Basic Calculus - 1

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Lecture 14 - Part 1

Derivative and Tangent - Part 1

Okay, this is lecture 14 of Basic Calculus 1. Recall that we had discussed about differentiability and the derivative of a function in our last lecture. Today we will be talking something about derivatives and tangents.

Physically, you see that the derivative is the speed, the instantaneous speed. But mathematically, it has something to do with the tangents. So, first of all, what do you mean by a tangent to a curve?
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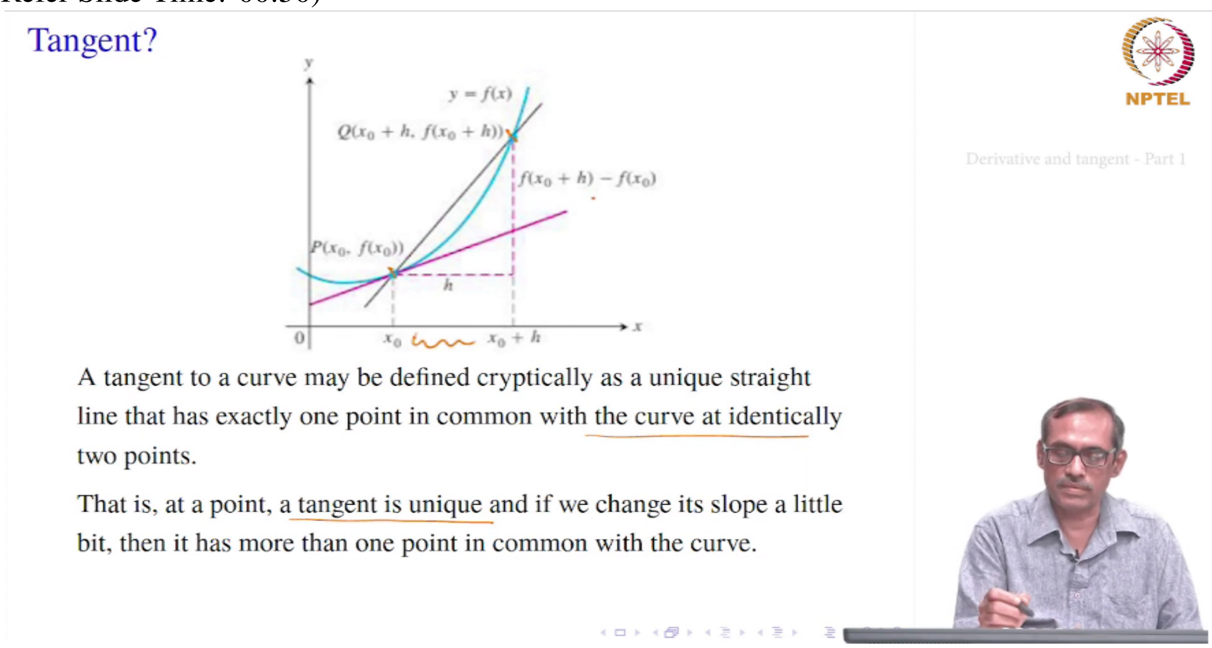

Let us take the curve, which is depicted in blue here. We consider a point, say, x_0 on the x-axis. Corresponding to this x_0, write the point on the curve as P with coordinates $(x_0, f(x_0)$. At this point P, we feel that the straight line in pink is the tangent to the curve. Intuitively, it is a straight line, a unique straight line, which touches the curve at that point. That is what we intuitively feel what a tangent should be. But there are some critical cases. Taking that into account, we may think of a tangent to a curve with a cryptic description such as "it is a unique straight line that has exactly one point in common with the curve at identically two points".

That is the cryptical matter here. What do you mean by identically two points? It would mean something like this: if you take the tangent, it is a unique straight line. And what sort of straight line is it? If you change its slope a little bit, then it has more than one point in common with the curve. That is what this means. It touches the curve at identically two points. If you change the direction of its slope a little bit, then there will be at least two points common to the curve; it will become a secant. That is what we mean by a tangent.

Of course, this is about the curves in the plane. We are only talking of curves, which are functions, specifically written as $y = f(x)$. For that, we want to see what could be the tangent at a point. As you see, the definition will be something like this: you take two points on the curve, say P and Q; let us say, the distance corresponding to that in the x-axis is h; one is at x_0 and another is at $x_0 + h$. Now we have two points P and Q; coordinates of P are $(x_0, f(x_0))$ and of Q is $(x_0 + h, f(x_0 + h))$. Here, h could be negative, so that Q could have been here on the left of P. We allow that and write Q at $x_0 + h$; so, it does not mean that h is always positive. Now we have the secant, the straight line joining these two points P and Q. Our feeling is that when h goes to 0, this secant becomes a tangent. That will be the notion of tangent for a curve.

Suppose $y = f(x)$ is the curve. We take two points P and Q; then we join these two points to get the secant. What is the slope of the secant? It is $(y_2 - y_1)/(x_2 - x_1)$, which is for Q and P, the numerator is $f(x_0 + h) - f(x_0)$ and the denominator simplifies to h. So, you get this ratio $[f(x_0 + h) - f(x_0)]/h$ of the increment in f divided by the increment in x.

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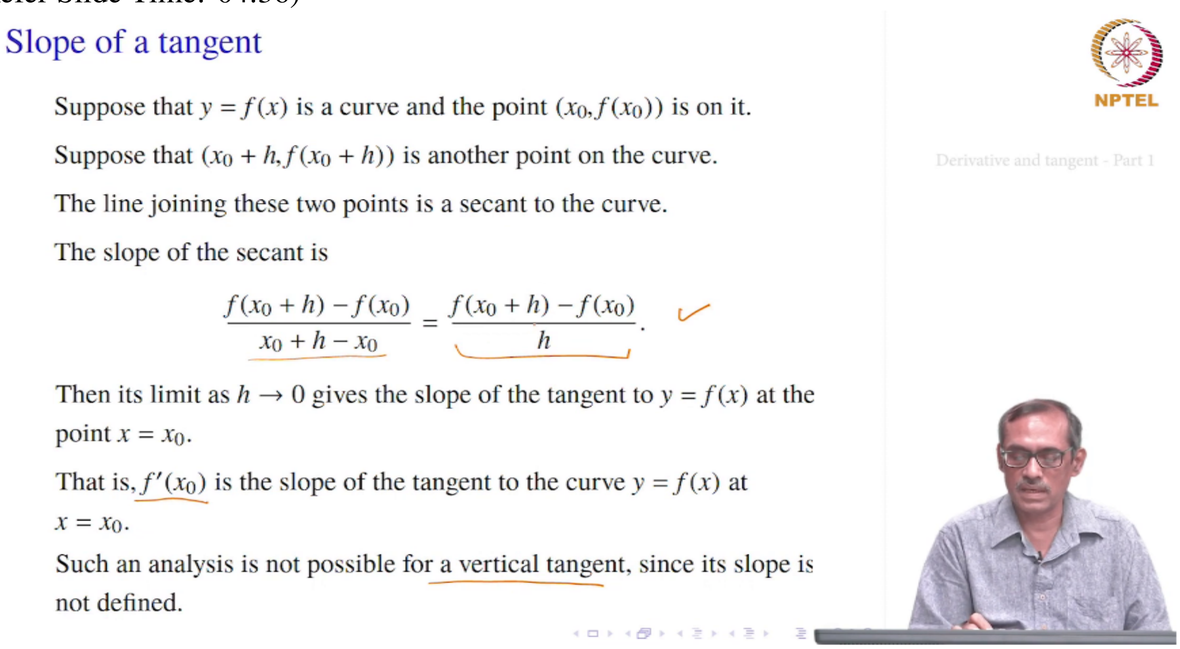

Our aim is when h goes to 0, this will be the slope of the tangent to $y = f(x)$ at the point $x = x_0$. This is exactly the definition of the derivative. In the limit as h goes to 0, the right hand side is the derivative at x_0. Therefore, we would say that $f'(x_0)$ is the slope of the tangent to the curve $y = f(x)$ at $x = x_0$.

This is so when everything goes well. There can be some cases where this fails. Like, when the limit does not exist, for example, or it blows up, becomes infinity; that is also one of the cases of nonexistence. Well, when it blows up, exactly what happens? The slope becomes infinity. That means, the straight line that we get from this slope is having an angle with the x-axis as $\pi/2$. That means a vertical tangent.

That is the case which is not here in the differentiability. For a vertical tangent this will fail and its slope is not defined. If you take the slope in this way, then it will not be defined. So, you have to take that into consideration while considering tangents. That is, differentiatiability has some more constraints than just having an existence of a tangent.

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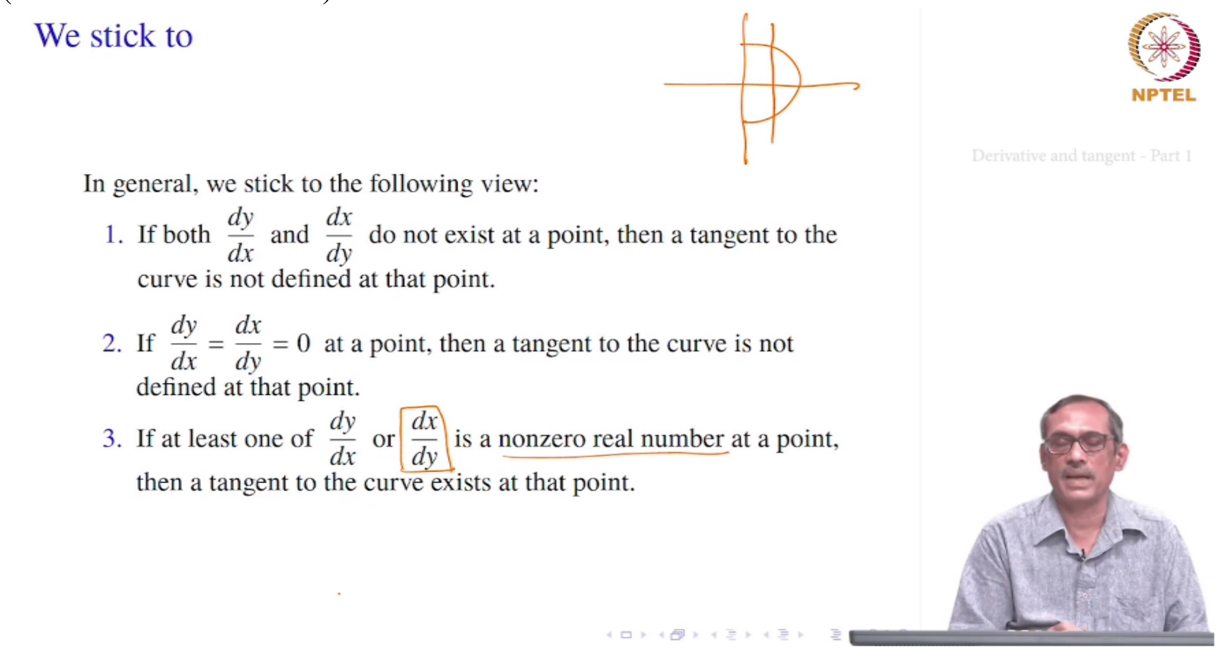

In general, we will stick to this view of a tangent, where we are taking a curve not necessarily as $y = f(x)$. It includes the cases where you might be able to express y as a function of x or x as a function of y. For example, take the vertical semicircle, say, on the right side. If you take this one, this is expressed as $x = f(y)$ not $y = f(x)$. This curve is not a function $y = f(x)$ because the vertical line can cross it in two parts. So, we are considering both dy/dx and dx/fy. If they do not exist at a point, then a tangent to the curve is not defined at that point. That is obvious from this.

If both of them are 0 at a point, then also a tangent to the curve is not defined at that point. As we have seen for the vertical tangent $y = f(x)$. If at least one of these two dx/dy or dy/dx exists and it is a nonzero real number at a point x_0, then the tangent to the curve exists at that point x_0. Since we are considering $y = f(x)$, our dx/dy goes away. We thus concentrate on this, that dy/dx is a nonzero real number. That would give rise to a tangent. Of course, it will exclude the case of a vertical tangent.

Let us see some examples to know how these critical cases are coming up and how they are covered. Suppose you take $y = |x|$ as the function. You know its graph; it look something like this. At x equal to 0, we cannot find a unique straight line which would touch the curve. Of course, one is x-axis; but we could have taken another, something like this. There are infinitely many straight lines, which we can draw to touch the curve at $x = 0$. Since uniqueness breaks down, we would say that it does not have a tangent at $x = 0$. Tangent should be a unique straight line. There are more than one straight line that touch the curve at $x = 0$. So, we say that it does not have a tangent at $x = 0$.

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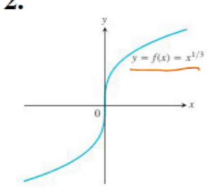
Three examples

1. The curve $y = |x|$ has no tangent at $x = 0$. There are more than one straight line that touches the curve at $x = 0$.

2.



The curve $y = x^{1/3}$ has a vertical tangent at $x = 0$, which also intersects the curve at $x = 0$.
But $y'(0)$ does not exist since

$$\lim_{h\to 0}\frac{h^{1/3}-0}{h} = \lim_{h\to 0} h^{-2/3} = \infty.$$

Let us consider the second example. Here, we have the curve as $y = f(x)$, which is $x^{1/3}$. Look at the point $x = 0$; that is the origin. This is a point on the curve. At that point, our definition would say that y-axis is a tangent to the curve. Because, if we change the slope slightly, it will intersect the curve in at least two points. And it touches (here crosses) the curve at 0, not at any other points. Though in the picture, you cannot see it because they are too close. That is how it goes. We would say that it is a vertical tangent. That is, the y-axis is a tangent to the curve. But here you see that the tangent itself intersects the curve. If you look at it from the upper side, it is touching the curve, and from downside also it is touching. If you vary the slope a little bit, then it will cross the curve in two points. So, it satisfies our notion of the tangent that it has only one point in common with the curve and it is a unique straight line where that happens. Other straight lines will not have that property; they will cross the curve at two points or even may not cross or may not have a common point with the curve at all. So, this is a tangent.

This is one type of thing that happens for a tangent. But here there is something else that also happens. If you take the derivative of this function, it does not exist at 0, why? The derivative will be the limit of $[f(0+h) - f(0)]/h$, which is the limit of $h^{1/3}/h$. That gives $h^{-2/3}$, whose limit does not exist as $h \to 0$. In fact, it is equal to infinity, it blows up, when h is 0+. And at 0−, since it is a square, it would give infinity again. Once it is infinity, the slope is infinity, it says that there is a vertical tangent. Intuitively, it says that there is a vertical tangent, because its limit is infinite. But it is not differentiable.

We will have another example. Let us take $y = x^{2/3}$. It looks like this. There is a cusp here, it comes down to this, and then goes up. It is not like the second one, not like $y = |x|$; there is a cusp type of thing. What happens here? When you take the derivative, try to compute its derivative at 0, it is $h^{2/3}$ by h. This is $[f(0+h) - f(0)]/h$, that is $h^{-1/3}$. When h is positive, it would give the limit as ∞, and the left side limit as h goes to 0 from the negative side, you would get $-\infty$.

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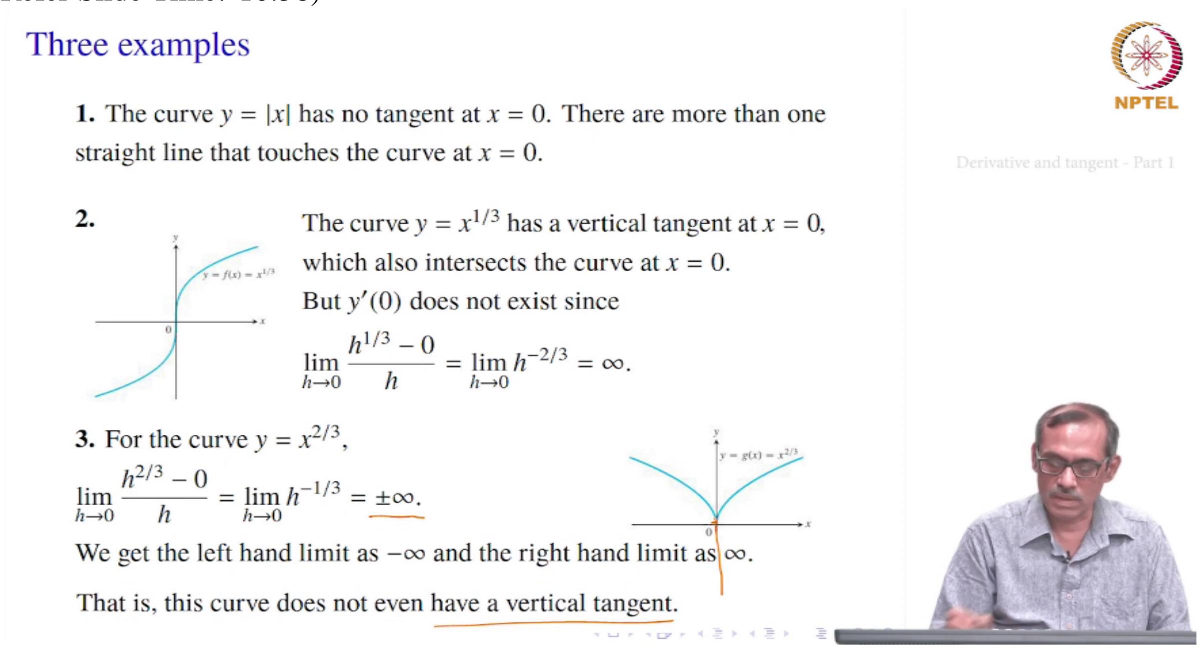

So again, the one sided limits will become $-\infty$ and ∞; so it will not be differentiable, the curve is not differentiable at $x = 0$. But then, what about the tangent? Again, you may say that a vertical tangent exists, but it is not differentiable at $x = 0$; there is a cusp at $x = 0$.

But how do you say that a vertical tangent exists or not? In order to say that a vertical tangent exists, we should have the slope as infinity, or minus infinity one of them; but it does not have a limit. So, we cannot say that this y-axis becomes a tangent to the curve. You would say that the curve does not even have a vertical tangent.

These are typical examples where there will be difference between the derivative and the slope of the tangent (which should be the derivative). So, there is a difference.

Now we will have another thing for differentiability. This we will also figure out in this discussion about tangents. What it says, if $y = f(x)$ is differentiable at $x = c$, then the same function $y = f(x)$ is continuous at that point. That is what you say this way: if a function is differentiable at a point then it is continuous. Of course, we can prove it very easily.

Let us assume that it is differentiable at a point, which is not an endpoint, but an interior point of the domain of $f(x)$. Let us take this case. In this case, we will have the limit of the increment in y divided by increment in x as that increment of x goes to 0 exists, that is your $f'(c)$ here. Suppose that exists. Now it is differentiable at $x = c$. That means this limit exists, and which we write as $f'(c)$. That is, the limit of $[f(c+h) - f(c)]/h$ as h goes to 0 exists.

Now what do we do? Because we want continuity, let us express $f(c+h) - f(c)$. Forget about the limit now, it is just the expression $f(c+h) - f(c)$. It is equal to $[f(c+h) - f(c)]/h$ times h. When you take the limit on the left side, and also limit on the right one, you see that limit of first expression exists, limit of the second expression also exists. So, limit of the product will be the product of the limits. The first one gives $f'(c)$, the second one gives 0. That is why it is equal to 0. So, what we get is: the limit of $f(c+h) - f(c)$ as h goes to 0, is equal to 0. And that satisfies the definition of continuity. Therefore, at c, f is continuous.

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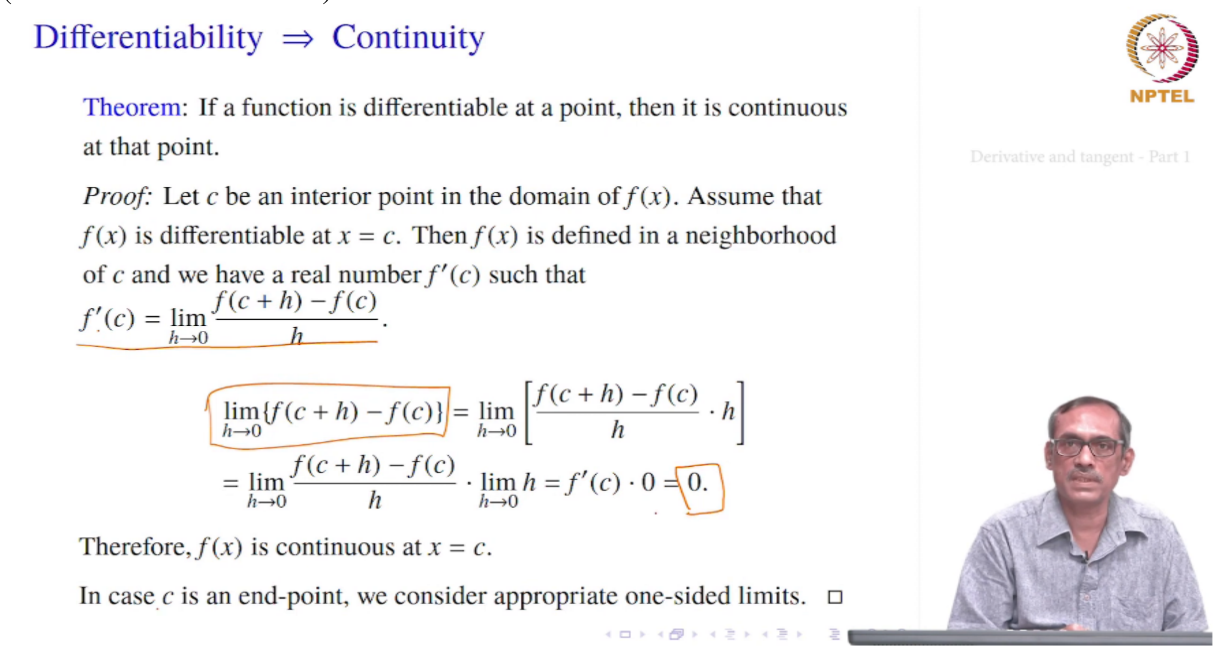

If it is left or right endpoint, then a similar thing happens. Only one sided limits will come here, and you would get the same result. All that we see is that differentiability at a point implies continuity at that point. Now, if you take stock, we have this one going along with the tangents.

When you say something is not differentiable, it can happen from the tangent side if you see geometrically, and also this condition that 'not continuous' also helps. So, these are the cases when a function is not differentiable at $x = c$.

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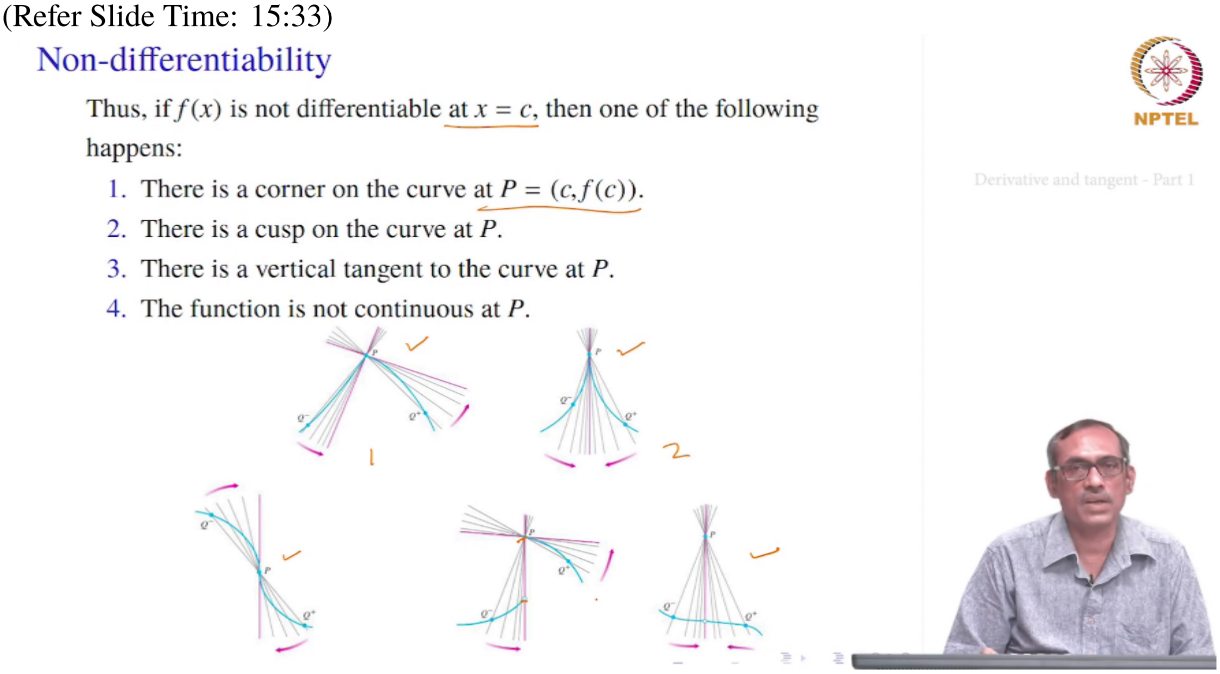

What happens is: there will be a corner at the curve at that point $(c, f(c))$. If you look at the curve it happens as in your $y = |x|$. There is a corner on the curve $y = |x|$ at $x = 0$.

Or there can be cusp, like your earlier curve $y = x^{2/3}$. What do you get there? Something like a cusp, which is this case. First case: there can be corner, or this second case: there can be cusp, as we have seen there.

Or there can be a vertical tangent to the curve as in $y = x^{1/3}$. This is the third case. Or we have this condition that the function is not continuous at $x = c$.

When it is not continuous, it can have two types of things. One, there is a jump so that the limit does not exist, limit of the function as x goes to c does not exist. Here, the left side limit is different from the right side limit.

Two, it can happen that the limit exists, the left side limit and the right side limit are equal to something some real number, but that is not the functional value. It is illustrated in this case.

These are the five possibilities that can happen when a function is not differentiable at a point; look at the geometries.

What happens is if it is not differentiable, then it can have jump discontinuity, it can have corners, it can have cusps and so on. What our result does not say is that, if the function is differentiable at a point, then that derivative is continuous. Our theorem does not say this. The derivative $f'(x)$ is a new function. This new function may not be continuous at that point. All that the theorem says that $f(x)$ is continuous at that point c, when $f'(c)$ exists; it says only that much. So, this is something else that the new function is continuous or not.

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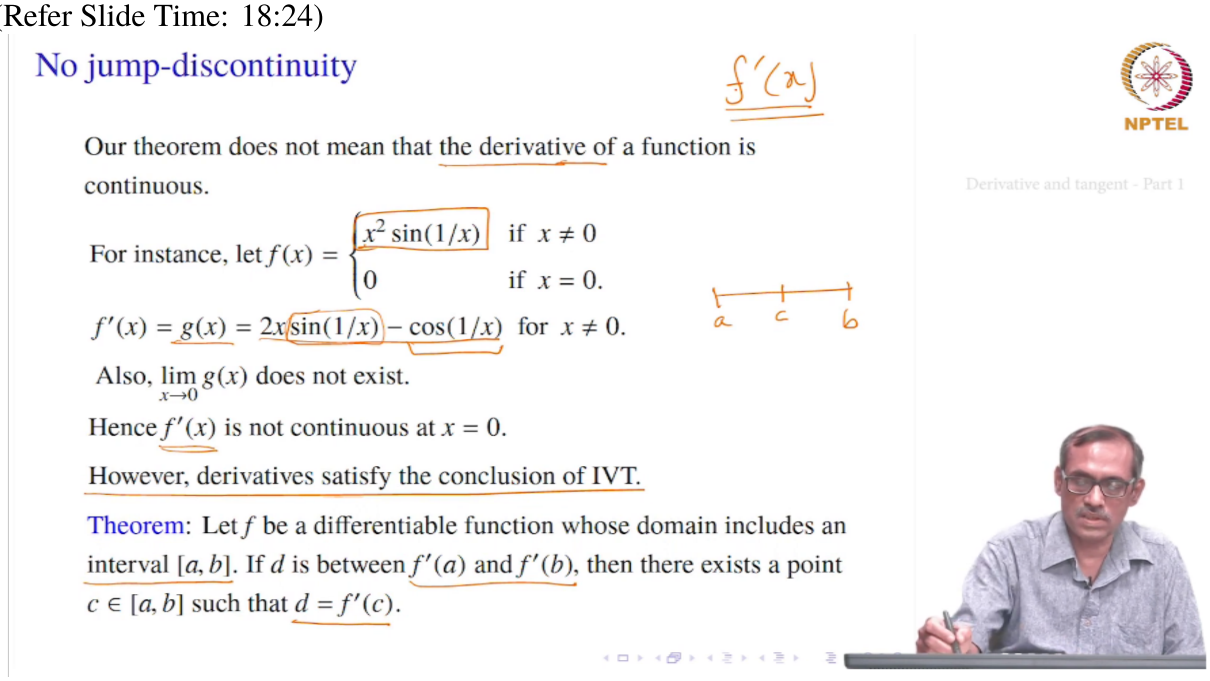

We can have examples for that. Let us take the function $f(x) = x^2 \sin(1/x)$ if $x \neq 0$, and $f(x) = 0$ if $x = 0$. It is not differentiable at $x = 0$, or is it differentiable at $x = 0$?

We have seen this earlier; we have found that at 0 it is differentiable. You can take $f(0 + h)$ and so on. This x^2 term dominates, $\sin(1/x)$ is like $1/x$ and that gives rise to x as one x cancels; then this gives rise to 0. And the functional value is 0. It can be shown that it is differentiable. But then what about the derivative? It is a new function. When $x \neq 0$, you will discover what it is. Certainly we have not done this yet. How to differentiate this product? It is a product of two functions. We know only for x^2; we know for $\sin x$. Using that we will find out soon that it is (or maybe in the next lecture) that the derivative will be this one: $2x\sin(1/x) - \cos(1/x)$, $x \neq 0$. That is how the derivative will look like.

You see that this new function $f'(x)$, which I write it as $g(x)$, its limit as x goes to 0 does not exist. Why is it so? Of course, this $2x\sin(1/x)$ goes to 0 as x goes to 0, since $\sin(1/x)$ is bounded by 1. So, this will give 0. But the limit of this factor $\cos(1/x)$ will not exist. This $1/x$ can take different types of values while going up to infinity, its limit goes to infinity. Near ∞, that is, in a neighborhood of ∞, you can find points so that cosine will become 0; you can find also points where cosine becomes 1 by taking suitable multiples of π or of $\pi/2$. So, that limits does not exist, it is not even ∞ or $-\infty$. It does not exist, and it is neither ∞ nor $-\infty$.

So, you see that for $g(x)$ or $f'(x)$, the limit does not exist at $x = 0$. So, it is not continuous at $x = 0$. But f is continuous, that is what our theorem says. Because it is differentiable at 0, the function f is continuous at 0.

But there is something else which happens for this derivative, that is, for $f'(x)$. It says that if a function is continuous on an interval, then it satisfies the conclusion of intermediate value theorem. That is, if you take an interval $[a.b]$, a closed interval, where f is continuous, then between $f(a)$ and $f(b)$ you take any value, that is achieved. That is, there is one x or c such that $f(c)$ will be equal to that value, which is in between $f(a)$ and $f(b)$. This is your intermediate value property or conclusion of intermediate value theorem.

If it is a derivative like your $g(x)$ (or $f'(x)$ here) then it satisfies this property. Though it is not continuous, this is satisfied. That is our theorem; it does not have jump discontinuity. There will not be a jump in $f'(x)$, though that may not be continuous.

We will not prove that theorem. It will again require our completeness property. We will see this: "Let f be a differential function whose domain includes an interval. That is important here because we want the conclusion of IVT. If d is between $f'(a)$ and $f'(b)$, then there exists a point $c \in [a, b]$ such that this d is achieved, that is, $d = f'(c)$." This is a nice property of derivatives. Sometimes we will be stating it as "the derivative does not have jump discontinuity." That is what it means.