Basic Calculus - 1

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Lecture 14 - Part 2

Derivative and Tangent - Part 2

Let us take some problems. Suppose I consider the function $y = (2+x)^{-1}$. You want to find whether this has a tangent at the point $x = 2$, and if so, what is the slope of this tangent?

As we see, if the function is differentiable at this point $x = 2$, then that derivative will become the slope of the tangent. If it is not differentiable, then we have to see what is the case. Out of those four cases, which case is it? Does it have a tangent or not? It may have a tangent, but it will be a vertical tangent, or it may not be differentiable because it does not have a tangent also. But here the things might be nice; it is $1/(2+x)$.

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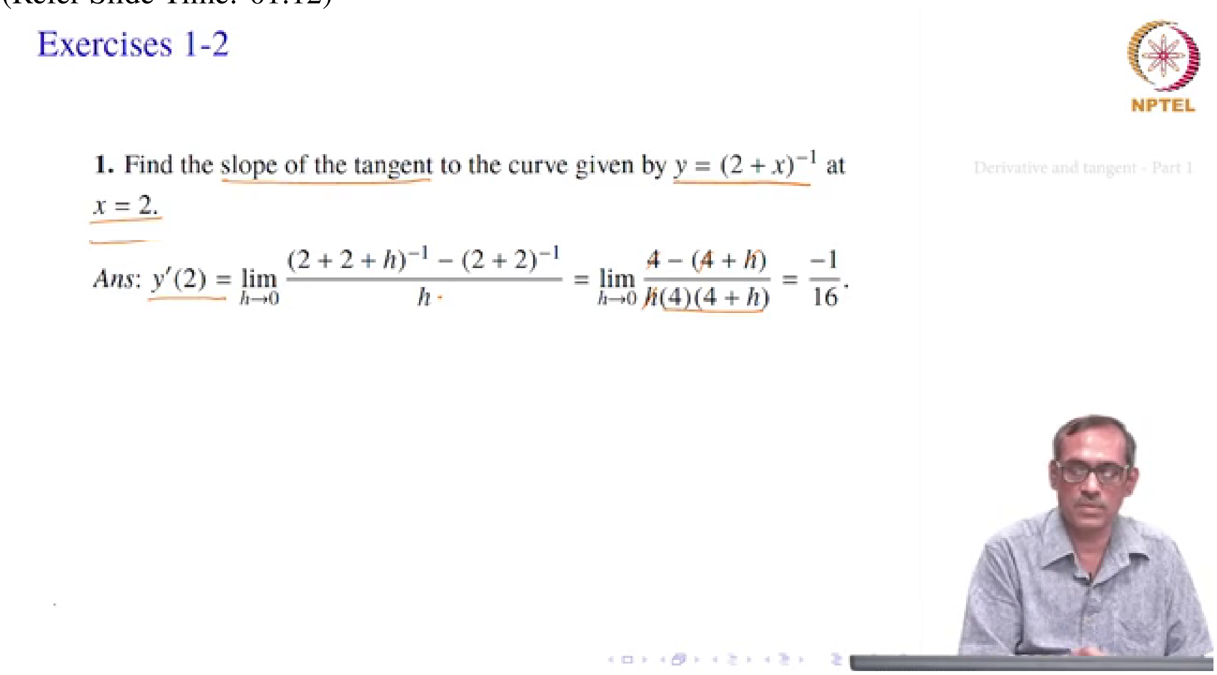

Let us see what happens. We will try to find this $f'(x)$ at $x = 2$, which we are writing as $y'(2)$. We do not know whether it exists or not. But this will be the limit. If it exists, it will be equal to that $y'(2)$. It is $[f(2+h) - f(2)]/h$, which is equal to $[1/(4+h) - 1/4]/h$. If you subtract it, it gives you on the numerator $4 - (4+h)$ and on the denominator you have $4(4+h)$. You have already one h; so that is what it is. The top one gives: 4 and -4 cancel, you get $-h$, that h cancels with this h; and now when h goes to 0, that gives you $-1/16$. So, this is the slope of the tangent to the curve $y = (2+x)^{-1}$ at $x = 2$, since this derivative exists.

We are taking another problem, where the function is given as $y = 1 + \sqrt{4-x}$. We are not asking anything directly for this function. We are telling that it gives a curve in the xy-plane. On that curve, you take the point $(3, 2)$. Is it a point on the curve? Here if I take $x = 3$, I would say

$1+\sqrt{4-3}$, which is $1+1=2$. So, this is a point on the curve. Now at this point, find the equation of the tangent.

If the problem is correct, we should get $f'(2)$, and that will be the slope of the tangent. Once we have the slope of a straight line, which is the tangent, and it passes through the point $(3,2)$, then we should be able to compute the equation of the straight line, or equation of the tangent at that point. That is what we will be doing.

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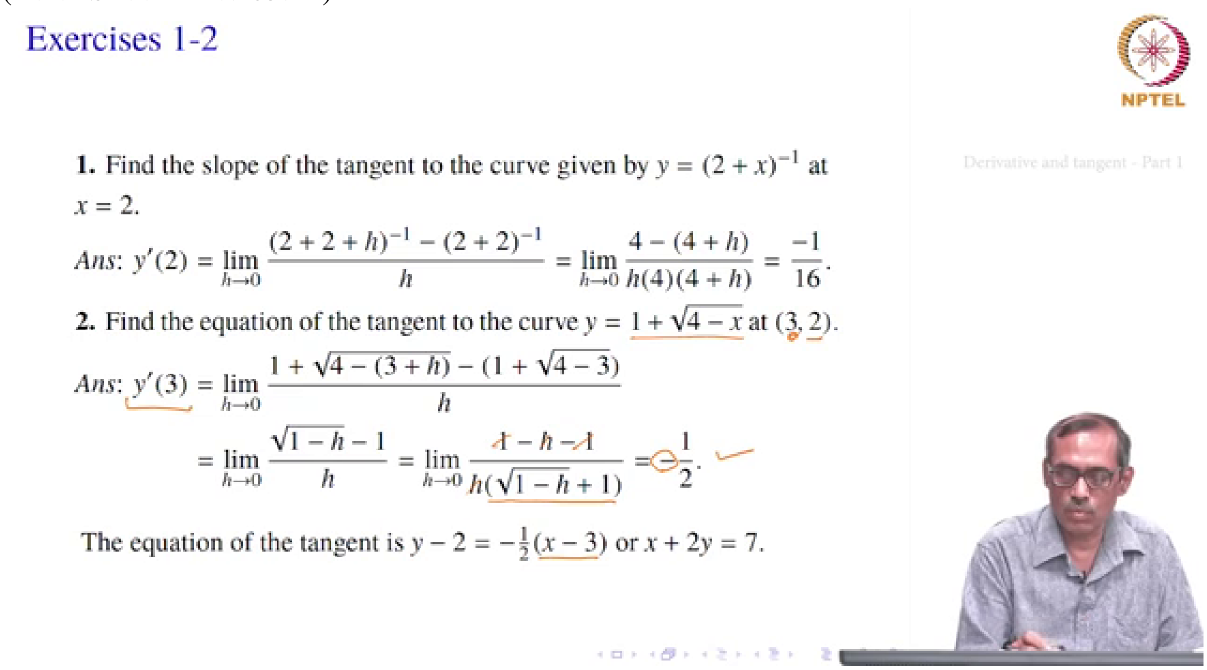

Okey. Let us consider the function $y=1+\sqrt{4-x}$ and compute its derivative at $x=3$. It is $y'(3)$, which is the limit of $[1+\sqrt{4-(3+h)}-(1+\sqrt{4-3})]/h$ as h goes to 0. This simplifies to $[\sqrt{1-h}-1]/h$.We have to take care of the square root. To cancel this h, what do we do? We multiply $\sqrt{1-h}+1$ in the numerator and in the denominator. When you multiply in the numerator, it gives $1-h-1=-h$. The denominator is $h(\sqrt{1-h}+1)$. One h cancels, and you get $-1/(\sqrt{1-h}+1)$. When h goes to 0, the denominator becomes $1+1=2$; so the limit is $-1/2$. We see that $y'(3)=-1/2$. That is, the slope of the tangent at $x=3$ is $-1/2$.

It passes through $(3,2)$. So its equation should be $y-y_1=$ the slope times $x-x_1$. That is, $y-2=(-1/2)(x-3)$. Simplify. That gives you $x+2y=7$. This 2 multiplies and 4 and 3, so, they become 7. So, $x+2y=7$. That is the equation of the tangent to the curve $y=1+\sqrt{4-x}$ at the point $(3,2)$.

Let us take another problem. Here we are asking something else. We have a function $y=2\sqrt{x}$. We look at it as a curve in the xy-plane. Does this curve have horizontal tangents?

What is the meaning of horizontal tangents? It means the slope must be 0, that is, along the x axis it makes the angle 0. As $\tan 0=0$, the slope of the tangent should be 0. And, you have to find the slopes and see.

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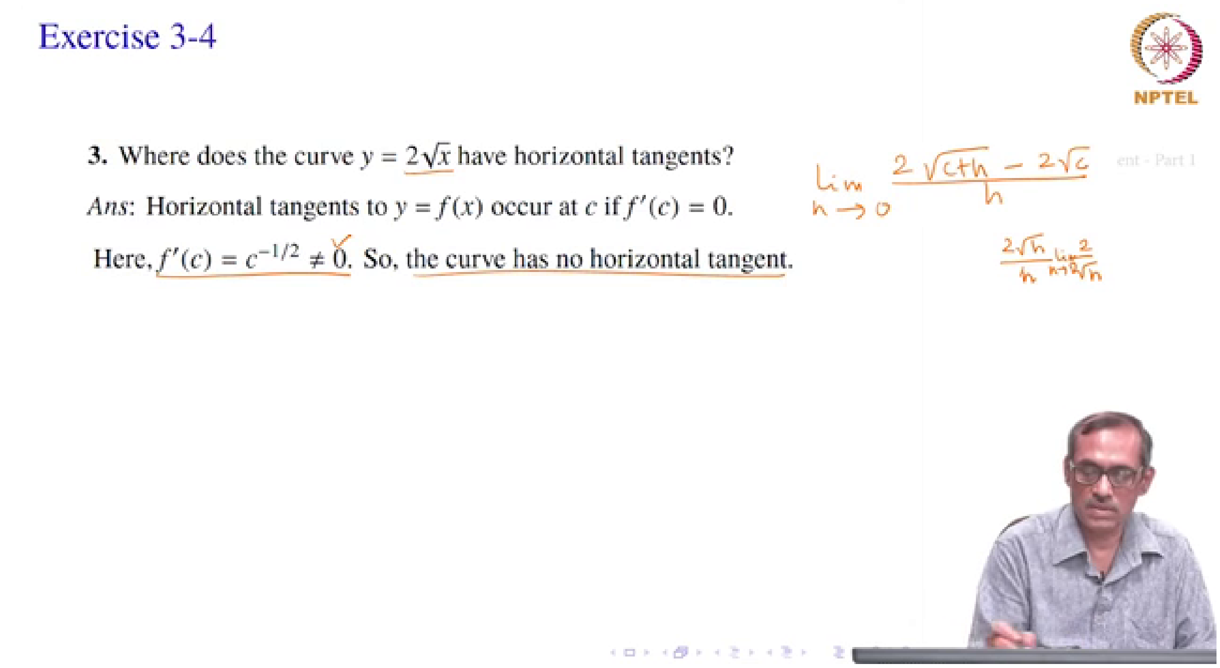

Suppose a horizontal tangent occurs at $x = c$. It is not the point really. At c means it is the point $(c, f(c))$, which is $(c, 2\sqrt{c})$; that is the point on the curve. This function $y = f(x)$ has a horizontal tangent means that it should have value 0 at that point $x = c$. So, we find $f'(x)$. As $f(x) = 2\sqrt{x}$, it is in the form $x^{m/n}$. Now, how do we differentiate $x^{m/n}$?

But here, if you do from the first principles, you can also find out. It will be $x^{-1/2}$. That will be the derivative for $x \neq 0$. You can see immediately. It is at c. So, it is the limit of $[2\sqrt{c+h} - 2\sqrt{c}]/h$ as h goes to 0. That is what you have to find. So, you multiply a similar expression with plus sign and divide. That gives you $2(c + h - c)$ on the top and $h(\sqrt{c+h} + \sqrt{c})$ on the bottom. So, h gets cancelled. As h goes to 0, the downside gives $2\sqrt{c}$. Again, 2 gets canceled and that gives you $c^{-1/2}$. That is how you would get $f'(c) = c^{-1/2}$ for $c \neq 0$. And we are not taking that at $c = 0$. Because this $c^{-1/2}$ is not be defined at $c = 0$.

So, what happens at 0? We have $y = 0$ and we would finf $f'(0)$. At 0, it will be something like $[2\sqrt{h} - 0]/h$. That gives $2/\sqrt{h}$. As h goes to 0, its limit blows up. We do not get a real number. Our requirement is 0, but it is not 0 anyway. So, at 0 we do not get a vertical tangent. It may happen at nonzero points only. And, at $c \neq 0$, $f'(c) = 2/\sqrt{c}$ is never 0. Therefore, the curve does not have any horizontal tangents. That is our conclusion.

Let us see the next one. It asks: Does there exist a function $f(x)$, which is not continuous. So, we are in search for a discontinuous function. That means on its domain, at least at one point continuity breaks down. But $|f(x)|$ will be differentiable. Had it been not differentiable you could have tried $|x|$ or something likt that, but it says $|f(x)|$ is differentiable.

We have a theorem which says that if $f(x)$ is differentiable at a point, then $f(x)$ is also continuous at that point. Here $f(x)$ is not differentiable; we want $|f(x)|$ to be differentiable; so they are different. All that we want is a function $f(x)$, whose absolute value is differentiable. So absolute value is differentiable but the function is not. It is a bit weird.

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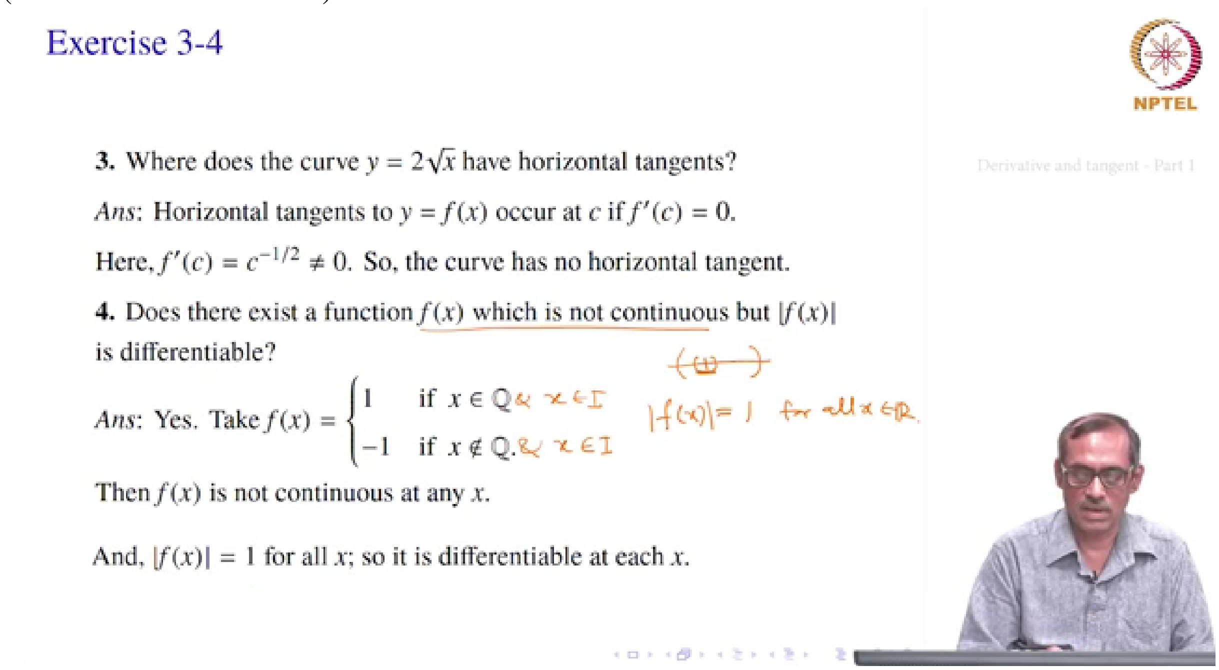

What do we do? We take something like this, where mod would make it a constant. We do like this, this is possible, we are generating an example. So, what do you say? Mod makes it a constant; let us try that.

We would define $f(x) = 1$, if x is rational and $f(x) = -1$, if x is irrational. Now, it is defined throughout the real numbers. When you take its absolute value it gives you 1. When x is rational, its absolute value is 1, and when x is irrational, its absolute value is also 1. So, $|f(x)|$ will be constant function; that is, $|f(x) = 1$ for all x.

Now, $|f(x)|$ is differentiable; it is a constant, we have seen that. And we have to see that $f(x)$ is not continuous. That is, this function is not continuous at at least one x; that should be sufficient. But here it says, even if you take the domain not to be the whole of $\mathbb{R}$, but some subset, say, the interval $[0, 1]$ or some interval, it will not be continuous there also. This really is a stronger thing, which we are providing here. Anywhere you take x, this function is not continuous at x. Why? Because, if you take any point on the real line, that may be a rational point or may be an irrational point, and if you take any neighborhood of that, any small neighborhood, the function will be having values 1 and -1. All these values are achieved everywhere because of denseness of $\mathbb{Q}$ and the denseness of irrationals.

In every neighborhood you will find the function achieves the value 1, and also achieves the value -1. If I take my ϵ to be, say, less than half or less than equal to half, then for that ϵ, I will not get any δ such that all points in $(x - \delta, x + \delta)$ the mod of the difference between that and the functional value, let us say 1 for a rational number, that will not be less than $\epsilon = 1/2$. Similarly, if you take an irrational instead of a rational here as this x, then you take the difference between $f(x)$ and -1, that is also bigger than half always. So, the function is not continuous at any x. That means, instead of defining on $\mathbb{R}$, where $f(x) = 1$ or $f(x) = -1$ with these conditions we could have defined on any interval also.

Take any interval. On that interval, define this function by "if $x \in \mathbb{Q}$ and x is in that interval, then $f(x) = 1$; and if $x \notin \mathbb{Q}$ and x is in that interval, then $f(x) = -1$. Then also this function satisfies our conditions. It is not continuous at any point in that interval, but $|f(x) = 1$ is differentiable. That is how this example has been generated. Let us stop here.