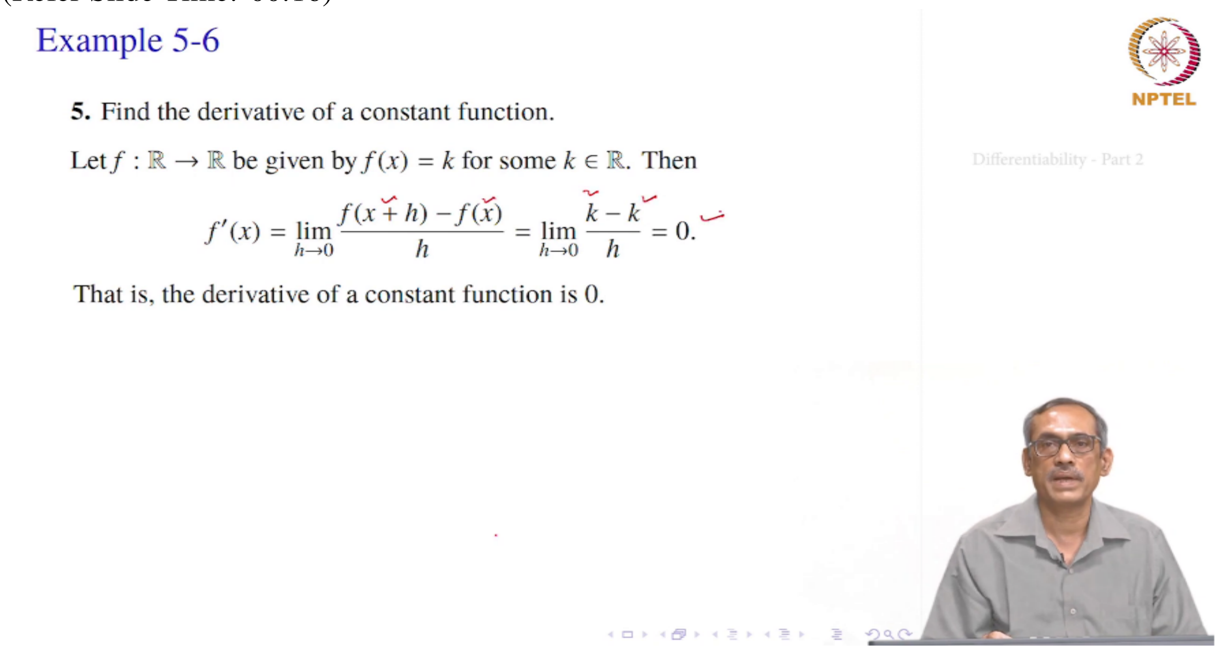
Basic Calculus - 1
Professor. Arindama Singh
Department of Mathematics
Indian Institute of Technology Madras
Lecture 13 - Part 2
Differentiability - Part 2

(Refer Slide Time: 00:16)



Let us consider another problem. We are asked to find the derivative of a constant function. That means $f(x)$ is defined as some k, a constant for all $x \in \mathbb{R}$; it is the constant function. Now, we go for the ratio, which is $[f(x+h) - f(x)]/h$. As $f(x+h)$ is k, and $f(x)$ is also k, that gives us 0. Therefore, its limit is 0, that is $f'(x)$, the new function which is the derivative of $f(x)$, is equal to 0 throughout, again, a constant function.

We consider the next problem. Suppose n is given to be a positive integer; one of $\{1, 2, 3, \ldots\}$, a natural number; and $f(x) = x^n$ is the power function defined throughout $\mathbb{R}$. We want to find its derivative at any x. It is claimed that the derivative will be nx^{n-1}. How do we show it?

Well, we start with the limit. We go on writing like this; it is a convenient way of writing; but actually, after this limit exists only, we can write this. As a trial, we go on doing it. If that exists, writing this is okay; if it does not exist, then we erase this and say that this $f'(x)$ does not exist.

So, let us try. The ratio is $[(x+h)^n - x^n]/h$. You can use the binomial theorem to expand this $(x+h)^n$, which is $x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + h^n$. Now, x^n and this x^n have opposite signs, so they cancel. And this h also gets canceled after that. Then, you have $nx^{n-1}nx$ plus h times this factor. As h goes to 0, this whole factor goes to 0. Therefore, the limit is nx^{n-1} as claimed. That is quite straightforward.

(Refer Slide Time: 01:30)

Example 5-6

5. Find the derivative of a constant function.

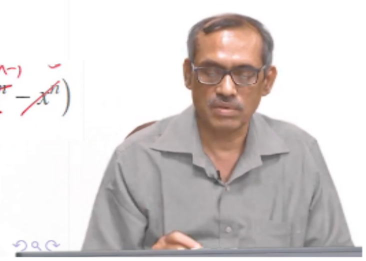
Let $f:\mathbb{R}\to\mathbb{R}$ be given by $f(x)=k$ for some $k\in\mathbb{R}$. Then

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{k-k}{h}=0.$$

That is, the derivative of a constant function is 0.

6. Let n be a positive integer, $f(x)=x^n$. Then $f'(x)=nx^{n-1}$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{(x+h)^n-x^n}{h} \\ &= \lim_{h\to 0}\frac{1}{h}\Big(x^n+nx^{n-1}h+\frac{1}{2!}n(n-1)x^{n-2}h^2+\cdots+nxh^{n-1}+h^n-x^n\Big) \\ &= \lim_{h\to 0}\Big[nx^{n-1}+h\Big(\frac{1}{2!}n(n-1)x^{n-1}+\cdots+h^{n-1}\Big)\Big] \end{aligned}$$



Fine, let us go to another problem. We want to find the derivatives of $\sin x$ and $\cos x$ if they exist. And here it is claimed that the derivative of $\sin x$ is $\cos x$, and the derivative of $\cos x$ is $\sin x$. We want to verify it. So, let us take the limit of the ratio as h goes to zero. That is $[\sin(x+h)-\sin x]/h$. We can use the sine half formula. It gives $2\cos(x+h/2)\sin(h/2)$. Then, what happens? It is $\sin(A+B)-\sin(A-B)$ formula; it gives you $2\cos[(A+B)/2]\sin[(A-B)/2]$. We do not know which one is bigger; that is why in we wrote the cosine form; you can also write in sine form, thinking that h is positive. Anyway, this is how it looks.

(Refer Slide Time: 03:15)

Example 7

The derivative of $\sin x$ is $\cos x$ and of $\cos x$ is $\sin x$.

$$\begin{aligned} (\sin x)' &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\ &= \lim_{h\to 0}\frac{2\cos(x+h/2)\sin(h/2)}{h} \\ &= \lim_{h\to 0}\cos(x+h/2)\lim_{h\to 0}\frac{\sin(h/2)}{h/2}=\cos x. \end{aligned}$$

$$\begin{aligned} (\cos x)' &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\ &= \lim_{h\to 0}\frac{-2\sin(x+h/2)\sin(h/2)}{h} \\ &= -\lim_{h\to 0}\sin(x+h/2)\lim_{h\to 0}\frac{\sin(h/2)}{h/2}=-\sin x. \end{aligned}$$

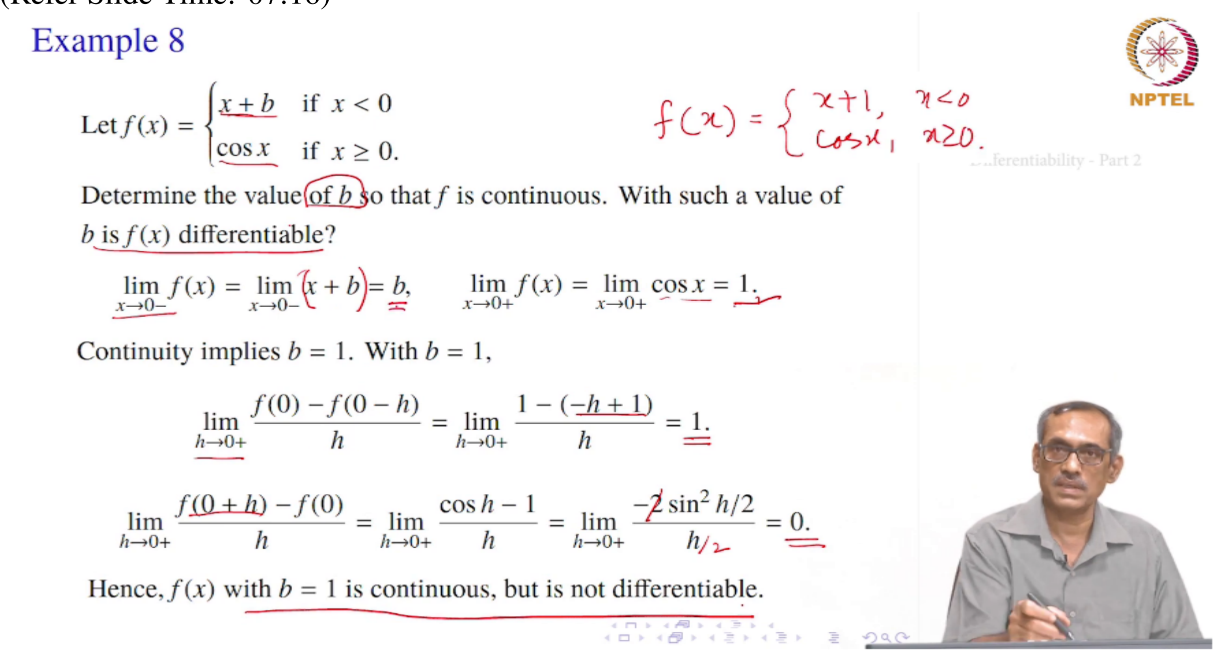
This gives rise to $\cos(x+h/2)$ which gives you $\cos x$ as $h\to 0$. Next, when h goes to 0, this is $\sin(h/2)$ by $h/2$ so that the limit is equal to 1. This h and this 2 came as $h/2$; and we know that limit to be equal to 1. So, when h goes to 0, we get $\cos x$. That is how we proceed.

Now for $\cos x$, similarly, we have $\cos(x+h) - \cos x$, which is $-2\sin(x+h/2)\sin(h/2)$; then divided by h. Again, $h/2$ comes here, $\sin(h/2)$ by $h/2$ becomes 1, and $\sin(x+h/2)$, as h goes to 0, will be $\sin x$. This minus sign remains, and you get $-\sin x$. That is a verification of the formulas for the derivatives of $\cos x$ and $\sin x$.

Let us take another problem. Here, the function is defined as $x + b$, b is an unknown here; we have not specified what this b is, for $x < 0$; and it is $\cos x$ for $x \geq 0$. It is defined for all real numbers x. The question is, this unknown b has to be determined so that f becomes a continuous function. We are using again continuity here. Once b is determined that so that f becomes continuous, we must check whether $f(x)$ is differentiable.

It looks that we have only one particular value of b that is assumed in the question. We have to see whether continuity of f gives one particular value of b or not. Or if it does not, then there will be a set of values. For all those values, we have to decide whether $f(x)$ is differentiable or not. So not only at one point but at any point, $f(x)$ is required to be differentiable. It means that we have to see whether it is differentiable at every point or not.

(Refer Slide Time: 07:16)



Our first job is to determine b so that f is continuous. And $f(x)$ is defined separately for $x < 0$ and for $x > 0$. We will take $x \to 0+$ and $x \to 0-$ separately. At 0 only there is a problem. As we have seen earlier that at the endpoints of the intervals, there can be possibly a break, a point of discontinuity. At other points the function $x + b$ is continuous; and $\cos x$ is also continuous. At the endpoint only we have to check.

When x is negative, you are taking $x \to 0-$ so that x is negative but near 0. You have the limit as x goes to $0-$ of this function $x + b$; it gives us b. And when $x > 0$, it is the other condition applicable. as $x \to 0+$, the limit of $f(x)$ will be the limit of $\cos x$ as x goes to $0+$; and that is equal to 1.

For $f(x)$ to be continuous, these two limits must exist and they should be equal, and they should be equal to the functional value. Now, what is the functional value $f(0)$? It is the second one which is applicable. It is $\cos 0 = 1$. All that we get from this is $b = 1$. Once $b = 1$, our f is continuous.

Now, with such a value of b we know what this $f(x)$ is. There is no unknown in $f(x)$. It is $x + 1$ if $x < 0$; and it is $\cos x$ is $x \geq 0$. With this $f(x)$, we have to see whether it is differentiable or not.

With this f, where $b = 1$, let us find out differentiability at $x = 0$. At other points it iss differentiable, because if $x < 0$, this is a function is known to be differentiable; this will be differentiable because of $x + b$. You can find it out directly also, because we have already known x^n. You can derive from there. (We need some algebra of differentiability here, like algebra of limit, it will follow from there.) And the next one $\cos x$ is also differentiable as you know. At the breakpoint only we have to check. It is the endpoint $x = 0$.

Then what will happen be at 0? We want to find out $f'(0)$. It will be the limit of the ratio. The left side limit will be $[f(0) - f(0 - h)]/h$ as h goes to 0+. That gives $f(0 - h) = f(-h)$ and $f(0) = 1$. When it is $-h$, that is negative. So, it is $x + 1$, that is $-h + 1$. So, this gives $-(-h) = h$, and h cancels. You get the limit as 1. That is the left side limit of the ratio for the derivative.

Similarly, we should get the right side also, which is $[f(0+h) - f(0)]/h$, and h is positive. The second one is applicable, which is $[\cos h - 1]/h$. And this limit, we know $\cos h - 1$ is $-2\sin^2(h/2)$. You bring this $h/2$ here, that 2 to this place. Now, $\sin(h/2)/(h/2)$ has limit 1. But another $\sin(h/2)$ remains, and its limt is 0. Therefore, the limit is 0. That means the function is not differentiable at 0. The left side limit of the ratio and right side limit of the ratio are not same. Therefore, limit does not exist.

Hence, $f'(0)$ does not exist. We cannot write even $f'(0)$ since the limit does not exist; the function is not differentiable at $x = 0$. At other places it is differentiable. So, it is continuous with $b = 1$ but not differentiable.

Let us take another problem. We are asked to find the derivative of this function $f(x) = (3x - 2)^{-1/2}$ from the first principles; that is, using the definitions. That is the only thing we have, of course.

Now, what is its domain? It is $(3x - 2)^{-1/2}$. At $x = 2/3$, it is not defined. And if $x < 2/3$, then it becomes negative. Negative to the power $-1/2$ is not meaningful. So, its domain is the open interval $(2/3, \infty)$. Choose any x between from this interval; it is an interior point of this. You have to get the limit. Now, $[f(x + h) - f(x)]/h$ is $[(3(x + h) - 2)^{-1/2} - (3x - 2)^{-1/2}]/h$. We will be computing its limit as $h \to 0$.

How do you compute this limit? Let us write it in a better way. This gives the square root of this; 1 divided by this; it goes to the numerator, this goes to the numerator on the other side; and divided by this factor; into this factor. So, this one is there; it is $1/h$; h is there. That is how it looks now. How do we get the limit of it?

(Refer Slide Time: 11:59)

Exercise 1

Find the derivative of $f(x) = (3x-2)^{-1/2}$ from the first principles.

Ans: Notice that $f(x)$ has domain as $(\frac{2}{3}, \infty)$. For such x

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(3(x+h)-2)^{-1/2} - (3x+2)^{-1/2}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3x-2} - \sqrt{3x+3h-2}}{h\sqrt{(3x+3h-2)(3x-2)}} \times \frac{\sqrt{3x-2} + \sqrt{3x+3h-2}}{\sqrt{3x-2} + \sqrt{3x+3h-2}} \\
&= \lim_{h\to 0} \frac{3x-2-(3x+3h-2)}{h\sqrt{(3x-2)(3x+3h-2)}(\sqrt{3x-2} + \sqrt{3x+3h-2})} \\
&= \frac{-3}{\sqrt{(3x-2)(3x-2)}(\sqrt{3x-2} + \sqrt{3x-2})} \\
&= \frac{-3}{2(3x-2)\sqrt{3x-2}}.
\end{aligned}
$$

Differentiability - Part 2

NPTEL

Well, when h goes to 0, we have to do something because this h has to be taken away to find the limit. You multiply with $\sqrt{3x-2} + \sqrt{3x+3h-2}$ and also divide it. It will be give us this expression without square root. At the top we find this; it cancels $3x$ with $-3x$ and 2 with 2, so you get $-3h$, and one h is there. This h cancels with $3h$, so we get -3 on the top. And on the down, we have the two other expressions, which when h goes to 0, gives rise to $3x-2$ into $3x-2$, here also $3x-2+3x-2$; and it is just a simplification of that. This gives -3 divided by $3x-2$, and this is $2\sqrt{3x-2}$.

So, that is the answer, that at every point of its domain, it is differentiable and its derivative is equal to this one: $-3/[2(3x-2)\sqrt{3x-2}]$.

(Refer Slide Time: 15:37)

Exercises 2-3

2. Find $dr/d\theta$ at $\theta = 0$, where $r = 2/\sqrt{4-\theta}$.

Ans: $$\frac{dr(0)}{d\theta} = \lim_{h\to 0} \frac{(2/\sqrt{4-h}) - 1}{h} = \lim_{h\to 0} \frac{2-\sqrt{4-h}}{h\sqrt{4-h}}$$

$$= \lim_{h\to 0} \frac{4-(4-h)}{h\sqrt{4-h}(2+\sqrt{4-h})} = 1/8.$$

$2+\sqrt{4-h}$

Differentiability - Part 2

NPTEL

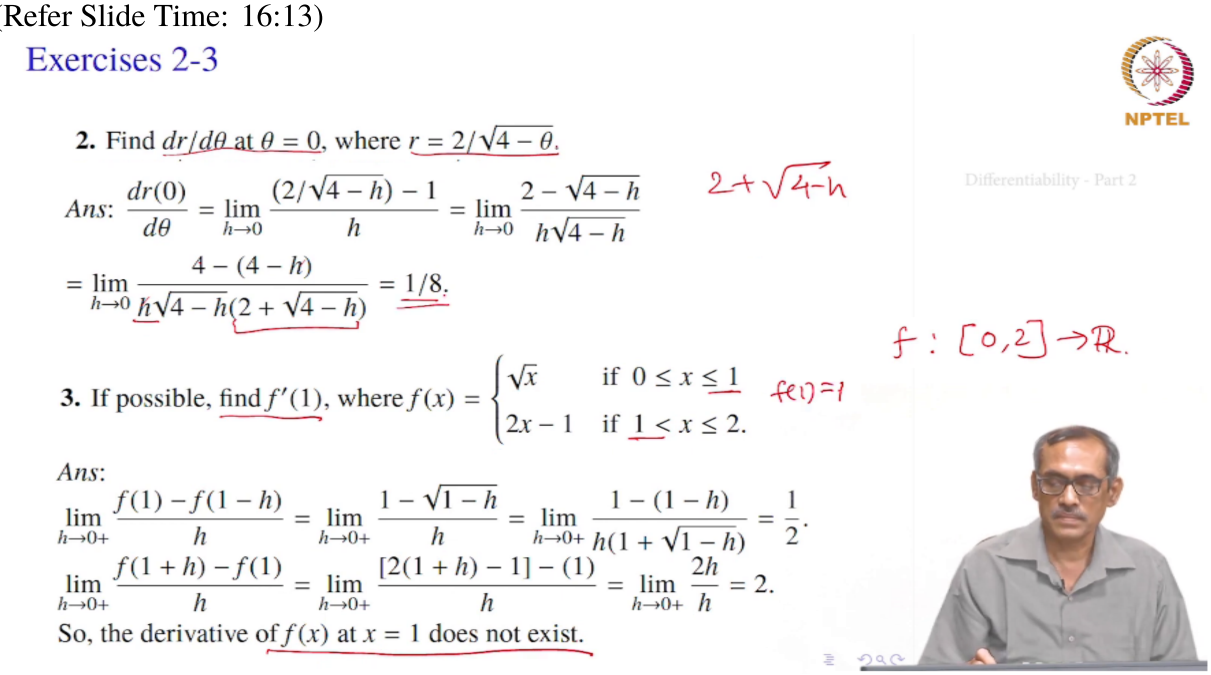
The next problem asks to find $dr/d\theta$ at $\theta = 0$, where r is equal to $2/\sqrt{4-\theta}$. So, r is a function of θ now. Instead of x and y, we have r and θ, and we want to find its derivative at $\theta = 0$. if possible.

The first thing is where it is not defined? It is the square root of $4-\theta$.At $\theta = 4$, it is not defined. And when it is negative also it is not defined, that is, when $4 < \theta$. For everything else, it is defined. But we are concerned about $\theta = 0$. So, we are safe in that sense as we can have always a neighborhood around 0, which is inside the domain.

So, we take the ratio. It is $[2/\sqrt{4-(\theta+h)} - 2/\sqrt{4-\theta}]/h$. At $\theta = 0$, it is $[2/\sqrt{4-h} - 2/\sqrt{4}]/h$ which is equal to $[2/\sqrt{4-h} - 1]/h$, and that is equal to $[2-\sqrt{4-h}]/[h\sqrt{4-h}]$. Again, we may have to multiply similar things, so that this h will be removed.

So, we multiply $2+\sqrt{4-h}$ on the top and on the bottom. The top becomes $4-(4-h) = h$ and the bottom becomes $h\sqrt{4-h}(2+\sqrt{4-h})$. That h cancels with this. Once that is canceled, you have 1 on the numerator, and the denominator gives you 2 as h goes to 0. The first factor into next is 2 plus 2; it is 4 into 2, that is, 8. So this answer is $1/8$. That is fine.

(Refer Slide Time: 16:13)



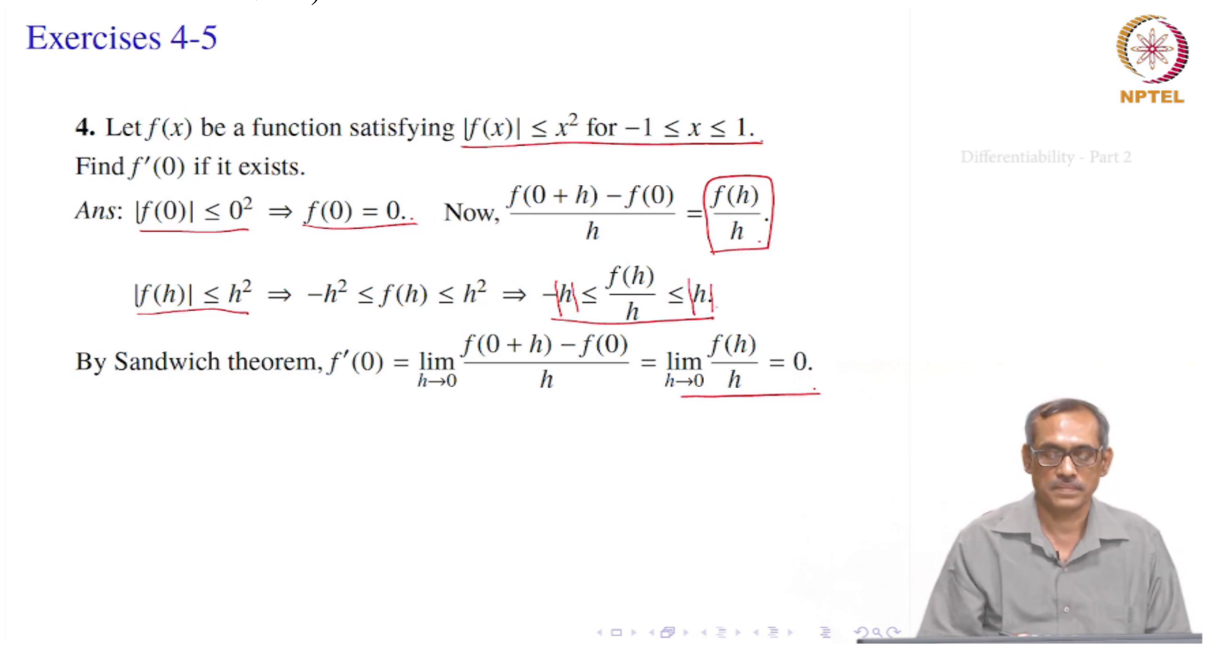
Let us go to the next problem. Here we want to find, if possible, the derivative of this function at $x = 1$. The function is given as: from 0 to 1, it is $\sqrt{x}$, and from 1 to 2, it is $2x-1$. The point 1 is included in the first one. So, the domain of definition of the function is $[0,2]$; $f : [0,2] \to \mathbb{R}$, given this way. On the left side of $x = 1$, there is one definition, on the right side there is another definition, and $f(1) = 1$. Let us try now. We have to get both sides limits.

Let us take the left side limit first. The left side limit is that of $[f(1-h) - f(1)]/h$ as h goes to 0+. Usually, instead of $h \to 0-$, this becomes convenient; and we take it this way. So, it is the limit of $[f(1) - f(1-h)]/h$ as h goes to 0+. That gives the limit of $[1-\sqrt{1-h}]/h$ as $h \to 0+$. Again, we multiply $1+\sqrt{1-h}$ on the numerator and on the denominator. The numerator becomes $1-(1-h)$. One 1 cancels, then plus h; again, h cancels; and as $h \to 0+$, it gives $1+\sqrt{1} = 2$. So, the answer is $1/2$.

Next, we take the right side limit. That gives $[f(1+h) - f(1)]/h$. This is $[2(1+h) - 1 - 1]/h$, which is equal to $2h$. Now the h cancels, and you get 2.

Now, the two limits are different. So, the function is not differentiable at $x = 1$. We cannot find $f'(1)$, because the derivative of $f(x)$ does not exist or the function is not differentiable at $x = 1$.

(Refer Slide Time: 19:56)



Let us take another problem. Let $f(x)$ be a function. It is a different kind of problem; it is not giving as $f(x)$ is equal to what, but only an inequality is given. It says that $f(x)$ is a function which satisfies $|f(x)| \le x^2$ for $-1 \le x \le 1$. Can we find $f'(0)$ by using this condition? Let us try.

First, to get the ratio; we must find $f(0)$. How to get $f(0)$? From this inequality, we have $|f(0)| \le 0^2$. That means $f(0) = 0$. Now we should take the ratio: $[f(0+h) - f(0)]/h$. Since $f(0) = 0$, it is just $f(h)/h$. Now, $|f(h)| \le h^2$. That gives rise to the inequality $-h^2 \le f(h) \le h^2$.

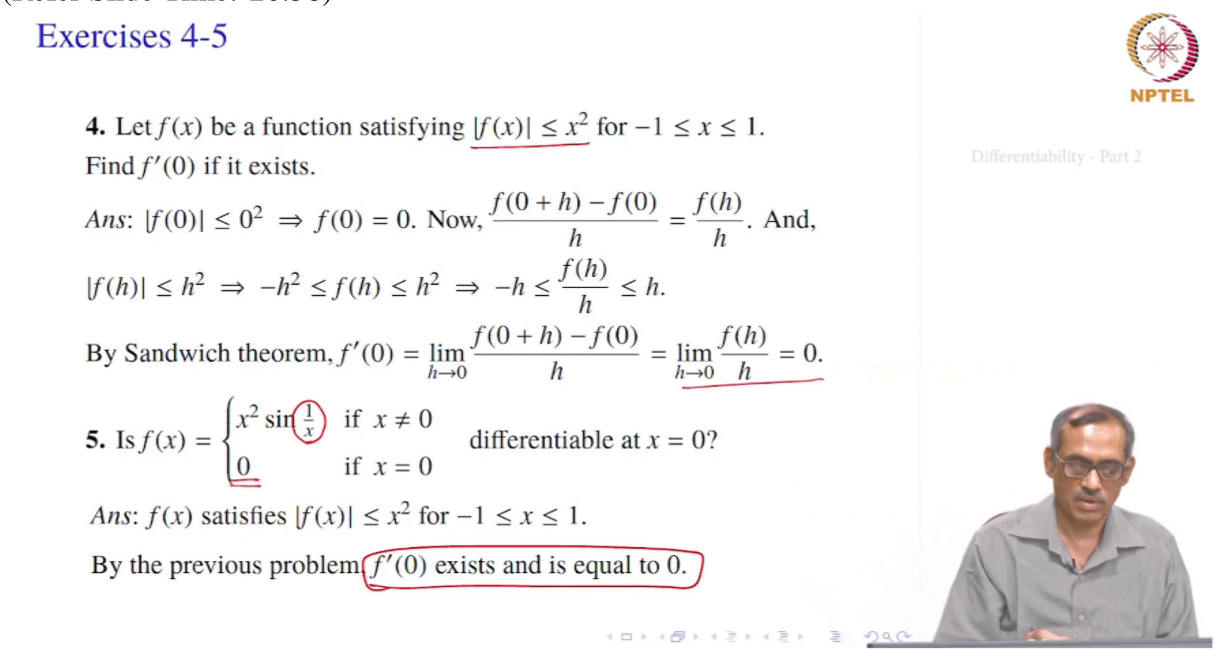
Think of h as a positive number here. That gives $-h \le f(h)/h \le h$. If h is negative, then of course, you get $f(h)/h$ lying between h and $-h$, the other way around. So, you may say that $f(h)/h$ is lying between $-|h|$ to $|h|$.

Now, you use sandwich theorem. Since it is $-|h|$ to $|h|$, we use sandwich theorem. As $h \to 0$, both $-|h|$ and $|h|$ go to 0. Therefore, $f(h)/h$ must go to 0 as h goes to 0. So, we have got $f'(0)$ this way.

Let us go to next problem. It is basing on this idea. Here, $f(x) = x^2 \sin(1/x)$, if x is non-zero, and it is 0 if $x = 0$. In $\sin(1/x)$ we have $1/x$, so it is not defined at $x = 0$; and that is why it is defined at 0 separately. Its value is 0 at $x = 0$. The question is whether $f(x)$ is differentiable at $x = 0$?

You see that it satisfies this inequality of the fourth problem. Because $\sin(1/x)$ is bounded by -1 and 1, $|f(x)| \le x^2$ from -1 to 1. We have seen that in this case, $f(0)$ must be 0 and it is already given as 0. That satisfies all the conditions of the fourth problem.

(Refer Slide Time: 20:58)



So, in this case also we conclude that it is differentiable. We have found out that $f(0) = 0$ and $f'(0) = 0$. That is, f is differentiable at $x = 0$ and the value of $f'(0)$ is 0. So, we just apply that and again conclude that $f'(0) = 0$.

Of course, you can do directly; the same method will be used to compute this; and find that $f'(0) = 0$. But, one matter of concern: since this is defined only for $x \neq 0$ this way, you can of course differentiate it at every point not equal to 0. That differentiation will give rise to some result. If you take the limit of that derivative, that is not the same thing as finding $f'(0)$. That would say whether the limit of $f'(x)$ as x goes to 0 exists or not. That is not same thing as $f'(0)$. $f'(0)$ is the limit of this ratio, which we computed to be equal to 0. Therefore, $f'(0)$ exists and is equal to 0. We stop here.