is positive, $y = x$. That is why you get the straight line on the right side. In the first quadrant, you get the straight line $y = x$. On the other side $y = -x$. Since x is negative here, $|x| = -x$. So, it is the line $y = -x$. This is how you will be plotting the value of $|x|$ with respect to each x.

We will see some properties of this absolute value or mod. The first property is $a \leq |a|$. That is very obvious because $|a|$ is equal to the maximum of a and $-a$. So whatever a is, it is the maximum; thus, a must be less than or equal to that.

(Refer Slide Time: 05:50)



Next, we have $|a| = a$ if and only if $a \geq 0$. How does it come through? See, $|a$ can either be equal to a or equal to $-a$. It is a when $a \geq 0$, and it is $-a$ when $a < 0$. That is what it says. So, we would say that mod a is equal to a if and only if a is greater than equal to 0. Similarly if it is less than 0 you get mod a equal to $-a$. But another point included here is $a = 0$. At 0, let us see the modulus from the definition; $|0|$ is equal to maximum of 0 and -0. Anyway it is 0. So we can include equality symbol here. That is, $|a| = -a$ if and only if $a \leq 0$.

Next We will see some properties of this absolute value or mod. The first property is $a \leq |a|$. That is very obvious because $|a|$ is equal to the maximum of a and $-a$. So whatever a is, it is the maximum; thus, a must be less than or equal to that.

By using these two, you see that whenever $a = 0$, its mod is equal to 0 from the definition. On the other side, suppose $|a| = 0$. Then, we have to show that $a = 0$, this a cannot be anything else. Let us see. If $a| = 0$, then this $|a|$ can be either a or $-a$; in either case, it is 0. That means $= 0$ or -0. In any case a must be equal to 0.

This is a very nice property. Because, sometimes we want to show that $a = 0$ but we are not able to show it directly. Then, we can do that for $|a|$, and show that $|a| = 0$. This will prove that $a = 0$. We will see such cases later.

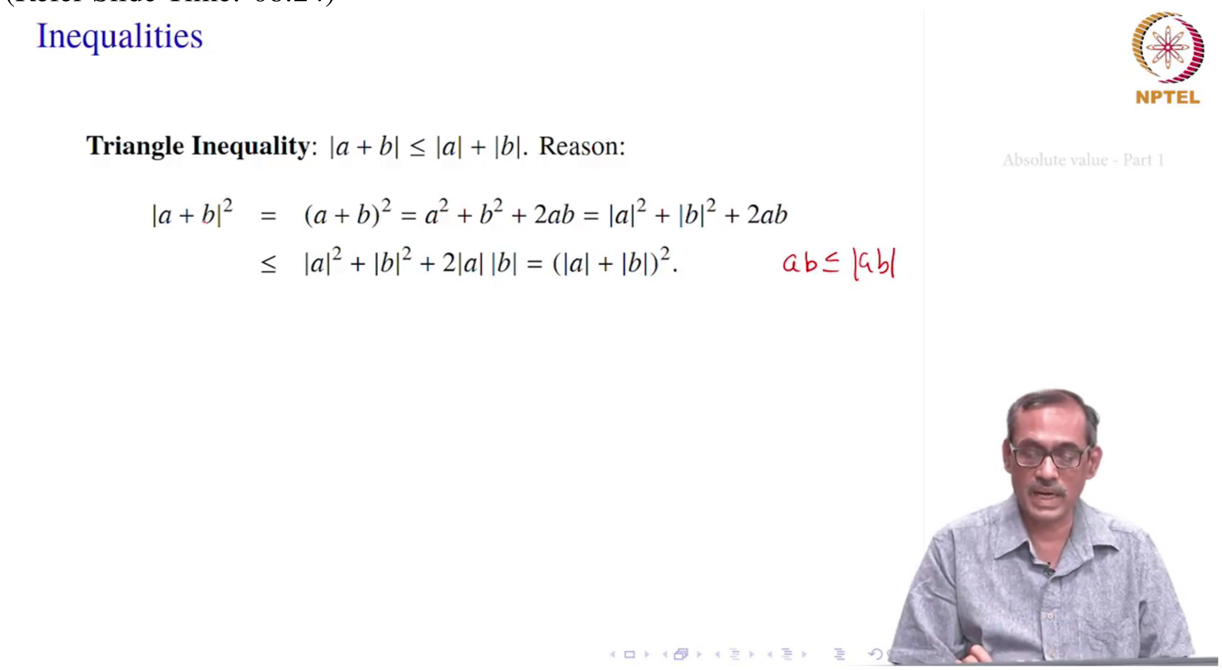
Now, the next property is very obvious: $|-a|$ is the distance of $-a$ from the origin. The distance of a from origin is equal to the distance of $-a$ from the origin. So, $|-a| = |a|$. Also, you can get it directly from the definition.

Well, how does it fare with the multiplication? If we take ab and take its absolute value, then it will be equal to $|a|$ into $|b|$. This is not very quick from the definition. But if you take the square root it will be easier to see. Because $|ab| = \sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a^2}\sqrt{b^2} = |a|\,|b|$.

And, instead of b, had I taken $1/b$, where $b \neq 0$, then I would reach at this: $|a/b| = |a|/|b|$ if $b \neq 0$.

These are very easy properties. But they are important and they will be used most often without telling that these properties hold. They will be used implicitly. That means we should be getting very familiar with them.

(Refer Slide Time: 08:24)



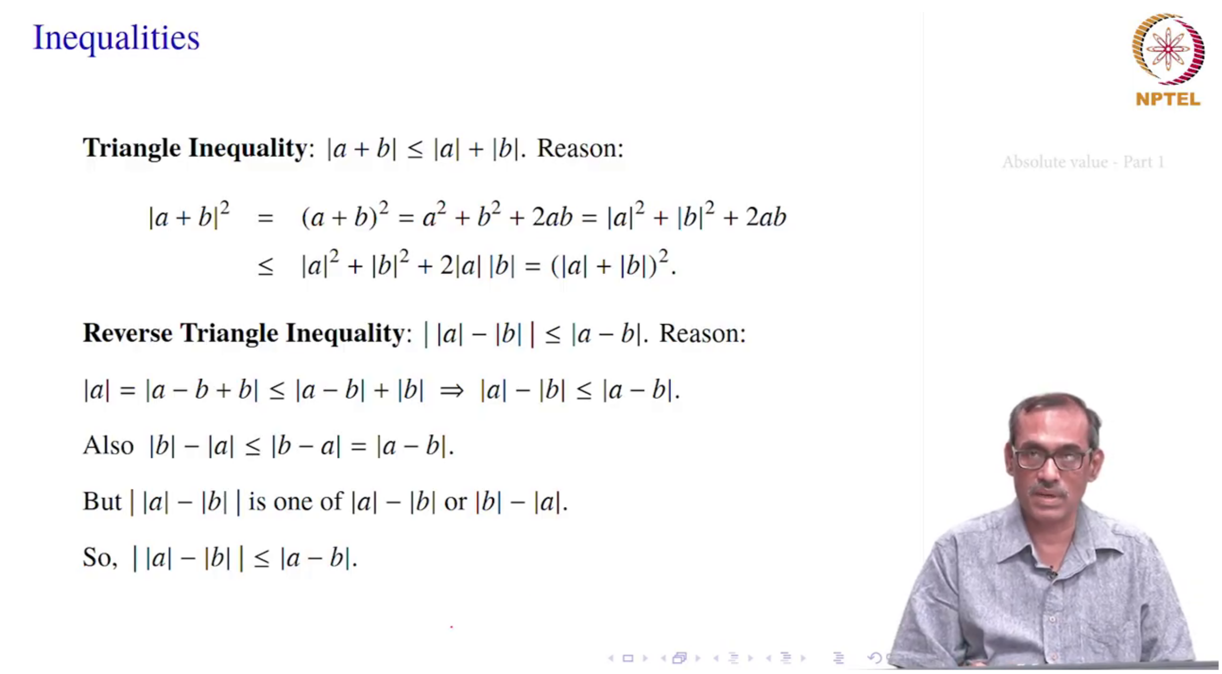
There is another. We have done for the multiplication. How about addition? If you take mod of $a + b$, then it is less than or equal to $|a| + |b|$. It may not be equal always; sometimes it is equal but sometimes it is not. For example, if you take one of them to be positive, one of them negative, say, $a = 2$ and $b = -1$. On the left side you get $|2 - 1| = 1$, but on right side you get $|2| + |1| = 3$. So, it can be less also, specifically when they have opposite signs. What happens if they have the same sign, that is, either both of them are to the right of 0 or both of them are to the left of 0? You can take, say, $|2 + 3| = 5 = 2 + 3$. Similarly, equality holds when both are negative.

But how do you show that for all real numbers a and b, $|a + b| \leq |a| + |b|$? Well, let us use that 'square' definition. Take the square of the left side. We will see how to manage the square of the right side. So, $|a + b|^2 = (a + b)^2 = a^2 + b^2 + 2ab$. And we know that $a^2 = |a^2| = |a|^2$; similarly, $b^2 = |b|^2$. We keep $2ab$ as it is and use another property. This ab is less than or equal to $|a|\,|b|$. It can be equal, but can it be less? Yes, because we are not taking absolute value of this. We are using here the propert that $x \leq |x|$. So, $ab \leq |a|\,|b|$. So,

$$a^2 + b^2 + 2ab \leq |a|^2 + |b|^2 + 2|a|\,|b| = (|a| + |b|)^2.$$

Once $x^2 \leq y^2$ and both x and y are non-negative, we must have $x \leq y$. And that gives us the triangle inequality: $|a + b| \leq |a| + |b|$.

(Refer Slide Time: 11:09)



We have something called reverse triangle inequality. It is not exactly the triangle inequality but it looks like that. What do we do here? Instead of plus, we are taking minus. But the inequality is also reversed here. You see, absolute value of a minus absolute value of b and then their absolute value is less than or equal to absolute value of $a - b$. Of course we can use the triangle inequality for proving it.

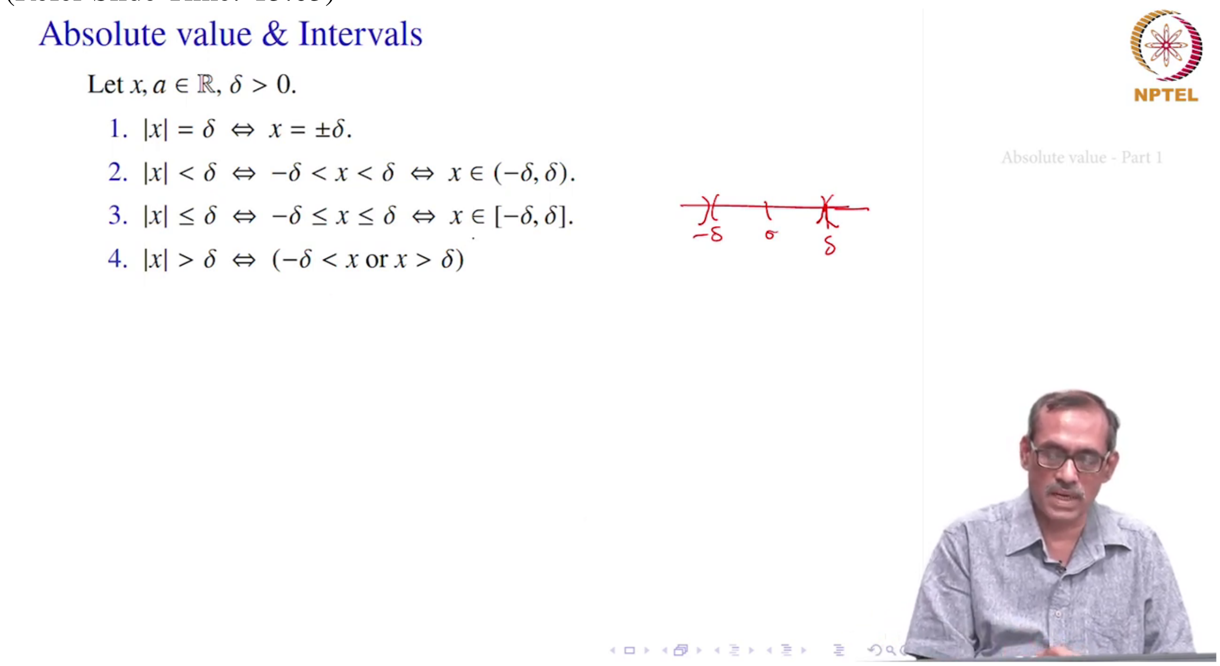
So let us start. We start with a, rather, $|a$. Inside that modulus sign we subtract one b and add one b; they are equal. Then we use the triangle inequality with $a - b$ as one factor and b as another. So, it is less than or equal to $|a - b|$ plus $|b$. That gives mod of $a - b$ is less than or equal to $|b| - |a|$.

Now if you reverse the roles of a and b, then you get the other inequality that $|b| - |a|$ is less than or equal to $|b - a|$, which is equal to $|a - b|$.

We get two inequalities now: $|a| - |b| \le |a - b|$ and $|b| - |a| \le |a - b|$. But what about the mod of $|a| - |b|$? It is either $|a| - |b|$ or $|b| - |a|$. So, in any case, the absolute value must be less than or equal to $|a - b|$.

That is how we have shown reverse triangle inequality. You can also square them both the sides and follow as we have done in triangle inequality. That will be an alternative proof.

(Refer Slide Time: 13:03)



Let us see how this absolute value plays with the intervals. In fact, they would give rise to the neighborhoods and let us see how. Suppose we fix two real numbers, x, a and then take another positive number $\delta > 0$. Now, $|x| = \delta$ if and only if $x = \delta$ or $x = -\delta$. That is obvious from the geometry. If you think of the real line and you take $\delta > 0$, then δ is here. If x is equal to δ or $-\delta$, then $|x|$ is $\delta = \delta$. So, one side is clear. What about the other side? If $|x| = \delta$, then look at $|x|$. It is either x or $-x$. If it is x, $x = \delta$. If it is $-x$, then $x = -\delta$. So, these are the two possibilities. Nothing else is possible.

In fact, if you take any other x, say, $x < 0$ and $x < -\delta$, then $|x| = x < \delta$. Similarly on the other side, if you take any x between $-\delta$ to 0, then $|x| < \delta$. If you take anything bigger than that or the left of $-\delta$, then its mod will be again bigger than δ. So, that is how this goes through. That is, $|x| = \delta$ means we have only two possibilities: either $x = \delta$ or $x = -\delta$. Now, that goes for the equality.

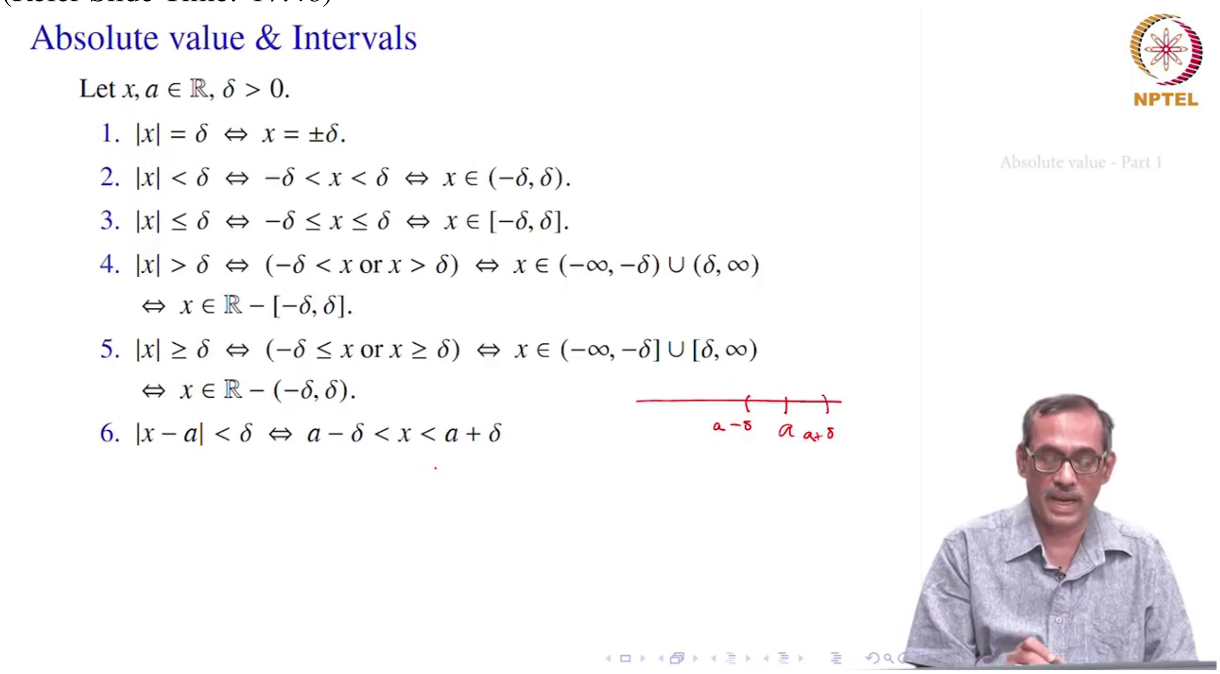
If it is less than, then what will happen? Again you go back to your drawing. I have 0 here, I have δ. Now x is a point such that $|x| < \delta$. That means the distance of x from 0 is less than δ. So, x can fall here or x can fall here, where this is $-\delta$. That means x can be anything between $-\delta$ to δ.

That is what the other side is telling: $|x| < \delta$ if and only if $-\delta < x < \delta$. That means x belongs to the δ-neighborhood of 0. This is the δ-neighborhood of 0. It says that $|x| < \delta$ is equivalent to telling in another way that x is an element of the δ-neighborhood of 0. That is what we express by telling x belongs to $(-\delta, \delta)$.

Well, if instead of less than, it is less than or equal to, then what happens? We are now combining both 1 and 2. Of course result will be combined again. It can be $(-\delta, \delta)$ or it can be equal to δ or equal to $-\delta$. So, all these things are possible. It gives the closed interval. That is x belongs to the closed interval $[-\delta, \delta]$.

And now, suppose it is bigger than delta. Then? That means we have to look at its complement. That is, x should not belong to $(-\delta, \delta)$. In other words, if $|x| > \delta$, then it is ot possible for x to be eqaual to δ, nor to $-\delta$, nor even in the interval $(-\delta, \delta)$. So, either $-\delta < x$ or $x > \delta$. That is what we express here; it is just the complement of the closed interval. We can write it as: $x \in (-\infty, -\delta) \cup (\delta, \infty)$. This is the complement of closed interval $-\delta, \delta]$.

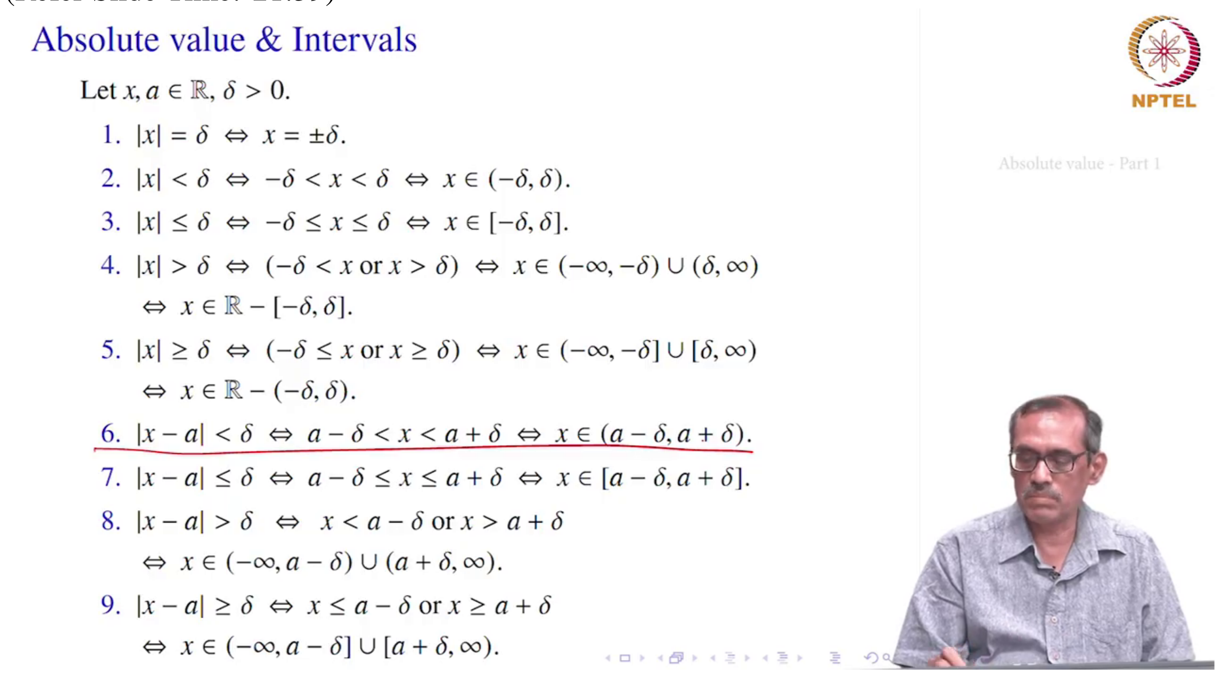
(Refer Slide Time: 17:46)



What happens when mod $|x| \ge \delta$? We have to include the delta now. That means instead of taking the complement of the closed interval $[-\delta, \delta]$, we should take the complement of the open interval $(-\delta, \delta)$. So, that is how we will get; it will be a semi-open interval. It is $\mathbb{R}$ minus the δ-neighborhood of 0; and x can be anywhere in that set. That is what we mean by $|x| \ge \delta$.

Now let us slightly shift our origin. Instead of considering from 0, let us consider from the point a which is also real number. That means we have a and we are now thinking of the number $x - a$. Now, $|x - a| < \delta$. So what does that mean? Instead of $|x| < \delta$ as in 2, we are thinking of $|x - a| < \delta$. As earlier, we will be translating, the origin will be shifted to a now. We can think of $-\delta$ to be here, δ to be here; so this point is really $a + \delta$, and this point is really $a - \delta$. As earlier, it is within that δ-neighborhood. This is the δ-neighborhood of a. We would get $|x - a| < \delta$ if and only if $a - \delta < x < a + \delta$. So, it is just another way of telling that x is an element of the δ-neighborhood of a. That is what it means.

(Refer Slide Time: 21:39)



Suppose it is less than or equal to delta. Then we will introduc equality to both the sides. Instead of the open interval it will be a closed interval. So, $x \in [a - \delta, a + \delta]$.

And, if $|x - a| > \delta$, then it will be the complement of the seventh one, that it cannot belong to the closed interval $[a - \delta, a + \delta]$. So, x is either to the left of $a - \delta$ or to the right of $a + \delta$. That is, $x < -\delta$ or it is possible that $x > a + \delta$. That is how it will look. We may say: it is $(-\infty, -\delta) \cup (\delta, \infty)$. So, it is really $a - \delta$; now x can be here or x can be here, which is $a + \delta$.

That is how it comes to: the inequality $|x - a| < \delta$ gives rise to the open interval $(a - \delta, a + \delta)$, if it is $\geq$ then the point δ will be included and we would get the semi-open intervals. That is, x must be in $(-\infty, a - \delta]$ where $a - \delta$ is included. And on the other side, $a + \delta$ can be included. So, these are various ways of interpreting the absolute value with inequalities.

Finally, one thing is enough to remember. It will give rise to other things by inclusion of a point or its exclusion. The earlier cases can be explained through that by taking specifically $a = 0$. It says that $|x - a| < \delta$ is same thing as telling x belongs to the δ-neighborhood of a.